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Integral group ring of the Suzuki sporadic simple group

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Abstract. Using the Luthar–Passi method, we investigate the classical Zassenhaus conjecture for the normalized unit group of the integral group ring of the Suzuki sporadic simple group Suz. As a consequence, for this group we confirm the Kimmerle's conjecture on prime graphs.

1. Introduction and main results

Let $V(\mathbb{Z}G)$ be the normalized unit group of the integral group ring $\mathbb{Z}G$ of a finite group G. A long-standing conjecture of H. ZASSENHAUS **(ZC)** says that every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G (see [32]).

For finite simple groups the main tool for the investigation of the Zassenhaus conjecture is the LUTHAR–PASSI method, introduced in [28] to solve it for A_5 and then applied in [29] for the case of S_5. Later M. HERTWECK extended the Luthar–Passi method and applied it for the investigation of the Zassenhaus conjecture for $PSL(2,p)$, $p = 7, 11, 13$ in [21], and for $PSL(2,9) \cong A_6$ in [25]. The Luthar–Passi method proved to be useful for groups containing non-trivial normal subgroups as well. For some recent results we refer to [7], [9], [22], [21], [23], [26]. Also, some related properties and some weakened variations of the Zassenhaus conjecture can be found in [3], [5], [29].

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We introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of G. The Gruenberg–Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labeled by the primes in $\#(G)$ and with an edge from p to q if there is an element of order pq in the group G. In [27] W. Kimmerle proposed the following weakened variation of the Zassenhaus conjecture:

(KC) If G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

In particular, in the same paper W. Kimmerle verified that **(KC)** holds for finite Frobenius and solvable groups. We remark that with respect to the so-called p-version of the Zassenhaus conjecture the investigation of Frobenius groups was completed by M. Hertweck and the first author in [6]. In [8], [9], [15], [16], [13], [14], [12], [11], [10] **(KC)** was confirmed for the Mathieu simple groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, sporadic Janko simple groups J_1, J_2 and J_3, the Higman–Sims group HS, the McLaughlin sporadic group McL and the Rudvalis sporadic group Ru.

Here we continue these investigations for the Suzuki simple group `Suz`. Although using the Luthar–Passi method we cannot prove the rational conjugacy for torsion units of its integral group ring, our main result gives a lot of information on partial augmentations of these units. In particular, we confirm the Kimmerle's conjecture for this group.

Let G be the Suzuki sporadic simple group `Suz` and let

$$\begin{aligned}\mathcal{C} = \{&C_1, C_{2a}, C_{2b}, C_{3a}, C_{3b}, C_{3c}, C_{4a}, C_{4b}, C_{4c}, C_{4d}, C_{5a}, C_{5b}, C_{6a},\\ &C_{6b}, C_{6c}, C_{6d}, C_{6e}, C_{7a}, C_{8a}, C_{8b}, C_{8c}, C_{9a}, C_{9b}, C_{10a},\\ &C_{10b}, C_{11a}, C_{12a}, C_{12b}, C_{12c}, C_{12d}, C_{12e}, C_{13a}, C_{13b}, C_{14a},\\ &C_{15a}, C_{15b}, C_{15c}, C_{18a}, C_{18b}, C_{20a}, C_{21a}, C_{21b}, C_{24a}\}\end{aligned}$$

be the collection of all its conjugacy classes, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order $k > 1$. Denote by $\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$ the partial augmentation of u with respect to C_{nt}. From the Berman–Higman Theorem (see [4] and [31], Ch. 5, p. 102) one knows that $\nu_1 = \alpha_1 = 0$, so

$$\sum_{C_{nt} \in \mathcal{C} \setminus C_1} \nu_{nt} = 1. \tag{1}$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt}\chi(h_{nt})$, where h_{nt} is a representative of the conjugacy class C_{nt}.

Our main result is the following theorem.

Theorem 1. *Let G denote the Suzuki sporadic simple group* Suz*. Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$ and denote by*

$$\begin{aligned}\mathfrak{P}(u) = (&\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{3c}, \nu_{4a}, \nu_{4b}, \nu_{4c}, \nu_{4d}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{6b}, \nu_{6c}, \nu_{6d}, \nu_{6e},\\ &\nu_{7a}, \nu_{8a}, \nu_{8b}, \nu_{8c}, \nu_{9a}, \nu_{9b}, \nu_{10a}, \nu_{10b}, \nu_{11a}, \nu_{12a}, \nu_{12b}, \nu_{12c}, \nu_{12d}, \nu_{12e},\\ &\nu_{13a}, \nu_{13b}, \nu_{14a}, \nu_{15a}, \nu_{15b}, \nu_{15c}, \nu_{18a}, \nu_{18b}, \nu_{20a}, \nu_{21a}, \nu_{21b}, \nu_{24a}\,) \in \mathbb{Z}^{42}\end{aligned}$$

the tuple of partial augmentations of u. The following properties hold.

(i) *There is no elements of orders* 22, 26, 33, 35, 39, 55, 65, 77, 91 *and* 143 *in $V(\mathbb{Z}G)$. Equivalently, if $|u| \notin \{28, 30, 36, 40, 42, 45, 56, 60, 63, 72, 84, 90, 120, 126, 168, 180, 252, 360, 504\}$, then $|u|$ coincides with the order of some element $g \in G$.*

(ii) *If $|u| \in \{7, 11\}$ then u is rationally conjugate to some $g \in G$.*

(iii) *If $|u| = 2$, then one has $(\nu_{2a}, \nu_{2b}) \in \{(4,-3), (3,-2), (2,-1), (1,0), (0,1), (-1,2), (-2,3), (-3,4)\,\}$.*

(iv) *If $|u| = 5$, then one has $(\nu_{5a}, \nu_{5b}) \in \{(5,-4), (4,-3), (3,-2), (2,-1), (1,0), (0,1), (-1,2), (-2,3), (-3,4)\}$.*

(v) *If $|u| = 13$, then one has $\{(\nu_{13a}, \nu_{13b}) \mid -8 \le \nu_{13a} \le 9,\ \nu_{13a} + \nu_{13b} = 1\}$.*

For the case of torsion units of order 3, using our implementation of the Luthar–Passi method, which we intend to make available in the GAP package LAGUNA [17], we are able to compute the set of 104 tuples containing (likely as a proper subset) possible tuples of partial augmentations, listed in the Appendix.

