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A note on the growth rate in the Fazekas–Klesov general law of large numbers and on the weak law of large numbers for tail series

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Abstract. Using the Hájek–Rényi type maximal inequality, FAZEKAS and KLESOV (2000) obtained the strong law of large numbers for sequences of random variables. Under the same conditions as those in FAZEKAS and KLESOV, HU and HU (2006) obtained the strong growth rate for sums of random variables which improves FAZEKAS and KLESOV's result. We further extend and improve these results. Next, the approach of using Hájek–Rényi type maximal inequality for proving limit theorems is applied to the weak law of large numbers for tail series.

1. Introduction

FAZEKAS and KLESOV (2000) gave a general method for obtaining the strong law of large numbers for sequences of random variables by using a Hájek–Rényi type maximal inequality. This general method, or better to say approach, of proving strong law of large numbers suggests to use directly a maximal inequality (the so-called Hájek–Rényi inequality) for a sequence of normed partial sums of dependent random variables. Under the same conditions as those in FAZEKAS and KLESOV (2000), HU and HU (2006) found the method for obtaining the strong growth rate for sums of random variables. Although the proof of HU and HU (2006) owes much to that of FAZEKAS and KLESOV (2000), their result is sharper.

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In this paper, we find a new method for obtaining the strong growth rate for sums of random variables by using the ideas of Fazekas and Klesov (2000). Our result generalizes and sharpens the results of Hu and Hu (2006). Moreover, our method can be applied to almost all cases of a dependence structure considered in Hu and Hu (2006), and we can get better results. We further extend and improve these results. Next, the approach of using Hájek–Rényi type maximal inequality for proving limit theorems is applied to the weak law of large numbers for tail series.

We use the following notation. Let $\{X_n,\ n \geq 1\}$ denote a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. The partial sums of the random variables are $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$ and $S_0 = 0$. Let $\varphi(x)$ be a positive function satisfying

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} < \infty \text{ and } 0 < \varphi(x) \uparrow \infty \text{ on } [c, \infty) \quad \text{for some } c > 0. \tag{1}$$

2. Main Results

The following lemma generalizes Dini's theorem for scalar series (cf. Lemma 1.1 in Hu and Hu (2006)).

Lemma 1. *Let $a_1, a_2, \ldots$ be a sequence of nonnegative real numbers such that $a_n > 0$ for infinitely many n. Let $v_n = \sum_{i=n}^{\infty} a_i$ for $n \geq 1$. Let $\varphi(x)$ be a positive function satisfying* (1). *If $\sum_{n=1}^{\infty} a_n < \infty$, then $\sum_{n=1}^{\infty} a_n \varphi(1/v_n) < \infty$.*

Proof. Without loss of generality, we may assume that $c = 1$ and $v_1 \leq 1$. For each $k \geq 0$, define n_k by $n_k = \min\{n : v_n \leq 2^{-k}\}$. It follows that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \varphi(1/v_n) &= \sum_{k=0}^{\infty} \sum_{n=n_k}^{n_{k+1}-1} a_n \varphi(1/v_n) \leq \sum_{k=0}^{\infty} \varphi(1/v_{n_{k+1}-1}) \sum_{n=n_k}^{n_{k+1}-1} a_n \\ &\leq \sum_{k=0}^{\infty} \varphi(1/v_{n_{k+1}-1}) v_{n_k} \leq \sum_{k=0}^{\infty} \frac{\varphi(2^{k+1})}{2^k},\end{aligned}$$

where we assume that in the case $n_{k+1} = n_k$, the sum $\sum_{n=n_k}^{n_{k+1}-1} = 0$.

Note that $\sum_{n=1}^{\infty} \varphi(n)/n^2 < \infty$ is equivalent to $\sum_{k=0}^{\infty} \varphi(2^k)/2^k < \infty$, since

$$\sum_{k=0}^{\infty} \frac{\varphi(2^k)}{2^k} \leq 4 \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} \frac{\varphi(n)}{n^2} = 4 \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^2} \leq 4 \sum_{k=0}^{\infty} \frac{\varphi(2^{k+1})}{2^k}.$$

The result follows by (1). $\square$

It is easy to find examples of functions $\varphi(x)$ that satisfy (1). Such functions are $|x|^\delta$ or $|x|^\delta(\log|x|)^\alpha$, where $0<\delta<1$ and α is any real number. Hu and Hu (2006) used Lemma 1 with $\varphi(x)=|x|^\delta$ $(0<\delta<1)$ to obtain the strong growth rate for sums of random variables.

