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(α, β)-metrics with relatively isotropic mean Landsberg curvature

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Abstract. (α, β)-metrics form an important class of computable Finsler metrics. In this paper, we obtain firstly a formula of mean Cartan torsion for (α, β)-metrics and characterize Riemann metrics among (α, β)-metrics. Further, we obtain a sufficient and necessary condition for an (α, β)-metric to be of relatively isotropic mean Landsberg curvature.

1. Introduction

In Finsler geometry, there are several very important non-Riemannian quantities. The Cartan torsion $\mathbf{C}$ is a primary quantity. There is another quantity which is determined by the Busemann–Hausdorff volume form, that is the so-called distortion τ. The vertical differential of τ on each tangent space gives rise to the mean Cartan torsion $\mathbf{I} := \tau_{y^k} dx^k$. $\mathbf{C}$, τ and $\mathbf{I}$ are the basic geometric quantities which characterize Riemannian metrics among Finslers metrics. Differentiating $\mathbf{C}$ along geodesics gives rise to the Landsberg curvature $\mathbf{L}$.The horizontal derivative of τ along geodesics is the so-called S-curvature $\mathbf{S} := \tau_{|k} y^k$. The horizontal derivative of $\mathbf{I}$ along geodesics is called the mean Landsberg curvature $\mathbf{J} := \mathbf{I}_{|k} y^k$. The Riemann curvature measures the shape of the space while the non-Riemannian quantities describe the change of the "color" on the space. Hence

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Finsler spaces are "colorful" geometric spaces. It is found that the flag curvature is closely related to these non-Riemannian quantities[3], [9], [10].

By the definition, $\mathbf{J}/\mathbf{I}$ can be regarded as the relative growth rate of the mean Cartan torsion along geodesic. We call a Finsler metric F is of *relatively isotropic mean Landsberg curvature* if F satisfies $\mathbf{J} + cF\mathbf{I} = 0$, where $c = c(x)$ is a scalar function on the Finsler manifold. In particular, when $c = 0$, Finsler metrics with $\mathbf{J} = 0$ are called *weakly Landsberg metrics*. Many known Finsler metrics satisfy $\mathbf{J} + cF\mathbf{I} = 0$ (cf. [3], [4], [9]). In [11], Z. Shen proves that a projectively flat Randers metric of constant flag curvature on an n-dimensional manifold is either locally Minkowskian or after a scaling, isometric to a Finsler metric on the unit ball $\mathbb{B}^n$ in the following form

$$F_a = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2} \pm \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \quad y \in T_x\mathbb{R}^n, \tag{1}$$

where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$. The Randers metric in (1) satisfies $\mathbf{J} \pm \frac{1}{2}F_a\mathbf{I} = 0$. In [4], the first author and Z. Shen classify Randers metrics of isotropic flag curvature $\mathbf{K} = \mathbf{K}(x)$ satisfying $\mathbf{J} + c(x)F\,\mathbf{I} = 0$ for some $c(x)$. Further, Cheng–Mo–Shen characterize flag curvature of Finsler metrics of scalar flag curvature with relatively isotropic mean Landsberg curvature [3].

In the past several years, we witness a rapid development in Finsler geometry. Various curvatures have been studied and their geometric meanings are better understood. This is partially due to the study of a special class of Finsler metrics. The special Finsler metrics we are going to discuss are expressed in terms of a Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_iy^i$. They are called (α, β)-metrics. The simplest (α, β)-metrics are the Randers metrics $F = \alpha + \beta$. (α, β)-metrics form an important class of Finsler metrics with many applications in physics and biology (cf. [1]). Most important, (α, β)-metrics are " computable" and they are of many interesting curvature properties.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemannian metric and $\beta = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M. Using α and β one can define a function F on TM as follows

$$F = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{2}$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_o, b_o)$. The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by

$$\|\beta_x\|_\alpha := \sup_{y \in T_xM} \frac{\beta(x, y)}{\alpha(x, y)} = \sqrt{a^{ij}(x)b_i(x)b_j(x)}.$$

We assume that

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\quad (|s|\le b<b_0),\tag{3}$$

so that $F=\alpha\phi(s)$, where $s=\beta/\alpha$, is a positive definite Finsler metric if and only if $b(x):=\|\beta_x\|_\alpha<b_0$ for all $x\in M$.