As an immediate consequence of part (i) of the Theorem we obtain

Corollary 1. *If $G =$* Suz *then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

2. Preliminaries

The following result is a reformulation of the Zassenhaus conjecture in terms of vanishing of partial augmentations of torsion units.

Proposition 1 ([28] and Theorem 2.5 in [30])**.** *Let $u \in V(\mathbb{Z}G)$ be of order k. Then u is conjugate in $\mathbb{Q}G$ to an element $g \in G$ if and only if for each d dividing k there is precisely one conjugacy class C with partial augmentation $\varepsilon_C(u^d) \neq 0$.*

The next result now yield that several partial augmentations are zero.

Proposition 2 ([22], Proposition 3.1; [21], Proposition 2.2)**.** *Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If x is an element of G whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

The key restriction on partial augmentations is given by the following result that is the cornerstone of the Luthar–Passi method.

Proposition 3 ([21], [28])**.** *Let either $p = 0$ or p a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G, then for every integer l the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d) z^{-dl}\}$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Finally, we will use the well-known bound for orders of torsion units.

Proposition 4 ([18])**.** *The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.*

3. Proof of the theorem

In this section we denote by G the Suzuki sporadic simple group `Suz`. The character table of G, as well as the p-Brauer character tables, which will be denoted by $\mathfrak{BCT}(p)$ where $p \in \{2, 3, 5, 7, 11, 13\}$, can be found using the computational algebra system GAP [19], which derives these data from [2], [1]. For the characters and conjugacy classes we will use throughout the paper the same notation, indexation inclusive, as used in the GAP Character Table Library.

It is well known (see [19], [20]) that $|G| = 448345497600 = 2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$ and $\exp(G) = 360360 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

Since the group G possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 20, 21 and 24, first of all we investigate units of some of these orders (except for the units of orders 4, 6, 8, 9, 10, 12, 14, 15, 18, 20, 21 and 24). After this, by Proposition 4, the order of each torsion unit divides the exponent of G, so to prove the Kimmerle's conjecture, it remains to consider units of orders 22, 26, 33, 35, 39, 55, 65, 77, 91 and 143. We will prove that no units of all these orders do appear in $V(\mathbb{Z}G)$. Since we omit orders 28, 30, 36, 40, 42, 45 and 63 that do not contribute to **(KC)**, we need to add to the list of exceptions in part (i) of Theorem also orders 56, 60, 72, 84, 90, 120, 126, 168, 180, 252, 360 and 504, but no more because of restrictions imposed by the exponent of G.

Now we consider separately each possible value of $|u|$.

• Let $|u| \in \{7, 11\}$. Since there is only one conjugacy class in G consisting of elements or order $|u|$, this case follows at once from Proposition 2. Thus, for units of orders 5 and 7 we obtained that there is precisely one conjugacy class with non-zero partial augmentation. Proposition 1 then yields part (ii) of the Theorem.

• Let u be an involution. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{2b} = 1$. Put $t_1 = 15\nu_{2a} - \nu_{2b}$ and $t_2 = 7\nu_{2a} - 3\nu_{2b}$. Applying Proposition 3 to the ordinary character χ_2 and 3-Brauer character χ_3 we get the following system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{2}(t_1 + 143) \geq 0; \qquad \mu_1(u, \chi_2, *) = \frac{1}{2}(-t_1 + 143) \geq 0;$$
$$\mu_0(u, \chi_3, 3) = \frac{1}{2}(2t_2 + 78) \geq 0; \qquad \mu_1(u, \chi_3, 3) = \frac{1}{2}(-2t_2 + 78) \geq 0.$$

From the requirement that all $\mu_i(u, \chi_j, p)$ must be non-negative integers it can be deduced that (ν_{2a}, ν_{2b}) satisfies the conditions of part (iii) of the Theorem.

• Let $|u| = 3$. By (1) and Proposition 2 we obtain that $\nu_{3a} + \nu_{3b} + \nu_{3c} = 1$. Put $t_1 = 35\nu_{3a} + 8\nu_{3b} - \nu_{3c}$ and $t_2 = 14\nu_{3a} - 13\nu_{3b} - 4\nu_{3c}$. Again applying Proposition 3 to the ordinary characters χ_2, χ_3 and 2-Brauer characters χ_2 and χ_7, we obtain the following system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{3}(2t_1 + 143) \geq 0; \qquad \mu_1(u, \chi_2, *) = \frac{1}{3}(-t_1 + 143) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \frac{1}{3}(-t_2 + 364) \geq 0; \qquad \mu_1(u, \chi_3, *) = \frac{1}{3}(t_2 + 364) \geq 0;$$
$$\mu_0(u, \chi_2, 2) = \frac{1}{3}(-50\nu_{3a} + 4\nu_{3b} + 4\nu_{3c} + 110) \geq 0;$$
$$\mu_0(u, \chi_7, 2) = \frac{1}{3}(142\nu_{3a} - 20\nu_{3b} + 16\nu_{3c} + 638) \geq 0,$$

that has only 101 non-trivial and three trivial solutions $(\nu_{3a}, \nu_{3b}, \nu_{3c})$ which are listed in the Appendix.