The following lemma is due to Fazekas and Klesov (2000), Theorem 2.1.

Lemma 2. *Let $\{b_n,\ n\ge 1\}$ be a nondecreasing unbounded sequence of positive numbers and $\{\alpha_n,\ n\ge 1\}$ be a sequence of nonnegative real numbers such that $\alpha_n>0$ for infinitely many n. Let r and C be fixed positive numbers. Assume that for each $n\ge 1$*

$$E\Big(\max_{1\le i\le n}|S_i|\Big)^r \le C\sum_{i=1}^n \alpha_i$$

and

$$\sum_{n=1}^\infty \alpha_n b_n^{-r}<\infty.$$

Then the strong law of large numbers holds, that is,

$$\lim_{n\to\infty} S_n/b_n=0 \text{ a.s.}$$

The following theorem gives a sharper result than Theorem 2.1 of Fazekas and Klesov (2000) (Lemma 2) and Lemma 1.2 of Hu and Hu (2006).

Theorem 1. *Assume that all the conditions of Lemma 2 are satisfied. Let $\varphi(x)$ be a positive function satisfying (1). Let*

$$\beta_n=\max_{1\le i\le n} b_i\varphi(1/v_i)^{-1/r} \quad \text{for } n\ge 1,$$

where $v_n=\sum_{i=n}^\infty \alpha_i b_i^{-r}$. Then the following statements hold.

(i) *If the sequence $\{\beta_n,\ n\ge 1\}$ is bounded, then $S_n/\beta_n=O(1)$ a.s.*

(ii) *If the sequence $\{\beta_n,\ n\ge 1\}$ is unbounded, then $S_n/\beta_n=o(1)$ a.s., i.e., $\lim_{n\to\infty} S_n/\beta_n=0$ a.s.*

Proof. It is easy to see that $\{\beta_n\}$ is a nondecreasing sequence of positive numbers. Since $\beta_n\ge b_n\varphi(\frac{1}{v_n})^{-1/r}$, we have by Lemma 1 that

$$\sum_{n=1}^\infty \alpha_n\beta_n^{-r}\le\sum_{n=1}^\infty \alpha_n\varphi(1/v_n)b_n^{-r}<\infty.$$

If the sequence $\{\beta_n,\ n\ge 1\}$ is unbounded, then $\lim_{n\to\infty} S_n/\beta_n=0$ a.s. by Lemma 2.

Now assume that $\{\beta_n\}$ is bounded by some constant $D > 0$. Then

$$\sum_{n=1}^{\infty} \alpha_n \leq D^r \sum_{n=1}^{\infty} \alpha_n \beta_n^{-r} < \infty.$$

It follows by the monotone convergence theorem that

$$E\Big(\sup_{n\geq 1} |S_n|\Big)^r = \lim_{n\to\infty} E\Big(\max_{1\leq i\leq n} |S_i|\Big)^r \leq C \sum_{n=1}^{\infty} \alpha_n < \infty.$$

Thus $\sup_{n\geq 1} |S_n| < \infty$ a.s. Since $0 < \beta_1 \leq \beta_n$ for all $n \geq 1$, we have that $S_n/\beta_n = O(1)$ a.s. □

Remarks. 1. In both cases either $S_n/\beta_n = O(1)$ a.s. or $S_n/\beta_n = o(1)$ a.s., it can be easily obtained that $\lim_{n\to\infty} S_n/b_n = 0$ a.s.

2. For the special case of $\varphi(x) = |x|^\delta$ $(0 < \delta < 1)$, in Lemma 1.2 of Hu and Hu (2006) it is proved that $S_n/\beta_n = O(1)$ a.s. under the same conditions as those in Theorem 1. We can safely state that Theorem 1 extends and sharpens the result of Hu and Hu (2006). Hence we can sharpen many results proved in Hu and Hu (2006).

