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Fixed points of isometries of a Finsler space

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Abstract. In this paper, we study the zero points of Killing vector fields of a Finsler space. It turns out that Kobayashi's result on Killing vector fields of a Riemannian manifold can be generalized to the Finslerian case. In particular, any connected component of the set of zero points of a Killing vector field in a Finsler space is a totally geodesic closed submanifold and the Euler number of the manifold is the sum of the Euler numbers of the connected components provided the manifold is compact. Some interesting corollaries are also given.

Introduction

Let (M, F) be a Finsler space, where F is positively homogeneous of degree one (but perhaps not absolutely homogeneous). A (smooth) vector field ξ on M is called a Killing vector field if any local one-parameter transformation group generated by ξ consists of local isometries of (M, F). Killing vector fields play a very important role in Riemannian geometry, see for example [7]. In Finsler geometry, it has also proven to be a useful tool to study some problems related to isometries, homogeneous spaces and curvatures. For example, it is the vital tool in our previous work [3] to prove that a homogeneous Finsler space with non-positive flag curvature and negative Ricci scalar is simply connected.

The purpose of this paper is to study the zero points of Killing vector fields in a Finsler space. Since a Killing vector field can be viewed as an infinitesimal

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isometry, we can also view the zero points as fixed points of isometries. The main result of this paper is the following

Theorem. *Let (M, F) be a (connected) Finsler space of dimension n and ξ be a Killing vector field of (M, F). Let V be the set of points in M where ξ vanishes and let $V = \cup V_i$, where the V_i's are the connected components of V. Assume that V is not empty. Then we have*

1. *Each V_i is a totally geodesic closed submanifold of M and the co-dimensions of the V_i's are even.*
2. *If (M, F) is forwardly complete and $x \in V_i$, $y \in V_j$ with $i \neq j$, then there is a one-parameter family of geodesics connecting x and y. In particular, x and y are conjugate to each other.*
3. *If M is compact, then the Euler number of M is the sum of Euler numbers of the V_i's, i.e.,*
$$\chi(M) = \sum \chi(V_i).$$

We remark that Kobayashi's original proof of the same theorem in the Riemannian case [6] does not apply to the Finslerian case. So we need some new technical arguments. In fact the proof of this theorem uses some deep results on maximal compact subgroups of a connected semisimple group and the structure theory of compact Lie groups. The theorem has the following corollaries:

Corollary 1. *Let $\mathfrak{k}$ be an abelian (finite-dimensional) Lie algebra consisting of Killing vector fields and V the set of points where every element of $\mathfrak{k}$ vanishes. Then the same conclusions as in the main theorem hold.*

Corollary 2. *Under the same assumptions as in Corollary 1, if (M, F) is a globally symmetric (resp. locally symmetric) Finsler space, then so is each V_i.*

Remark. A Finsler space (M, F) is called globally symmetric if each point is the isolated fixed point of an involutive isometry. It is called locally symmetric if for any $x \in M$ there exists a neighborhood U of x such that the local geodesic symmetry is an isometry on U (see [4]).

Corollary 3. *Under the same assumptions as in Corollary 1, if (M, F) is forwardly complete and the flag curvature is non-positive, then V is either empty or connected.*

Corollary 4. *Let (M, F) be a compact Finsler space of dimension $2m$. Suppose a torus group of dimension m acts on M differentiably and effectively. Then the Euler number of M is zero or positive according as the fixed point set V is* *empty or not. If M is orientable and V is non-empty, then the Euler number of M is ≥ 2.*

1. The proof

First we give the proof of the theorem.

(1) Suppose $x \in V$. Then we can take a neighborhood U of x such that ξ generates a local one-parameter transformation group from U into M. That is, there is a set of mappings $\{\varphi_t \mid |t| < \varepsilon,\ \varepsilon > 0\}$ such that each φ_t is a diffeomorphism from U onto $\varphi_t(U)$ and

$$\varphi_t(\varphi_s(p)) = \varphi_{t+s}(p), \quad \text{for } p \in U,\ \varphi_s(p) \in U,\ |t+s| < \varepsilon.$$