• Let u be a unit of order 5. By (1) and Proposition 2 we have $\nu_{5a} + \nu_{5b} = 1$. Put $t_1 = 8\nu_{5a} + 3\nu_{5b}$. From the ordinary character table and Brauer character tables for $p = 2, 3$ we obtain the following system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{5}(4t_1 + 143) \geq 0; \qquad \mu_1(u, \chi_2, *) = \frac{1}{5}(-t_1 + 143) \geq 0;$$
$$\mu_0(u, \chi_2, 2) = \frac{1}{5}(-20\nu_{5a} + 110) \geq 0; \quad \mu_0(u, \chi_2, 3) = \frac{1}{5}(16\nu_{5a} - 4\nu_{5b} + 64) \geq 0.$$

From the requirement that all $\mu_i(u, \chi_j, p)$ must be non-negative integers it can be deduced that (ν_{5a}, ν_{5b}) satisfies the condition of part (iv) of the Theorem.

• Let u be a unit of order 13. By (1) and Proposition 2 we have $\nu_{13a} + \nu_{13b} = 1$. Put $t_1 = 7\nu_{13a} - 6\nu_{13b}$. Applying Proposition 3 to the Brauer character tables for $p = 2, 3$, we get the following system of inequalities

$$\mu_1(u, \chi_2, 2) = \frac{1}{13}(t_1 + 110) \geq 0; \quad \mu_1(u, \chi_{10}, 3) = \frac{1}{13}(-t_1 + 5103) \geq 0;$$
$$\mu_2(u, \chi_2, 2) = \frac{1}{13}(-6\nu_{13a} + 7\nu_{13b} + 110) \geq 0,$$

which has integral solution (ν_{13a}, ν_{13b}) listed in part (v) of the Theorem.

• Let $|u| = 22$. By (1) and Proposition 2 we have $\nu_{2a} + \nu_{2b} + \nu_{11a} = 1$. Put $t_1 = 15\nu_{2a} - \nu_{2b}$ and $t_2 = 44\nu_{2a} + \nu_{11a}$. Since $|u^{11}| = 2$, by part (iii) of the Theorem we have eight cases, which we consider separately.

Case 1. Let $\chi(u^{11}) = \chi(2a)$. Using Proposition 3 for the characters χ_2, χ_3 and χ_4 of G, we get the following system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{22}(10t_1 + 158) \geq 0; \quad \mu_{11}(u, \chi_2, *) = \frac{1}{22}(-10t_1 + 128) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \frac{1}{22}(10t_2 + 418) \geq 0; \quad \mu_{11}(u, \chi_3, *) = \frac{1}{22}(-10t_2 + 330) \geq 0;$$
$$\mu_{11}(u, \chi_4, *) = \frac{1}{22}(-120\nu_{2a} - 200\nu_{2b} + 10\nu_{11a} + 758) \geq 0;$$
$$\mu_1(u, \chi_2, *) = \frac{1}{22}(15\nu_{2a} - \nu_{2b} + 128) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \frac{1}{22}(44\nu_{2a} + \nu_{11a} + 319) \geq 0.$$

Case 2. Let $\chi(u^{11}) = \chi(2b)$. Using Proposition 3 for the characters χ_2, χ_3 and χ_4 of G, we obtain the following system

$$\mu_0(u, \chi_2, *) = \frac{1}{22}(10t_1 + 142) \geq 0; \quad \mu_{11}(u, \chi_2, *) = \frac{1}{22}(-10t_1 + 144) \geq 0;$$

$$\mu_1(u,\chi_2,*) = \frac{1}{22}(t_1+144) \geq 0; \qquad \mu_1(u,\chi_3,*) = \frac{1}{22}(t_2+363) \geq 0;$$

$$\mu_0(u,\chi_3,*) = \frac{1}{22}(10t_2+374) \geq 0; \quad \mu_{11}(u,\chi_3,*) = \frac{1}{22}(-10t_2+374) \geq 0;$$

$$\mu_0(u,\chi_4,*) = \frac{1}{22}(10(12\nu_{2a}+20\nu_{2b}-\nu_{11a})+790) \geq 0;$$

$$\mu_{11}(u,\chi_4,*) = \frac{1}{22}(-10(12\nu_{2a}+20\nu_{2b}-\nu_{11a})+750) \geq 0.$$

Case 3. Let $\chi(u^{11}) = 4\chi(2a) - 3\chi(2b)$. Using Proposition 3 for the characters χ_2 and χ_3 of G, we obtain the following system

$$\mu_0(u,\chi_2,*) = \frac{1}{22}(10t_1+206) \geq 0; \quad \mu_1(u,\chi_2,*) = \frac{1}{22}(t_1+80) \geq 0;$$

$$\mu_{11}(u,\chi_2,*) = \frac{1}{22}(-10t_1+80) \geq 0; \quad \mu_0(u,\chi_3,*) = \frac{1}{22}(10t_2+550) \geq 0;$$

$$\mu_1(u,\chi_3,*) = \frac{1}{22}(t_2+187) \geq 0; \qquad \mu_{11}(u,\chi_3,*) = \frac{1}{22}(-10t_2+198) \geq 0.$$

Case 4. Let $\chi(u^{11}) = 3\chi(2a) - 2\chi(2b)$. Using Proposition 3 for the characters χ_2 and χ_3 of G, we obtain the following system

$$\mu_0(u,\chi_2,*) = \frac{1}{22}(10t_1+190) \geq 0; \quad \mu_1(u,\chi_2,*) = \frac{1}{22}(t_1+96) \geq 0;$$

$$\mu_{11}(u,\chi_2,*) = \frac{1}{22}(-10t_1+96) \geq 0; \quad \mu_0(u,\chi_3,*) = \frac{1}{22}(10t_2+506) \geq 0;$$

$$\mu_1(u,\chi_3,*) = \frac{1}{22}(t_2+231) \geq 0; \qquad \mu_{11}(u,\chi_3,*) = \frac{1}{22}(-10t_2+242) \geq 0.$$