Recently, B. Li and Z. Shen characterize weakly Landsberg metrics in (α,β)-metrics and show that there exist weakly Landsberg metrics which are not Landsberg metrics in dimension greater than two [8].

In this paper, we study (α,β)-metrics with relatively isotropic mean Landsberg curvature and prove the following

Theorem 1.1. *Let $F=\alpha\phi(\beta/\alpha)$ be an (α,β)-metric on an n-dimensional manifold M $(n\ge 3)$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form. Then F is of relatively isotropic mean Landsberg curvature, i.e. there exists a scalar function $c=c(x)$ on M such that $\mathbf{J}+c(x)F\,\mathbf{I}=0$, if and only if β satisfies*

$$s_{ij}=0,\tag{4}$$

$$r_{ij}=k(b^2a_{ij}-b_ib_j)+\sigma b_ib_j,\tag{5}$$

where $k=k(x)$ and $\sigma=\sigma(x)$ are scalar functions on M and $\phi=\phi(s)$ satisfies the following ODE:

$$\big\{\Psi_1k+s\sigma\Psi_3\big\}+c(x)\Phi(\phi-s\phi')=0,\tag{6}$$

where Φ, Ψ_1, Ψ_3 are defined as follows

$$Q:=\frac{\phi'}{\phi-s\phi'},\quad \Delta:=1+sQ+(b^2-s^2)Q',\tag{7}$$

$$\Phi:=-(n\Delta+1+sQ)(Q-sQ')-(b^2-s^2)(1+sQ)Q'',\tag{8}$$

$$\Psi_1:=\sqrt{b^2-s^2}\,\Delta^{1/2}\left[\frac{\sqrt{b^2-s^2}\,\Phi}{\Delta^{3/2}}\right]',\tag{9}$$

$$\Psi_2:=2(n+1)(Q-sQ')+3\frac{\Phi}{\Delta}\tag{10}$$

and

$$\Psi_3:=\frac{s}{b^2-s^2}\Psi_1+\frac{b^2}{b^2-s^2}\Psi_2.\tag{11}$$

By Theorem 1.1, we can see that $\mathbf{J} = 0$ if and only if β satisfies

$$s_{ij} = 0, \quad r_{ij} = k(b^2 a_{ij} - b_i b_j) + \sigma b_i b_j$$

and $\phi = \phi(s)$ satisfies

$$\Psi_1 k + s\sigma\Psi_3 = 0.$$

This is just the result of Proposition 3.1 in [8].

Example 1.2. Let $\phi(s) = 1 + s$. Then $F = \alpha\phi(\beta/\alpha) = \alpha + \beta$ is a Randers metric on the manifold. By a direct computation, we can prove that F is of relatively isotropic mean Landsberg curvature, $\mathbf{J} + c(x)F\mathbf{I} = 0$, if and only if β satisfies (4) and (5) with $k = 2c/b^2$ and $\sigma = 2c(1 - b^2)/b^2$, that is, β is closed and $r_{ij} = 2c(a_{ij} - b_i b_j)$. This result is first given by the first author and Z. Shen in [4].

2. Preliminaries

Let $F = F(x, y)$ be a Finsler metric on an n-dimensional manifold M. Let

$$g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}(x, y)$$

and $(g^{ij}) := (g_{ij})^{-1}$. For a non-zero vector $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, F induces an inner product on T_xM

$$g_y(u, v) = g_{ij} u^i v^j,$$

where $u = u^i \frac{\partial}{\partial x^i}$, $v = v^j \frac{\partial}{\partial x^j} \in T_xM$. $g = \{g_y\}$ is called the *fundamental tensor* of F.