Moreover, for any $p \in U$, $\xi(p)$ is just the initial vector of the curve $\varphi_t(p)$. Since $\xi(x) = 0$, any φ_t keeps x fixed. Therefore the differentials $(\mathrm{d}\varphi_t)|_x$, $|t| < \varepsilon$, are linear automorphisms of the tangent space $T_x(M)$. Since φ_t is an (local) isometry, $(\mathrm{d}\varphi_t)|_x$ keeps the length of any vector in $T_x(M)$. This implies that $(\mathrm{d}\varphi_t)|_x$ is a linear isometry of the Minkowski space $(T_x(M), F|_x)$. Let L denote the group of linear isometries of $(T_x(M), F|_x)$. Then a result of H. C. Wang asserts that L is a compact subgroup of the general linear group $\mathrm{GL}(T_x(M))$ ([8]). Fix a basis $\{e_1, e_2, \ldots, e_n\}$ of $T_x(M)$. Then $\mathrm{GL}(T_x(M))$ is identified with the group $\mathrm{GL}(n, \mathbb{R})$ of all $n \times n$ real invertible matrices. Since L is a compact subgroup of $\mathrm{GL}(n, \mathbb{R})$, the unit component L_0 of L must be contained in the subgroup $\mathrm{SL}(n, \mathbb{R})$ of matrices of determinant 1. Now $\mathrm{SL}(n, \mathbb{R})$ is a connected semisimple Lie group. Therefore any two maximal compact subgroups of it are conjugate [5]. Since $\mathrm{SO}(n)$ is a maximal subgroup of $\mathrm{SL}(n, \mathbb{R})$ [5], there exists $g \in \mathrm{SL}(n, \mathbb{R})$ such that $g^{-1}L_0 g \subset \mathrm{SO}(n)$. Now consider the subset $\{(\mathrm{d}\varphi_t)|_x\}$ of L. It is obvious that it is contained in L_0. Thus $g^{-1}\{(\mathrm{d}\varphi_t)|_x\}g$ is contained in a one-parameter subgroup of $\mathrm{SO}(n)$. Since any two maximal connected commutative subgroups (i.e., the maximal tori) are conjugate [5] and the matrices of the form

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & \\ & & \ddots & & & \\ & & & \cos\theta_r & \sin\theta_r & \\ & & & -\sin\theta_r & \cos\theta_r & \\ & & & & & \mathrm{I}_{n-2r} \end{pmatrix}, \quad r \geq 0,\ \theta_i \in \mathbb{R}$$

constitute a maximal torus of $\mathrm{SO}(n)$, there is $g_1 \in \mathrm{SO}(n)$, $s \geq 0$ and $\gamma_i \in \mathbb{R}$, $\gamma_i \neq 0$ such that

$$g_1^{-1}(g^{-1}\{(\mathrm{d}\varphi_t)|_x\}g)g_1 = \begin{pmatrix} \cos t\gamma_1 & \sin t\gamma_1 & & & & \\ -\sin t\gamma_1 & \cos t\gamma_1 & & & & \\ & & \ddots & & & \\ & & & \cos t\gamma_s & \sin t\gamma_s & \\ & & & -\sin t\gamma_s & \cos t\gamma_s & \\ & & & & & \mathrm{I}_{n-2s} \end{pmatrix}$$

Thus there is a basis $\{w_1, w_2, \ldots, w_n\}$ of $T_x(M)$ such that the matrix of $(\mathrm{d}\varphi_t)|_x$ under this basis is as the right hand of the above equation.

If $n - 2s = 0$, then n is even and x is an isolated fixed point of φ_t, hence an isolated zero point of ξ. Suppose $n - 2s > 0$. Then for any vector $v \in S$, where S is the span of $w_{2s+1}, w_{2s+2}, \ldots, w_n$, the geodesic emanating from x with the direction v must be kept fixed by the one-parameter group φ_t. This can be easily seen from the obvious fact that in a Finsler space an isometry sends a geodesic to a geodesic and the uniqueness of the (constant speed) geodesics. Now we assert that there exists a neighborhood W of x such that the set of such geodesics form a $n - 2s$-dimensional submanifold W' of W. This is not obvious as in the Riemannian case because in a general Finsler space the exponential mapping is only C^1 at the zero section (but it is smooth away from the zero section, cf. [1]). Let c be a positive number and define as in [1]

$$\mathcal{B}_x^+(c) = \{y \in M \mid d(x,y) < c\}, \quad \mathcal{B}_x^-(c) = \{y \in M \mid d(y,x) < c\}.$$