Case 5. Let $\chi(u^{11}) = 2\chi(2a) - \chi(2b)$. Using Proposition 3 for the characters χ_2 and χ_3 of G, we obtain the following system

$$\mu_0(u,\chi_2,*) = \frac{1}{22}(10t_1+174) \geq 0; \qquad \mu_1(u,\chi_2,*) = \frac{1}{22}(t_1+112) \geq 0;$$

$$\mu_{11}(u,\chi_2,*) = \frac{1}{22}(-10t_1+112) \geq 0; \quad \mu_0(u,\chi_3,*) = \frac{1}{22}(10t_2+462) \geq 0;$$

$$\mu_1(u,\chi_3,*) = \frac{1}{22}(t_2+275) \geq 0; \qquad \mu_{11}(u,\chi_3,*) = \frac{1}{22}(-10t_2+286) \geq 0.$$

Case 6. Let $\chi(u^{11}) = -\chi(2a) + 2\chi(2b)$. Using Proposition 3 for the characters χ_2, χ_3 and χ_4 of G, we obtain the following system

$$\mu_0(u,\chi_2,*) = \frac{1}{22}(10t_1+126) \geq 0; \quad \mu_{11}(u,\chi_2,*) = \frac{1}{22}(-10t_1+160) \geq 0;$$

$$\mu_0(u,\chi_3,*) = \frac{1}{22}(10t_2+330) \geq 0; \quad \mu_{11}(u,\chi_3,*) = \frac{1}{22}(-10t_2+418) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \frac{1}{22}(t_1+160) \geq 0; \quad \mu_1(u,\chi_3,*) = \frac{1}{22}(t_2+407) \geq 0;$$
$$\mu_0(u,\chi_4,*) = \frac{1}{22}(120\nu_{2a}+200\nu_{2b}-10\nu_{11a}+798) \geq 0.$$

Case 7. Let $\chi(u^{11}) = -2\chi(2a)+3\chi(2b)$. Using Proposition 3 for the characters χ_2 and χ_3 of G, we obtain the following system

$$\mu_0(u,\chi_2,*) = \frac{1}{22}(10t_1+110) \geq 0; \quad \mu_{11}(u,\chi_2,*) = \frac{1}{22}(-10t_1+176) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \frac{1}{22}(10t_2+286) \geq 0; \quad \mu_{11}(u,\chi_3,*) = \frac{1}{22}(-10t_2+462) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \frac{1}{22}(t_1+176) \geq 0; \quad \mu_1(u,\chi_3,*) = \frac{1}{22}(t_2+451) \geq 0.$$

Case 8. Let $\chi(u^{11}) = -3\chi(2a)+4\chi(2b)$. Using Proposition 3 for the characters χ_2 and χ_3 of G, we obtain the following system

$$\mu_0(u,\chi_2,*) = \frac{1}{22}(10t_1+110) \geq 0; \quad \mu_{11}(u,\chi_2,*) = \frac{1}{22}(-10t_1+192) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \frac{1}{22}(10t_2+242) \geq 0; \quad \mu_{11}(u,\chi_3,*) = \frac{1}{22}(-10t_2+506) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \frac{1}{22}(t_1+192) \geq 0; \quad \mu_1(u,\chi_3,*) = \frac{1}{22}(t_2+495) \geq 0.$$

In all eight cases we obtained systems of inequalities that have no solutions.

• Let $|u| = 26$. By (1) and Proposition 2 we have $\nu_{2a}+\nu_{2b}+\nu_{13a}+\nu_{13b} = 1$. Put $t_1 = 15\nu_{2a}-\nu_{2b}$, $t_2 = \nu_{2a}$ and $t_3 = 243\nu_{2a}+35\nu_{2b}-6\nu_{13a}+7\nu_{13b}$. Since $|u^{13}| = 2$ and $|u^2| = 13$, by parts (iii) and (v) of the Theorem we need to consider $8 \cdot 18 = 144$ cases. We will index them by $\chi(u^{13})$ and consider two possibilities.

First, let $\chi(u^{13}) \in \{\chi(2a), 4\chi(2a)-3\chi(2b), 3\chi(2a)-2\chi(2b), 2\chi(2a)-\chi(2b), -\chi(2a)+2\chi(2b), -2\chi(2a)+3\chi(2b), -3\chi(2a)+4\chi(2b)\}$ and put

$$(\alpha,\beta) = \begin{cases} (408,320), & \text{if } \chi(u^{13}) = \chi(2a); \\ (540,188), & \text{if } \chi(u^{13}) = 4\chi(2a)-3\chi(2b); \\ (496,232), & \text{if } \chi(u^{13}) = 3\chi(2a)-2\chi(2b); \\ (452,276), & \text{if } \chi(u^{13}) = 2\chi(2a)-\chi(2b); \\ (320,408), & \text{if } \chi(u^{13}) = -\chi(2a)+2\chi(2b); \\ (276,452), & \text{if } \chi(u^{13}) = -2\chi(2a)+3\chi(2b); \\ (232,496), & \text{if } \chi(u^{13}) = -3\chi(2a)+4\chi(2b). \end{cases}$$

Now using Proposition 3 for the character χ_3, we obtain the following system

$$\mu_0(u,\chi_3,*) = \frac{1}{26}(528t_2+\alpha) \geq 0; \quad \mu_{13}(u,\chi_3,*) = \frac{1}{26}(-528t_2+\beta) \geq 0,$$

which has no integral solution.

Second, suppose that $\chi(u^{13}) = \chi(2b)$. Using Proposition 3 for the characters χ_2, χ_3, χ_4 and χ_{31}, we obtain the following system

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \frac{1}{26}(12t_1+142) \geq 0; & \mu_1(u,\chi_2,*) &= \frac{1}{26}(-12t_1+144) \geq 0;\\
\mu_0(u,\chi_3,*) &= \frac{1}{26}(528t_2+364) \geq 0; & \mu_{13}(u,\chi_3,*) &= \frac{1}{26}(-528t_2+364) \geq 0;\\
\mu_1(u,\chi_{31},*) &= \frac{1}{26}(t_3+93512) \geq 0; & \mu_4(u,\chi_{31},*) &= \frac{1}{26}(-t_3+93583) \geq 0;\\
\mu_{13}(u,\chi_2,*) &= \frac{1}{26}(-180\nu_{2a}+12\nu_{2b}+144) \geq 0; & &\\
\mu_0(u,\chi_4,*) &= \frac{1}{26}(144\nu_{2a}+240\nu_{2b}+800) \geq 0, & &
\end{aligned}$$

which has no integral solution.