Let

$$C_{ijk} := \frac{1}{4}\left[F^2\right]_{y^i y^j y^k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}.$$

Define symmetric trilinear form $\mathbf{C} := C_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k$ on $TM\backslash\{0\}$. We call $\mathbf{C}$ the *Cartan torsion.* The *mean Cartan torsion* $\mathbf{I} = I_i dx^i$ is defined by

$$I_i := g^{jk} C_{ijk}.$$

Further, we have ([3], [7], [9])

$$I_i = g^{jk} C_{ijk} = \frac{\partial}{\partial y^i}\left[\ln\sqrt{\det(g_{jk})}\right]. \tag{12}$$

For a Finsler metric F, the geodesics are characterized locally by a system of 2nd ODEs:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where

$$G^i = \frac{1}{4}g^{il}\Big\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big\}. \tag{13}$$

G^i are called the *geodesic coefficients* of F. The *Landsberg curvature* $\mathbf{L} = L_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k$ is a horizontal tensor on $TM \setminus \{0\}$ defined by (cf. [9], [10])

$$L_{ijk} := -\frac{1}{2}FF_{y^m}\big[G^m\big]_{y^i y^j y^k}. \tag{14}$$

F is called a *Landsberg metric* if $\mathbf{L} = 0$. The *mean Landsberg curvature* $\mathbf{J} = J_i dx^i$ is defined by

$$J_i := g^{jk}L_{ijk}. \tag{15}$$

We call F a *weakly Landsberg metric* if $\mathbf{J} = 0$. We say that F is of *relatively isotropic mean Landsberg curvature* if $J_i + c(x)\mathrm{F}I_i = 0$ for a scalar function $c(x)$ on M.

Now we consider an (α, β)-metric on an n-dimensional manifold M, $F = \alpha\phi(s)$, $s = \beta/\alpha$. By a linear algebra technique, one obtains (cf. [2], [7], [9])

$$\det(g_{ij}) = \phi^{n+1}(\phi - s\phi')^{n-2}\big[(\phi - s\phi') + (b^2 - s^2)\phi''\big]\det(a_{ij}), \tag{16}$$

where $b(x) := \|\beta_x\|_\alpha$.

In order to study the geometric properties of (α, β)-metrics, one needs a formula for the geodesic coefficients of an (α, β)-metric. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$
$$s^i{}_j := a^{ih}s_{hj}, \quad r_j := b^m r_{mj}, \quad s_j := b_i s^i{}_j = b^m s_{mj},$$

where "|" denotes the covariant derivative with respect to the Levi–Civita connection of α. We will denote $r_{00} := r_{ij}y^i y^j, s^i{}_0 := s^i{}_j y^j$, etc. Let G^i and $\bar{G}^i$ denote the geodesic coefficients of F and α respectively in the same coordinate system. By a direct computation, one gets the following formula [7], [9]:

$$G^i = \bar{G}^i + \alpha Q s^i{}_0 + \Theta\big\{-2\alpha Q s_0 + r_{00}\big\}\left\{\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'}b^i\right\}, \tag{17}$$

where

$$\Theta := \frac{Q - sQ'}{2\Delta}.$$

The Landsberg curvature of (α, β)-metrics is given by Z. Shen [12] as follows

$$L_{jkl} = -\frac{\rho}{6\alpha^3}\big\{h_j h_k C_l + h_j h_l C_k + h_k h_l C_j + 3(E_j h_{kl} + E_k h_{jl} + E_l h_{jk})\big\}, \quad (18)$$

where

$$\begin{aligned}
\rho &= \phi(\phi - s\phi'),\\
h_j &= b_j - \alpha^{-1} s y_j,\\
h_{jk} &= a_{j\ k} - \alpha^{-2} y_j y_k,\\
C_j &= \alpha(X_4 r_{00} + Y_4 \alpha s_0) h_j - 3Q'' D_j,\\
E_j &= \alpha(X_6 r_{00} + Y_6 \alpha s_0) h_j - (Q - sQ') D_j,\\
D_j &= \frac{\alpha^2}{\Delta}(\Delta s_{j0} + r_{j0} - Q\alpha s_j) - \frac{1}{\Delta}(r_{00} - Q\alpha s_0) y_j,
\end{aligned}$$