It is interesting to investigate the cases in which the sequence $\{\beta_n,\ n \geq 1\}$ is unbounded. But first we derive a useful condition on $\varphi(x)$.

Lemma 3. *If $\varphi(x)$ is a positive function satisfying* (1), *then*

$$\lim_{n\to\infty} \varphi(n)/n = 0.$$

Proof. Without loss of generality, we may assume that $\varphi(x)$ is a non-decreasing on $[1, \infty)$. According to the proof of Lemma 1, we have that $\sum_{k=0}^{\infty} \varphi(2^k)/2^k < \infty$ and hence $\lim_{k\to\infty} \varphi(2^k)/2^k = 0$. For $2^k \leq n < 2^{k+1}$,

$$\frac{\varphi(2^k)}{2^{k+1}} \leq \frac{\varphi(n)}{n} \leq \frac{\varphi(2^{k+1})}{2^k}.$$

Thus we have that $\lim_{n\to\infty} \varphi(n)/n = 0$. □

The following lemma shows that $\{\beta_n\}$ defined in Theorem 1 is unbounded when $\alpha_n = 1$ for all $n \geq 1$.

Lemma 4. *Let $b_1, b_2, \ldots$ be a nondecreasing unbounded sequence of positive numbers and $\varphi(x)$ be a positive function satisfying* (1). *Assume that*

$$\sum_{n=1}^{\infty} b_n^{-r} < \infty \quad \textit{for some } r > 0.$$

Let $\beta_n = \max_{1\le i\le n} b_i \varphi(1/v_i)^{-1/r}$ for $n \ge 1$, where $v_n = \sum_{i=n}^{\infty} b_i^{-r}$. Then $\{\beta_n\}$ is unbounded.

Proof. For each $k \ge 0$, define n_k by

$$n_k = \min\{n : b_n \ge 2^k\}.$$

Let $d_k = n_k - n_{k-1}$ for $k \ge 1$. Then we have that

$$v_{n_k} \ge \sum_{n=n_k}^{n_{k+1}-1} b_n^{-r} \ge d_{k+1} b_{n_{k+1}-1}^{-r} \ge d_{k+1} 2^{-r(k+1)} \ge d_{k+1} 2^{-r} b_{n_k}^{-r}. \tag{2}$$

It follows that for all large k

$$b_{n_k} \varphi(1/v_{n_k})^{-1/r} \ge b_{n_k} \varphi\left(\frac{2^r b_{n_k}^r}{d_{k+1}}\right)^{-1/r} = \left[\frac{2^r b_{n_k}^r / d_{k+1}}{\varphi(2^r b_{n_k}^r / d_{k+1})}\right]^{1/r} \frac{d_{k+1}^{1/r}}{2}. \tag{3}$$

Since $\lim_{n\to\infty} v_n = 0$, (2) implies that $\lim_{d_{k+1}\ne 0, k\to\infty} b_{n_k}^r / d_{k+1} = \infty$. By (3) and Lemma 3, we obtain that $\lim_{d_{k+1}\ne 0, k\to\infty} b_{n_k} \varphi(1/v_{n_k})^{-1/r} = \infty$. Hence $\{\beta_n\}$ is unbounded. □

As a consequence of Theorem 1 and Lemma 4, we obtain the following result.

Theorem 2. *Let $b_1, b_2, \ldots$ be a nondecreasing unbounded sequence of positive numbers and $\varphi(x)$ be a positive function satisfying* (1). *Let r and C be fixed positive numbers. Assume that for each $n \ge 1$*

$$E\Big(\max_{1\le i\le n} |S_i|\Big)^r \le Cn$$

and

$$\sum_{n=1}^{\infty} b_n^{-r} < \infty.$$

Let $\beta_n = \max_{1\le i\le n} b_i \varphi(1/v_i)^{-1/r}$ for $n \ge 1$, where $v_n = \sum_{i=n}^{\infty} b_i^{-r}$. Then $\lim_{n\to\infty} S_n/\beta_n = 0$ a.s.