• Let $|u| = 33$. By (1) and Proposition 2 we have $\nu_{3a}+\nu_{3b}+\nu_{3c}+\nu_{11a} = 1$. Since $|u^{11}| = 3$, we have to consider 104 cases accordingly to the Appendix. Put $t_1 = 35\nu_{3a}+8\nu_{3b}-\nu_{3c}$, $t_2 = 14\nu_{3a}-13\nu_{3b}-4\nu_{3c}-\nu_{11a}$ and $t_3 = 105\nu_{3a}-3\nu_{3b}+6\nu_{3c}-\nu_{11a}$. First, when $\chi(u^{11})$ takes values from the first column of the following table, we have appropriate coefficients α_i given in the second column

$\chi(u^{11})$	$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6)$
$\chi(3a)$	$(213,108,346,388,980,665)$
$\chi(3b)$	$(159,135,361,400,764,773)$
$2\chi(3a)-14\chi(3b)+13\chi(3c)$	$(33,33,47,58,1430,440)$
$2\chi(3a)-13\chi(3b)+12\chi(3c)$	$(51,189,65,76,1412,449)$
$2\chi(3a)-12\chi(3b)+11\chi(3c)$	$(69,180,83,94,1394,458)$
$\chi(3a)-11\chi(3b)+11\chi(3c)$	$(105,162,496,130,1358,476)$
$\chi(3a)-10\chi(3b)+10\chi(3c)$	$(33,33,478,166,1160,575)$
$\chi(3a)-10\chi(3b)+10\chi(3c)$	$(123,153,487,148,1340,485)$

From the last table we obtain the system

$$\mu_0(u,\chi_2,*) = \frac{1}{33}(20t_1+\alpha_1) \geq 0; \qquad \mu_{11}(u,\chi_2,*) = \frac{1}{33}(-10t_1+\alpha_2) \geq 0;$$

$$\mu_0(u,\chi_3,*) = \frac{1}{33}(-20t_2+\alpha_3) \geq 0; \qquad \mu_{11}(u,\chi_3,*) = \frac{1}{33}(10t_2+\alpha_4) \geq 0;$$

$$\mu_0(u,\chi_4,*) = \frac{1}{33}(20t_3+\alpha_5) \geq 0; \qquad \mu_{11}(u,\chi_4,*) = \frac{1}{33}(-10t_3+\alpha_6) \geq 0,$$

which has no integral solution.

Finally, if $\chi(u^{11}) \in \{\chi(3c), 2\chi(3a) - 15\chi(3b) + 14\chi(3c)\}$, then we get

$$\mu_0(u,\chi_2,*) = \frac{1}{33}(20t_1+15) \geq 0; \qquad \mu_3(u,\chi_2,*) = \frac{1}{33}(-2t_1+15) \geq 0,$$

and if $\chi(u^{11}) = 2\chi(3a) - 11\chi(3b) + 10\chi(3c)$, we get

$$\mu_0(u,\chi_2,*) = \frac{1}{33}(20t_1+87) \geq 0; \qquad \mu_{11}(u,\chi_2,*) = \frac{1}{33}(-10t_1+171) \geq 0,$$

both of which have no integer solutions.

• Let $|u| = 35$. By (1) and Proposition 2 we have $\nu_{5a} + \nu_{5b} + \nu_{7a} = 1$. Put $t_1 = 8\nu_{5a} + 3\nu_{5b} + 3\nu_{7a}$ and $t_2 = \nu_{5a} - 4\nu_{5b}$. Since $|u^7| = 5$, by part (vi) of the Theorem we have to consider nine cases.

Case 1. Let $\chi(u^7) = \chi(5a)$, then

$$\mu_0(u,\chi_2,*) = \frac{1}{35}(24t_1+193) \geq 0; \qquad \mu_7(u,\chi_2,*) = \frac{1}{35}(-6t_1+153) \geq 0;$$

$$\mu_0(u,\chi_3,*) = \frac{1}{35}(-24t_2+360) \geq 0; \qquad \mu_7(u,\chi_3,*) = \frac{1}{35}(6t_2+365) \geq 0;$$

$$\mu_7(u,\chi_2,2) = \frac{1}{35}(30\nu_{5a}+12\nu_{7a}+103) \geq 0.$$

Case 2. Let $\chi(u^7) = \chi(5b)$, then

$$\mu_0(u,\chi_2,*) = \frac{1}{35}(24t_1+173) \geq 0; \qquad \mu_7(u,\chi_2,*) = \frac{1}{35}(-6t_1+158) \geq 0;$$

$$\mu_0(u,\chi_3,*) = \frac{1}{35}(-24t_2+380) \geq 0; \qquad \mu_7(u,\chi_3,*) = \frac{1}{35}(6t_2+360) \geq 0;$$

$$\mu_0(u,\chi_4,*) = \frac{1}{35}(240\nu_{5a}+72\nu_{7a}+798) \geq 0.$$

Case 3. Let $\chi(u^7) = 5\chi(5a) - 4\chi(5b)$, then

$$\mu_0(u,\chi_2,*) = \frac{1}{35}(24t_1+273) \geq 0; \qquad \mu_7(u,\chi_2,*) = \frac{1}{35}(-6t_1+133) \geq 0;$$

$$\mu_0(u,\chi_3,*) = \frac{1}{35}(-24t_2+280) \geq 0; \qquad \mu_7(u,\chi_3,*) = \frac{1}{35}(6t_2+385) \geq 0.$$