Then for sufficiently small c the exponential mapping is a C^1 diffeomorphism from $B_x(c) = \{v \in T_x(M) \mid F(v) < c\}$ onto $\mathcal{B}_x^+(c)$. Thus the exponential mapping is a C^1 diffeomorphism from $S \cap B_x(c)$ onto $\exp(S \cap B_x(c))$. Hence the latter is a $n - 2s$-dimensional C^1 submanifold of $\mathcal{B}_x^+(c)$. To prove our assertion we only need to proceed to take c so small so that every pair of points in $\mathcal{B}_x^+(c) \cap \mathcal{B}_x^-(c)$ can be joint by a unique minimizing geodesic (see [1]). Fix an arbitrary point z in $\exp(S \cap B_x(c)) \cap (\mathcal{B}_x^+(c) \cap \mathcal{B}_x^-(c))$ and consider the exponential mapping at z. Since each φ_t (We can take c small enough so that $\mathcal{B}_x^+(c) \subset U$) keeps z fixed it is easily seen that there exists a $n - 2s$-dimensional subspace S' of $T_z(M)$ such that exp is a (smooth) diffeomorphism from $S' \cap B_z(c) - \{0\}$ onto $\exp(S' \cap B_z(c) - \{0\})$, which is a neighborhood of x in $\exp(S \cap B_x(c))$. This proves our assertion. Now we show that any zero point of ξ in $\mathcal{B}_x^+(c) \cap \mathcal{B}_x^-(c)$ must be in $\exp(S \cap B_x(c))$. Let y be a zero point of ξ in $\mathcal{B}_x^+(c) \cap \mathcal{B}_x^-(c)$. Select a geodesic τ joining x and y. Since

x and y are zero points of ξ, φ_t keeps x and y fixed. Hence φ_t must keep the geodesic τ point-wise fixed. Therefore the initial vector of τ at x must be in S. This proves that y is in $\exp(S \cap B_x(c))$. Hence each V_i is a closed submanifold and the codimension is even. From the above arguments we easily see that each V_i must be a totally geodesic submanifold.

(2) Suppose $x \in V_i$, $y \in V_j$, $i \neq j$ and (M, F) is forwardly complete. By the Hopf–Rinow theorem ([1]), there is a geodesic τ joining x and y. The restriction of the Killing vector field ξ to τ is a Jacobi field along τ (see [3]). Hence $\xi|_\tau$ is the variation vector field of a one-parameter family geodesics [1]. Since x and y are zero points of ξ, this family of geodesics can be taken to start from x and to end at y. However, since $i \neq j$, this family of geodesics can not be left fixed by the local transformation group generated by ξ. Otherwise there would be a curve consisting of zero points of ξ and joining V_i and V_j, contradicting the definition of V_i.

(3) Since M is compact, the Killing vector field ξ is complete. Hence ξ generates a global one-parameter group of transformations φ_t, $-\infty < t < \infty$. Any φ_t is an isometry of the Finsler space (M, F). Let G be the full group of isometries of (M, F). Then G is a Lie transformation group of M and for any $x \in M$, the isotropic subgroup G_x of G at x is a compact subgroup of G [2]. Since M is compact, G is a compact Lie group (this can be seen from the observation that the orbit of G at a point $x \in M$ is a closed submanifold of M (see [5]) and $G \cdot x = G/G_x$). Let G_0 be the unit component of G and $\mathrm{d}\mu$ be the standard normalized bi-invariant Haar measure of G_0. Fix any Riemannian metric h on M and for $x \in M$, $v_1, v_2 \in T_x(M)$ define

$$h_1(x)(v_1, v_2) = \int_{G_0} h(g(x))(\mathrm{d}g|_x(v_1), \mathrm{d}g|_x(v_2))\mathrm{d}\mu(g).$$

Then it is easily seen that h_1 is a Riemannian metric on M. In fact, the only thing we need to check is the smoothness of h_1. But this follows easily from the smoothness of the action of G on M. Now by the definition of h_1, we see that h_1 is invariant under the action of G_0. Thus every element of G_0 is an isometry of h_1. It is easily seen that the one-parameter subgroup $\{\varphi_t\}$ must be contained in G_0. Thus any φ_t is an isometry of the Riemannian metric h_1. This means that the vector field ξ is also a Killing vector field with respect to the Riemannian metric h_1. Consequently the conclusion of (3) follows from the Riemannian case in [6]. $\square$

It is obvious that the above argument can also be used to prove Corollary 4. Other corollaries can be proved similarly as in the Riemannian case. We omit the details here. See [6].

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