where $y_j := a_{jk} y^k$ and

$$\begin{aligned}
X_4 &= \frac{1}{2\Delta^2}\big\{-2\Delta Q''' + 3[(Q - sQ') + (b^2 - s^2)Q'']Q''\big\},\\
X_6 &= \frac{1}{2\Delta^2}\big\{(Q - sQ')^2 + [2(s + b^2 Q) - (b^2 - s^2)(Q - sQ')]Q''\big\},\\
Y_4 &= -2QX_4 + \frac{3Q'Q''}{\Delta},\\
Y_6 &= -2QX_6 + \frac{(Q - sQ')Q'}{\Delta}.
\end{aligned}$$

Then the mean Landsberg curvature of (α, β)-metrics is given by B. Li and Z. Shen [8] as follows

$$\begin{aligned}
J_j = -\frac{1}{2\alpha^4\Delta}\Bigg\{ &\frac{2\alpha^3}{b^2 - s^2}\left[\frac{\Phi}{\Delta} + (n+1)(Q - sQ^{'})\right](s_0 + r_0) h_j\\
&+ \frac{\alpha^2}{b^2 - s^2}\left[\Psi_1 + s\frac{\Phi}{\Delta}\right](r_{00} - 2\alpha Q s_0) h_j\\
&+ \alpha\big[-\alpha^2 Q' s_0 h_j + \alpha Q(\alpha^2 s_j - y_j s_0) + \alpha^2 \Delta s_{j0}\\
&+ \alpha^2 (r_{j0} - 2\alpha Q s_j) - (r_{00} - 2\alpha Q s_0) y_j\big]\frac{\Phi}{\Delta}\Bigg\}. \quad (19)
\end{aligned}$$

For our aim, we need the following formula for the mean Cartan torsion of (α, β)-metrics.

Lemma 2.1. *For an (α,β)-metric $F=\alpha\phi(\beta/\alpha)$, the mean Cartan torsion is given by*

$$I_i=-\frac{1}{2F}\frac{\Phi}{\Delta}(\phi-s\phi')h_i. \tag{20}$$

Proof. By use of (12) and (16), after a direct computation, one can obtain

$$I_i=\frac{1}{2\alpha}\left\{(n+1)\frac{\phi'}{\phi}-(n-2)\frac{s\phi''}{\phi-s\phi'}+\frac{(b^2-s^2)\phi'''-3s\phi''}{\phi-s\phi'+(b^2-s^2)\phi''}\right\}h_i. \tag{21}$$

Further, by use of a Maple programm, one can get (20). □

By Deicke's theorem, a Finsler metric is Riemannian metric if and only if $\mathbf{I}=0$ [2]. By (3) and the assumption $\phi(s)>0$, we have $\phi(s)-s\phi'(s)>0$, $|s|\le b<b_0$ (cf. [7]). Thus, from Lemma 2.1, we have the following

Proposition 2.2. *An (α,β)-metric F is a Riemannian metric if and only if $\Phi=0$.*

In the following we always assume that F is not a Riemannian metric, that is, $\Phi\neq 0$. From (19) and (20), we have the following

$$\begin{aligned}J_j+c(x)FI_j=-\frac{1}{2\alpha^4\Delta}\Big\{&\frac{2\alpha^3}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](s_0+r_0)h_j\\&+\frac{\alpha^2}{b^2-s^2}\Big[\Psi_1+s\frac{\Phi}{\Delta}\Big](r_{00}-2\alpha Qs_0)h_j\\&+\alpha\big[-\alpha^2Q's_0h_j+\alpha Q(\alpha^2s_j-y_js_0)+\alpha^2\Delta s_{j0}\\&+\alpha^2(r_{j0}-2\alpha Qs_j)-(r_{00}-2\alpha Qs_0)y_j\big]\frac{\Phi}{\Delta}\\&+c(x)\alpha^4\Phi(\phi-s\phi')h_j\Big\}.\end{aligned} \tag{22}$$