Case 4. Let $\chi(u^7) = 4\chi(5a) - 3\chi(5b)$, then

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(24t_1 + 253) \geq 0; \qquad \mu_7(u, \chi_2, *) = \frac{1}{35}(-6t_1 + 138) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \frac{1}{35}(-24t_2 + 300) \geq 0; \quad \mu_7(u, \chi_3, *) = \frac{1}{35}(6t_2 + 380) \geq 0;$$
$$\mu_0(u, \chi_2, 3) = \frac{1}{35}(96\nu_{5a} - 24\nu_{5b} + 24\nu_{7a} + 146) \geq 0.$$

Case 5. Let $\chi(u^7) = 3\chi(5a) - 2\chi(5b)$, then

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(24t_1 + 233) \geq 0; \qquad \mu_7(u, \chi_2, *) = \frac{1}{35}(-6t_1 + 143) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \frac{1}{35}(-24t_2 + 320) \geq 0; \quad \mu_7(u, \chi_3, *) = \frac{1}{35}(6t_2 + 375) \geq 0;$$
$$\mu_0(u, \chi_2, 3) = \frac{1}{35}(96\nu_{5a} - 24\nu_{5b} + 24\nu_{7a} + 126) \geq 0.$$

Case 6. Let $\chi(u^7) = 2\chi(5a) - \chi(5b)$, then

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(24t_1 + 213) \geq 0; \qquad \mu_7(u, \chi_2, *) = \frac{1}{35}(-6t_1 + 148) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \frac{1}{35}(-24t_2 + 340) \geq 0; \quad \mu_7(u, \chi_3, *) = \frac{1}{35}(6t_2 + 370) \geq 0;$$
$$\mu_0(u, \chi_7, 2) = \frac{1}{35}(72\nu_{5a} - 48\nu_{5b} + 24\nu_{7a} + 676) \geq 0.$$

Case 7. Let $\chi(u^7) = -\chi(5a) + 2\chi(5b)$, then

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(24t_1 + 153) \geq 0; \qquad \mu_7(u, \chi_2, *) = \frac{1}{35}(-6t_1 + 163) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \frac{1}{35}(-24t_2 + 400) \geq 0; \quad \mu_7(u, \chi_3, *) = \frac{1}{35}(6t_2 + 355) \geq 0.$$

Case 8. Let $\chi(u^7) = -2\chi(5a) + 3\chi(5b)$, then

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(24t_1 + 133) \geq 0; \qquad \mu_5(u, \chi_2, *) = \frac{1}{35}(-6t_1 + 112) \geq 0;$$
$$\mu_0(u, \chi_3, *) = \frac{1}{35}(-24t_2 + 420) \geq 0; \quad \mu_7(u, \chi_3, *) = \frac{1}{35}(6t_2 + 350) \geq 0;$$
$$\mu_0(u, \chi_2, 3) = \frac{1}{35}(96\nu_{5a} - 24\nu_{5b} + 24\nu_{7a} + 26) \geq 0.$$

Case 9. Let $\chi(u^7) = -3\chi(5a) + 4\chi(5b)$, then

$$\mu_0(u, \chi_2, *) = \frac{1}{35}(24t_1 + 113) \geq 0; \qquad \mu_5(u, \chi_2, *) = \frac{1}{35}(-6t_1 + 92) \geq 0;$$

$$\mu_0(u, \chi_3, *) = \frac{1}{35}(-24t_2 + 440) \geq 0; \quad \mu_7(u, \chi_3, *) = \frac{1}{35}(6t_2 + 345) \geq 0;$$

$$\mu_0(u, \chi_2, 3) = \frac{1}{35}(96\nu_{5a} - 24\nu_{5b} + 24\nu_{7a} + 6) \geq 0.$$

In all of the above cases we obtained systems that have no integer solutions.

• Let u be a unit of order 39. By (1) and Proposition 2 we have

$$\nu_{3a} + \nu_{3b} + \nu_{3c} + \nu_{13a} + \nu_{13b} = 1.$$

Put $t_1 = 35\nu_{3a} + 8\nu_{3b} - \nu_{3c}$, $t_2 = 14\nu_{3a} - 13\nu_{3b} - 4\nu_{3c}$, $t_3 = 35\nu_{3a} - \nu_{3b} + 2\nu_{3c}$ and $t_4 = 9\nu_{3c} - 7\nu_{13a} + 6\nu_{13b}$. Since $|u^{13}| = 3$ and $|u^3| = 13$, by part (vi) of the Theorem and the Appendix we have to consider 1872 cases. Using the LAGUNA package [17], we find out that in all cases we have the system of inequalities which has no integral solutions

$$\mu_0(u, \chi_2, *) = \frac{1}{39}(24t_1 + 213) \geq 0; \qquad \mu_1(u, \chi_2, *) = \frac{1}{39}(t_1 + 108) \geq 0;$$

$$\mu_{13}(u, \chi_2, *) = \frac{1}{39}(-12t_1 + 108) \geq 0; \quad \mu_0(u, \chi_3, *) = \frac{1}{39}(-24t_2 + 336) \geq 0;$$

$$\mu_1(u, \chi_3, *) = \frac{1}{39}(-t_2 + 378) \geq 0; \qquad \mu_{13}(u, \chi_3, *) = \frac{1}{39}(12t_2 + 378) \geq 0;$$

$$\mu_0(u, \chi_4, *) = \frac{1}{39}(72t_3 + 990) \geq 0; \qquad \mu_1(u, \chi_{31}, *) = \frac{1}{39}(-t_4 + 93548) \geq 0;$$

$$\mu_3(u, \chi_{31}, *) = \frac{1}{39}(2t_4 + 93548) \geq 0.$$

• Let $|u| = 55$. By (1) and Proposition 2 we have $\nu_{5a} + \nu_{5b} + \nu_{11a} = 1$. Put $t_1 = 8\nu_{5a} + 3\nu_{5b}$ and $t_2 = \nu_{5a} - 4\nu_{5b} - \nu_{11a}$. Since $|u^{11}| = 5$, by part (iv) of the Theorem we have to consider nine cases that we collect into four groups.