3. Application to associated random variables

By using Theorem 1 and Theorem 2, we can extend and sharpen many results from Hu and Hu (2006). For an illustration we show how Theorem 3.2 of Hu and Hu (2006) for (positively) associated random variables can be improved. The interested reader could consider an extension of other results of Hu and Hu (2006) devoted to negatively associated random variables and martingale differences.

The concept of associated random variables was introduced by Esary *et al.* (1967) in the following way. A finite family of random variables $\{X_i,\ 1 \le i \le n\}$ with finite second moments is said to be *associated* if for any real coordinate-wise nondecreasing scalar functions f and g on $\mathbf{R}^n$,

$$\mathrm{Cov}\,(f(X_1,\ldots,X_n), g(X_1,\ldots,X_n)) \ge 0.$$

An infinite family of random variables $\{X_i,\ i \ge 1\}$ is *associated* if every finite subfamily is associated.

Theorem 3. *Let $\{X_n,\ n \ge 1\}$ be a sequence of associated random variables with mean zero and finite variance, and $\{b_n,\ n \ge 1\}$ be a nondecreasing unbounded sequence of positive numbers. Assume that*

$$\sum_{n=1}^{\infty} \frac{ES_n^2 - ES_{n-1}^2}{b_n^2} < \infty.$$

Let $\beta_n = \max_{1\le i\le n} b_i \varphi(1/v_i)^{-1/2}$ for $n \ge 1$, where $v_n = \sum_{i=n}^{\infty}(ES_i^2 - ES_{i-1}^2)/b_i^2$. Then the following statements hold.

(i) *If $\{\beta_n,\ n \ge 1\}$ is bounded, then $S_n/\beta_n = O(1)$ a.s.*

(ii) *If $\{\beta_n,\ n \ge 1\}$ is unbounded, then $S_n/\beta_n = o(1)$ a.s., i.e., $\lim_{n\to\infty} S_n/\beta_n = 0$ a.s.*

Proof. The proof is similar to that of Theorem 3.2 in Hu and Hu (2006). Since $\{X_n\}$ is a sequence of associated random variables, $\{-X_n\}$ is also a sequence of associated random variables. From Theorem 2 of Newman and Wright (1981), we have

$$E\Big(\max_{1\le i\le n} S_i^2\Big) \le E\Big(\max_{1\le i\le n} S_i\Big)^2 + E\Big(\max_{1\le i\le n} (-S_i)\Big)^2 \le 2ES_n^2.$$

By the definition of associated random variables, we get

$$ES_n^2 = ES_{n-1}^2 + EX_n^2 + 2\mathrm{Cov}\,(S_{n-1}, X_n) \ge ES_{n-1}^2.$$

Let $\alpha_n = ES_n^2 - ES_{n-1}^2$ for $n \geq 1$. Then

$$E\Big(\max_{1\leq i\leq n} S_i^2\Big) \leq 2\sum_{i=1}^{n} \alpha_i \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\alpha_n}{b_n^2} < \infty.$$

Thus the result follows from Theorem 1. □

Remark 3. Under the same conditions of Theorem 3 with $\varphi(x) = |x|^\delta$ $(0 < \delta < 1)$, Hu and Hu (2006) proved that $S_n/\beta_n = O(1)$ a.s. which improves Theorem 3.3 of Prakasa Rao (2002). Thus Theorem 3 extends the result of Hu and Hu (2006). In particular, we point out the fact that Theorem 3 sharpens the result of Hu and Hu (2006) when $\{\beta_n,\ n \geq 1\}$ is unbounded. An example of a situation is which $\{\beta_n,\ n \geq 1\}$ is unbounded can be easily obtained. For example, if $ES_n^2 - ES_{n-1}^2 = 1$ for all $n \geq 1$, then $\{\beta_n,\ n \geq 1\}$ is unbounded by Lemma 4.