3. Necessary conditions

We have known that $\mathbf{J}$ can be expressed in terms of α, β and $\phi(s)$, $\phi'(s)$ and etc, where $s=\beta/\alpha$. But the formula (19) is very complicated. So the equation $\mathbf{J}+c(x)F\mathbf{I}=0$ is complicated too because one has to deal with the terms $\phi(s),\phi'(s)$, etc. To overcome this difficulty, a useful technique is to take a special local coordinate system at a point x as in [12] such that

$$\alpha_x=\sqrt{\sum_{i=1}^n(y^i)^2},\quad \beta_x=by^1,$$

where $b = \|\beta_x\|_\alpha$. Then we take another special coordinate: $(s, u^A) \to (y^i)$ given by

$$y^1 = \frac{s}{\sqrt{b^2 - s^2}}\,\bar{\alpha}, \quad y^A = u^A,$$

where $\bar{\alpha} = \sqrt{\sum_{A=2}^n (y^A)^2}$. We have

$$\alpha = \frac{b}{\sqrt{b^2 - s^2}}\,\bar{\alpha}, \quad \beta = \frac{bs}{\sqrt{b^2 - s^2}}\,\bar{\alpha}.$$

Because the expression (22) involves r_{ij}, s_{ij} etc, one needs the following expressions:

$$r_1 = br_{11}, \quad r_A = br_{1A}, \quad s_1 = 0, \quad s_A = bs_{1A},$$

$$r_{00} = \frac{s^2\bar{\alpha}^2}{b^2 - s^2}\, r_{11} + 2\frac{s\bar{\alpha}}{\sqrt{b^2 - s^2}}\,\bar{r}_{10} + \bar{r}_{00},$$

$$r_{10} = \frac{s\bar{\alpha}}{\sqrt{b^2 - s^2}}\, r_{11} + \bar{r}_{10}, \quad s_{10} = \bar{s}_{10},$$

where $\bar{r}_{10} = r_{1A}u^A$, $\bar{s}_{10} = s_{1A}u^A$, $\bar{r}_{00} = r_{AB}u^Au^B$. We have $\bar{r}_0 = r_Au^A = b\bar{r}_{10}$, $\bar{s}_0 = s_Au^A = b\bar{s}_{10}$.

By a direct computation and using the formula (22), one can show that $J_1 + c(x)FI_1 = 0$ is equivalent to that

$$\big\{\Psi_3[s^2r_{11}\bar{\alpha}^2 + (b^2 - s^2)\bar{r}_{00}] - b^2\Psi_2\bar{r}_{00}\big\} + c(x)sb^2\Phi(\phi - s\phi')\bar{\alpha}^2 = 0 \tag{23}$$

and

$$(2s\Psi_1 + b^2\Psi_2)\bar{r}_{10} + b^2(\Psi_2 - 2Q\Psi_1)\bar{s}_{10} = 0, \tag{24}$$

$J_A + c(x)FI_A = 0 \ \ (A = 2, \cdots, n)$ is equivalent to that

$$\begin{aligned}\big\{\Psi_3[s^2r_{11}\bar{\alpha}^2 + (b^2 - s^2)\bar{r}_{00}] - b^2\Psi_2\bar{r}_{00}\big\}y_A + b^2\frac{\Phi}{\Delta}\big[\bar{r}_{00}y_A - (\bar{r}_{A0} + \Delta\bar{s}_{A0})\bar{\alpha}^2\big]\\ + c(x)sb^2\Phi(\phi - s\phi')\bar{\alpha}^2y_A = 0\end{aligned} \tag{25}$$

and

$$\begin{aligned}s\big\{2s\Psi_1 + b^2\Psi_2\big\}\bar{r}_{10}y_A + sb^2\big\{\Psi_2 - 2Q\Psi_1\big\}\bar{s}_{10}y_A\\ + b^2\frac{\Phi}{\Delta}\big\{s(\bar{r}_{10}y_A - r_{1A}\bar{\alpha}^2) - (b^2Q + \Delta s)(\bar{s}_{10}y_A - s_{1A}\bar{\alpha}^2)\big\} = 0.\end{aligned} \tag{26}$$