Case 1. When $\chi(u^{11}) \in \{\chi(5a), \chi(5b), 4\chi(5a) - 3\chi(5b), 3\chi(5a) - 2\chi(5b), 2\chi(5a) - \chi(5b), , -\chi(5a) + 2\chi(5b)\}$ then we put

$$(\alpha, \beta) = \begin{cases} (175, 135) & \text{if} \quad \chi(u^{11}) = \chi(5a); \\ (155, 140) & \text{if} \quad \chi(u^{11}) = \chi(5b); \\ (235, 120) & \text{if} \quad \chi(u^{11}) = 4\chi(5a) - 3\chi(5b); \\ (215, 125) & \text{if} \quad \chi(u^{11}) = 3\chi(5a) - 2\chi(5b); \\ (195, 130) & \text{if} \quad \chi(u^{11}) = 2\chi(5a) - \chi(5b); \\ (135, 145) & \text{if} \quad \chi(u^{11}) = -\chi(5a) + 2\chi(5b). \end{cases}$$

Using Proposition 3 for the character χ_2 of G, we get the system

$$\mu_0(u,\chi_2,*) = \frac{1}{55}(40t_1+\alpha) \geq 0; \qquad \mu_{11}(u,\chi_2,*) = \frac{1}{55}(-10t_1+\beta) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \frac{1}{55}(t_1+\beta) \geq 0,$$

which has no integral solution.

In the remaining three cases below we have no integral solutions as well.

Case 2. Let $\chi(u^{11}) = 5\chi(5a) - 4\chi(5b)$, then

$$\mu_0(u,\chi_2,*) = \frac{1}{55}(40t_1+255) \geq 0; \quad \mu_{11}(u,\chi_2,*) = \frac{1}{55}(-10t_1+115) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \frac{1}{55}(t_1+115) \geq 0; \quad \mu_0(u,\chi_3,*) = \frac{1}{55}(-40t_2+290) \geq 0;$$
$$\mu_{11}(u,\chi_3,*) = \frac{1}{55}(10t_2+395) \geq 0; \quad \mu_1(u,\chi_3,*) = \frac{1}{55}(-t_2+384) \geq 0.$$

—it Case 3. Let $\chi(u^{11}) = -2\chi(5a) + 3\chi(5b)$, then

$$\mu_0(u,\chi_2,*) = \frac{1}{55}(40t_1+115) \geq 0; \quad \mu_{11}(u,\chi_2,*) = \frac{1}{55}(-10t_1+150) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \frac{1}{55}(t_1+150) \geq 0; \quad \mu_0(u,\chi_3,*) = \frac{1}{55}(-t_2+430) \geq 0;$$
$$\mu_1(u,\chi_3,*) = \frac{1}{55}(-t_2+349) \geq 0; \quad \mu_{11}(u,\chi_3,*) = \frac{1}{55}(10t_2+360) \geq 0.$$

Case 4. Let $\chi(u^{11}) = -3\chi(5a) + 4\chi(5b)$, then

$$\mu_0(u,\chi_2,*) = \frac{1}{55}(40t_1+95) \geq 0; \quad \mu_{11}(u,\chi_2,*) = \frac{1}{55}(-10t_1+155) \geq 0;$$
$$\mu_1(u,\chi_2,*) = \frac{1}{55}(t_1+155) \geq 0; \quad \mu_{11}(u,\chi_3,*) = \frac{1}{55}(10t_2+355) \geq 0;$$
$$\mu_0(u,\chi_3,*) = \frac{1}{55}(-40t_2+450) \geq 0; \quad \mu_1(u,\chi_3,*) = \frac{1}{55}(-t_2+344) \geq 0.$$

• Let $|u| = 65$. By (1) and Proposition 2 we have $\nu_{5a}+\nu_{5b}+\nu_{13a}+\nu_{13b} = 1$. Put $t_1 = 8\nu_{5a}+3\nu_{5b}$, $t_2 = \nu_{5a}-4\nu_{5b}$ and $t_3 = 10\nu_{5a}-6\nu_{13a}+7\nu_{13b}$. Since $|u^{13}| = 5$ and $|u^5| = 13$, by parts (iv) and (v) of the Theorem we have to consider 162 cases.

$$\text{First, let } (\alpha,\beta) = \begin{cases} (175,135) & \text{if } \ \chi(u^{13}) = \chi(5a); \\ (155,40) & \text{if } \ \chi(u^{13}) = \chi(5b); \\ (255,115) & \text{if } \ \chi(u^{13}) = 5\chi(5a) - 4\chi(5b); \\ (135,145) & \text{if } \ \chi(u^{13}) = -\chi(5a) + 2\chi(5b); \\ (115,150) & \text{if } \ \chi(u^{13}) = -2\chi(5a) + 3\chi(5b); \\ (95,155) & \text{if } \ \chi(u^{13}) = -3\chi(5a) + 4\chi(5b). \end{cases}$$

Then the following pair of inequalities has no integral solution

$$\mu_0(u,\chi_2,*) = \frac{1}{65}(48t_1+\alpha) \geq 0; \qquad \mu_{13}(u,\chi_2,*) = \frac{1}{65}(-12t_1+\beta) \geq 0.$$

In the remaining cases we put

$$(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \begin{cases} (235,120,380,300), & \text{if } \chi(u^{13}) = 4\chi(5a)-3\chi(5b); \\ (215,125,375,320), & \text{if } \chi(u^{13}) = 3\chi(5a)-2\chi(5b); \\ (195,130,370,340), & \text{if } \chi(u^{13}) = 2\chi(5a)-\chi(5b), \end{cases}$$

and also we parametrize (α_5,α_6) by values of $\chi(u^{13})$ and $\chi(u^5)$ accordingly to the following table