4. Weak law of large numbers for tail series

The rate of convergence for an almost surely convergent series $S_n = \sum_{j=1}^{n} X_j$ of variables $\{X_n,\ n \geq 1\}$ is studied in this section. More specifically, if S_n converges almost surely to a random variable S, then the tail series $T_n \equiv S - S_{n-1} = \sum_{j=n}^{\infty} X_j$ is a well-defined sequence of random variable (referred to as the *tail series*) with $T_n \to 0$ almost surely. The main result provides conditions for

$$\sup_{k\geq n} |T_k|/b_n \to 0 \ \text{ in probability} \tag{4}$$

to hold for a given numerical sequence $0 < b_n = o(1)$. These results are, of course, of greatest interest when $b_n = o(1)$. Nam and Rosalsky (1996) provided an example showing *inter alia* that a.s. convergence to 0 does not necessarily hold for the expression in (4).

Theorem 4 is a very general result and we will see that some previously obtained results are immediate corollaries of it. In Theorem 4, a condition is imposed in general on the joint distributions of the random variables $\{X_n,\ n \geq 1\}$. However, in the Corollary $\{X_n,\ n \geq 1\}$ is a martingale difference sequence of random variables. Certainly, the result is true when $\{X_n,\ n \geq 1\}$ is a sequence of independent random variables. In the Corollary a moment condition on the $|X_n|$ and a limit behavior of b_n are imposed.

Theorem 4. *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables, $\{b_n,\ n \geq 1\}$ be a sequence of nonnegative numbers, $\{\alpha_n,\ n \geq 1\}$ be a sequence of positive* *numbers, and* $r > 0$. *Let moreover*

$$\sum_{j=1}^{\infty} \alpha_j < \infty.$$

If for all natural numbers $n < m$

$$E\left(\max_{n \le k \le m} \left| \sum_{j=n}^{k} X_j \right| \right)^r \le \sum_{j=n}^{m} \alpha_j,$$

then the series $\sum_{n=1}^{\infty} X_n$ *converges a.s. and the tail series* $\{T_n = \sum_{j=n}^{\infty} X_j,\ n \ge 1\}$ *is a well-defined sequence of random variables. Next, if*

$$\sum_{j=n}^{\infty} \alpha_j = o(b_n^r) \text{ as } n \to \infty,$$

then the tail series obeys the limit law

$$\frac{\sup_{k \ge n} |T_k|}{b_n} \to 0 \text{ in probability.}$$

Proof. For arbitrary $\varepsilon > 0$ and $n \ge 1$

$$\begin{aligned}
& P\left\{ \sup_{m > n} \left| \sum_{j=1}^{m} X_j - \sum_{j=1}^{n} X_j \right| > \varepsilon \right\} \\
& \le \varepsilon^{-r} E\left(\sup_{m > n} \left| \sum_{j=1}^{m} X_j - \sum_{j=1}^{n} X_j \right|^r \right) \text{ (by the Markov inequality)} \\
& = \varepsilon^{-r} \lim_{m \to \infty} E\left(\max_{n+1 \le k \le m} \left| \sum_{j=n+1}^{k} X_j \right|^r \right) \\
& \qquad \text{(by the Lebesgue monotone convergence theorem)} \\
& \le \varepsilon^{-r} \lim_{m \to \infty} \sum_{j=n}^{m} \alpha_j = o(1).
\end{aligned}$$

Then by Corollary 3.3.4 of Chow and Teicher (1997), p. 68, $\sum_{n=1}^{\infty} X_n$ converges a.s. Thus, the tail series $\{T_n = \sum_{j=n}^{\infty} X_j,\ n \ge 1\}$ is a well-defined sequence of random variables.

Next, for arbitrary $\varepsilon > 0$

$$P\left\{ \frac{\sup_{k \ge n} |T_k|}{b_n} > \varepsilon \right\} \le (\varepsilon b_n)^{-r} E\left(\sup_{k \ge n} |T_k|^r \right) \text{ (by the Markov inequality)}$$

$$= (\varepsilon b_n)^{-r} \lim_{N\to\infty} E\left(\max_{n\le k\le N} \left| \lim_{m\to\infty} \sum_{j=k}^{m} X_j \right|^r \right)$$