Lemma 3.1. *($n \geq 3$) For an (α,β)-metric F, if $J_j + c(x)FI_j = 0$ at a point x, then we have*

$$s_{AB} = 0, \tag{27}$$

$$r_{AB} = kb^2\delta_{AB}, \tag{28}$$

$$r_{11} = \sigma b^2, \tag{29}$$

where $k = k(x)$ and $\sigma = \sigma(x)$ are scalar functions on M.

Proof. It follows from (23) and (25) that

$$\bar{r}_{00}y_A - (\bar{r}_{A0} + \Delta\bar{s}_{A0})\bar{\alpha}^2 = 0. \tag{30}$$

Since $n \geq 3$, (30) implies (27) and (28). Letting $\sigma := r_{11}/b^2$, we obtain (29). □

Lemma 3.2. *($n \geq 3$) For an (α,β)-metric F, if $J_i + c(x)FI_i = 0$ at a point x, then*

$$s_{1A} = 0, \quad r_{1A} = 0. \tag{31}$$

Proof. It follows from (24) and (26) that

$$\big\{s\bar{r}_{10} - (b^2Q + \Delta s)\bar{s}_{10}\big\}y_A - \big\{r_{1A} - (b^2Q + \Delta s)s_{1A}\big\}\bar{\alpha}^2 = 0. \tag{32}$$

Since $n \geq 3$, we obtain from (32) that

$$sr_{1A} - (b^2Q + \Delta s)s_{1A} = 0. \tag{33}$$

Then we can claim that $s_{1A} = 0$ and $r_{1A} = 0$. See Lemma 4.2 in [8] for more details. □

From Lemma 3.1 and 3.2, one obtains the following

Corollary 3.3. *For an arbitrary (α,β)-metric F on an n-dimensional manifold M ($n \geq 3$), if $J_i + c(x)FI_i = 0$, then β must be closed.*

Now, plugging (28) and (29) into (23) yield

$$\big\{\Psi_1 k + s\sigma\Psi_3\big\} + c(x)\Phi(\phi - s\phi') = 0. \tag{34}$$

Let us summarize what we have proved.

Proposition 3.4. *Let $F = \alpha\phi(s)$ be an (α,β)-metric on an n-dimensional manifold ($n \geq 3$). Suppose that $J_i + c(x)FI_i = 0$ at a point x. Then β satisfies*

$$s_{ij} = 0, \tag{35}$$

$$r_{ij} = k(b^2a_{ij} - b_ib_j) + \sigma b_ib_j \tag{36}$$

and $\phi = \phi(s)$ satisfies

$$\big\{\Psi_1 k + s\sigma\Psi_3\big\} + c(x)\Phi(\phi - s\phi') = 0.$$

4. Sufficient conditions

In this section, we are going to prove the sufficient conditions for an (α,β)-metric $F=\alpha\phi(\beta/\alpha)$ to be of relatively isotropic mean Landsberg curvature. Assume that α and β satisfy (4) and (5), we have

$$s_{j0}=0,\quad s_j=0,\quad s_0=0, \tag{37}$$

$$r_{j0}=k(b^2y_j-\beta b_j)+\sigma\beta b_j,\quad r_0=\sigma\beta b^2, \tag{38}$$

$$r_{00}=k(b^2\alpha^2-\beta^2)+\sigma\beta^2. \tag{39}$$

Substituting them into (22), we obtain

$$J_j+c(x)FI_j=-\frac{1}{2\Delta}\big\{\Psi_1k+s\sigma\Psi_3+c(x)\Phi(\phi-s\phi')\big\}h_j.$$

By our assumption (6) on ϕ, we have

$$J_j+c(x)FI_j=0.$$

This completes the proof of the sufficient conditions.

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