	$\chi(u^{13}) =$ $4\chi(5a){-}3\chi(5b)$	$\chi(u^{13}) =$ $3\chi(5a){-}2\chi(5b)$	$\chi(u^{13}) =$ $2\chi(5a){-}\chi(5b)$	$\chi(u^5)$
(α_5,α_6)	$(93508,93708)$	$(93518,93668)$	$(93528,93628)$	$\chi(13a)$
	$(93521,93721)$	$(93531,93681)$	$(93541,93641)$	$\chi(13b)$
	$(93404,93604)$	$(93414,93564)$	$(93424,93524)$	$9\chi(13a)-8\chi(13b)$
	$(93417,93617)$	$(93427,93577)$	$(93437,93537)$	$8\chi(13a)-7\chi(13b)$
	$(93430,93630)$	$(93440,93590)$	$(93450,93550)$	$7\chi(13a)-6\chi(13b)$
	$(93443,93643)$	$(93453,93603)$	$(93463,93563)$	$6\chi(13a)-5\chi(13b)$
	$(93456,93656)$	$(93466,93616)$	$(93476,93576)$	$5\chi(13a)-4\chi(13b)$
	$(93469,93669)$	$(93479,93629)$	$(93489,93589)$	$4\chi(13a)-3\chi(13b)$
	$(93482,93682)$	$(93492,93642)$	$(93502,93602)$	$3\chi(13a)-2\chi(13b)$
	$(93495,93695)$	$(93505,93655)$	$(93515,93615)$	$2\chi(13a)-\chi(13b)$
	$(93534,93734)$	$(93544,93694)$	$(93554,93654)$	$-\chi(13a)+2\chi(13b)$
	$(93547,93747)$	$(93557,93707)$	$(93567,93667)$	$-2\chi(13a)+3\chi(13b)$
	$(93560,93760)$	$(93570,93720)$	$(93580,93680)$	$-3\chi(13a)+4\chi(13b)$
	$(93573,93773)$	$(93583,93733)$	$(93593,93693)$	$-4\chi(13a)+5\chi(13b)$
	$(93586,93786)$	$(93596,93746)$	$(93606,93706)$	$-5\chi(13a)+6\chi(13b)$
	$(93599,93799)$	$(93609,93759)$	$(93619,93719)$	$-6\chi(13a)+7\chi(13b)$
	$(93612,93812)$	$(93622,93772)$	$(93632,93732)$	$-7\chi(13a)+8\chi(13b)$
	$(93625,93825)$	$(93635,93785)$	$(93645,93745)$	$-8\chi(13a)+9\chi(13b)$

In all of these cases we obtain the following system

$$\mu_0(u,\chi_2,*) = \frac{1}{65}(48t_1+\alpha_1) \geq 0; \qquad \mu_{13}(u,\chi_2,*) = \frac{1}{65}(-12t_1+\alpha_2) \geq 0;$$

$$\mu_0(u,\chi_3,*) = \frac{1}{65}(-48t_2+\alpha_3) \geq 0; \quad \mu_{13}(u,\chi_3,*) = \frac{1}{65}(12t_2+\alpha_4) \geq 0;$$

$$\mu_1(u,\chi_{31},*) = \frac{1}{65}(t_3+\alpha_5) \geq 0; \qquad \mu_{10}(u,\chi_{31},*) = \frac{1}{65}(-4t_3+\alpha_6) \geq 0,$$

that has no integral solutions.

• Let u be a unit of order 77. By (1) and Proposition 2 we have $\nu_{7a} + \nu_{11a} = 1$. Then we obtain the following unsolvable system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{77}(180\nu_{7a} + 161) \geq 0; \quad \mu_0(u, \chi_2, 2) = \frac{1}{77}(-120\nu_{7a} + 98) \geq 0.$$

• Let $|u| = 91$. By (1) and Proposition 2 we have $\nu_{7a} + \nu_{13a} + \nu_{13b} = 1$. Since $|u^7| = 13$, by part (v) of the Theorem we have to consider 18 cases. But in all cases we obtain the same non-compatible system of inequalities

$$\mu_0(u, \chi_2, *) = \frac{1}{91}(216\nu_{7a} + 161) \geq 0; \qquad \mu_{13}(u, \chi_2, *) = \frac{1}{91}(-36\nu_{7a} + 140) \geq 0.$$

• Let $|u| = 143$. By (1) and Proposition 2 we have $\nu_{11a} + \nu_{13a} + \nu_{13b} = 1$. Since $|u^{11}| = 13$, by part (v) of the Theorem we have to consider 18 cases. But in all cases we obtain the same unsolvable system of inequalities

$$\mu_0(u, \chi_3, 3) = \frac{1}{143}(120\nu_{11a} + 88) \geq 0; \quad \mu_0(u, \chi_4, *) = \frac{1}{143}(-120\nu_{11a} + 770) \geq 0.$$

Appendix. Possible partial augmentations $(\nu_{3a}, \nu_{3b}, \nu_{3c})$ for units of order 3:

$$\begin{aligned}
&\{(-3, \nu_{3b}, \nu_{3c}) \mid 5 \leq \nu_{3b} \leq 7,\ \nu_{3a} + \nu_{3b} + \nu_{3c} = 1\} \\
&\quad \cup \{(-2, \nu_{3b}, \nu_{3c}) \mid 1 \leq \nu_{3b} \leq 11,\ \nu_{3a} + \nu_{3b} + \nu_{3c} = 1\} \\
&\quad \cup \{(-1, \nu_{3b}, \nu_{3c}) \mid -3 \leq \nu_{3b} \leq 14,\ \nu_{3a} + \nu_{3b} + \nu_{3c} = 1\} \\
&\quad \cup \{(0, \nu_{3b}, \nu_{3c}) \mid -7 \leq \nu_{3b} \leq 16,\ \nu_{3a} + \nu_{3b} + \nu_{3c} = 1\} \\
&\quad \cup \{(1, \nu_{3b}, \nu_{3c}) \mid -11 \leq \nu_{3b} \leq 12,\ \nu_{3a} + \nu_{3b} + \nu_{3c} = 1\} \\
&\quad \cup \{(2, \nu_{3b}, \nu_{3c}) \mid -15 \leq \nu_{3b} \leq 8,\ \nu_{3a} + \nu_{3b} + \nu_{3c} = 1\}.
\end{aligned}$$

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