(by the Lebesgue monotone convergence theorem)

$$= (\varepsilon b_n)^{-r} \lim_{N\to\infty} E\left(\max_{n\le k\le N} \lim_{m\to\infty} \left| \sum_{j=k}^{m} X_j \right|^r \right)$$

$$= (\varepsilon b_n)^{-r} \lim_{N\to\infty} E\left(\lim_{m\to\infty} \max_{n\le k\le N} \left| \sum_{j=k}^{m} X_j \right|^r \right)$$

$$\le (\varepsilon b_n)^{-r} \lim_{N\to\infty} \liminf_{m\to\infty} E\left(\max_{n\le k\le N} \left| \sum_{j=k}^{m} X_j \right|^r \right) \text{ (by Fatou's lemma)}$$

$$\le (\varepsilon b_n)^{-r} \liminf_{m\to\infty} E\left(\max_{n\le k\le m} \left| \sum_{j=k}^{m} X_j \right|^r \right) \le (\varepsilon b_n)^{-r} \liminf_{m\to\infty} \sum_{j=n}^{m} \alpha_j = o(1)$$

The conclusion of the theorem now follows easily. □

The next Corollary was obtained by Rosalsky and Rosenblatt (1998). Our proof seems to be much simpler.

Corollary. *Let $\{S_n = \sum_{j=1}^{n} X_j,\ n \ge 1\}$ be a martingale and $1 \le r \le 2$. If $\sum_{j=n}^{\infty} E|X_j|^r = \mathcal{O}(b_n^r)$, then the series $\sum_{n=1}^{\infty} X_n$ converges a.s.*
If $\sum_{j=n}^{\infty} E|X_j|^r = o(b_n^r)$, then the tail series obeys the limit law

$$\frac{\sup_{k\ge n} |T_k|}{b_n} \to 0 \quad \textit{in probability.}$$

Proof. By the Burkholder-Davis-Gundy inequality (cf., for example Chow and Teicher (1997)) we have that for all $m \ge n \ge 1$

$$E\left(\max_{n\le k\le m} \left| \sum_{j=n}^{k} X_j \right|^r \right) \le C \sum_{j=n}^{m} E|X_j|^r.$$

We have

$$\lim_{m\to\infty} E\left(\max_{n\le k\le m} \left| \sum_{j=n}^{k} X_j \right|^r \right) \le C \sum_{j=n}^{\infty} E|X_j|^r = o(1) \text{ as } n \to \infty$$

under the conditions of the Corollary.

The Corollary follows immediately from Theorem 4. □

Certainly, Theorem 4 could be generalized on the Banach space setting. We refer the interested reader to the papers Rosalsky and Volodin (2001) and (2003) for such type of generalizations.

References

[1] Y. S. Chow and H. Teicher, Probability Theory: Independence, Interchangeability, Martingales, 3rd ed., *Springer-Verlag*, *New York*, 1997.

[2] J. D. Esary, F. Proschan, and D. W. Walkup, Association of random variables, with applications, *Ann. Math. Statist.* **38** (1967), 1466–1474.

[3] I. Fazekas and O. Klesov, A general approach to the strong law of large numbers, *Theory Probab. Appl.* **45** (2000), 436–449.

[4] S. Hu and M. Hu, A general approach rate to the strong law of large numbers, *Statist. Probab. Lett.* **76** (2006), 843–851.

[5] E. Nam and A. Rosalsky, On the rate of convergence of series of random variables, *Theory Probab. Math. Statist.* **52** (1996), 129–140.

[6] C. M. Newman and A. L. Wright, An invariance principle for certain dependent sequences, *Ann. Probab.* **9** (1981), 671–675..

[7] B. L. S. Prakasa Rao, Hájek–Rényi type inequality for associated sequences, *Statist. Probab. Lett.* **57** (2002), 139–143.

[8] A. Rosalsky and J. Rosenblatt, On convergence of series of random variables with applications to martingale convergence and convergence of series with orthogonal summands, *Stochastic Anal. Appl.* **16** (1998), 553–566.

[9] A. Rosalsky and A. Volodin, On convergence of series of random elements via maximal moment relations with applications to martingale convergence and to convergence of series with p-orthogonal summands, *Georgian Mathematical Journal* **8** (2001), 377–388.

[10] A. Rosalsky and A. Volodin, On convergence of series of random elements via maximal moment relations with applications to martingale convergence and to convergence of series with p-orthogonal summands. Correction., *Georgian Mathematical Journal* **10** (2003), 799–802.

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