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Generators of topological rings

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Dedicated to Professor Adalbert Bovdi on his 70th birthday

Abstract. H. Fujita and D. Shakhmatov, in [FS], have introduced the concept of a finitely generated – modulo open sets – topological group. In this paper we transfer their results to topological rings which are finitely generated modulo open sets. Moreover, we are introducing the concept of the free compact associative ring $c\mathbb{F}_p\langle X\rangle$ of prime characteristic p over a set X. For a countable set X, the free ring $c\mathbb{F}_p\langle X\rangle$ is universal in the following two means: (i) that it contains an isomorphic copy of every compact second countable associative ring of characteristic p; (ii) every compact second countable associative ring with 1 of characteristic p is a continuous homomorphic image of $c\mathbb{F}_p\langle X\rangle$. We introduce also the concept of a free topologically nilpotent compact ring of prime characteristic over a set. We give a realization of the free compact topologically nilpotent ring with a countable set of generators and prove that it is a domain. It is obtained that every compact second countable topologically nilpotent compact ring is a continuous homomorphic image of a compact domain. There are compact rings of prime characteristic which are not continuous images of compact reduced rings.

1. Introduction

H. Fujita and D. Shakhmatov have considered in a recent paper [FS] some topological generalizations of finitely generated groups. In this paper we prove

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analogous results for topological rings, where the methods of [FS] are adapted in Section 3.

In [U1], it was proved that a linearly compact ring having a set of topological generators of cardinality less than 2^ω is strictly linearly compact. A generalization of this result will be given in Theorem 4.1, from which follows that every left linearly compact finitely generated modulo open sets ring R having a fundamental system of neighborhoods of 0 consisting of ideals is strictly linearly compact.

In [Mv], A. I. Mal'cev, has proved that every countable or finite associative ring can be embedded in a two-generated associative ring. Also, we should mension the well known result that a free associative algebra of countably rank is a subalgebra of rank two [Co, § 0.7, Ex. 6]. While in this context, we consider the following question: can an associative compact separable ring be embedded in a compact associative ring with two topological generators? A negative answer is given in Example 5.1.

In Section 5, we prove that for a prime p the two free rings $c\mathbb{F}_p\langle 2\rangle$ and $c\mathbb{F}_p\langle\omega\rangle$ are universal in the sense that every compact associative second countable ring of characteristic p can be embedded in $c\mathbb{F}_p\langle 2\rangle$ and $c\mathbb{F}_p\langle\omega\rangle$, respectively. In addition, every compact associative second countable ring with 1 of characteristic p is a continuous homomorphic image of $c\mathbb{F}_p\langle\omega\rangle$. We introduce the notion of a free compact topologically nilpotent ring of prime characteristic. We prove that the free compact topologically nilpotent ring with a countable set of generators is a domain. This implies that every second countable compact topologically nilpotent ring of prime characteristic is a continuous homomorphic image of a compact domain. Not every compact ring of prime characteristic is a continuous homomorphic image of a compact reduced ring.

In Section 6, we outline the construction of a compact right and left Noetherian ring without left classical quotient ring in abstract sense.

2. Notation

All topological rings are assumed to be T_2, not necessarily associative. We use the terminology on topological rings as it is given in [W]. We will identify ω with the set of natural numbers, where ω is the first infinite ordinal. In the set of natural numbers $\mathbb{N}$, if $m, n \in \mathbb{N}$, $m \leq n$, then $[m, n]$ will denote the set $\{m, m+1, \dots, n\}$ of naturals. If X is a subset of a ring R then $\langle X\rangle$ stands for the subring generated by X and (X) for the ideal generated by X; and if R is an associative ring, then $J(R)$ stands for the Jacobson radical of R. A Galois field containing p^n elements, where p is prime, is denoted by $\mathbb{F}_{p^n}$. For each $n \in \mathbb{N}$ and each ring $R, M(n, R)$ stands for the ring of $n \times n$-matrices over R. In a topological space X, the closure of a subset A of X is denoted by $\overline{A}$; and the weight of X is denoted by wX.

3. Topological rings finitely generated modulo open sets

Recall that a topological ring R is said to be:

(i) *compactly generated* provided that $R = \langle K \rangle$ for some compact subspace K of R,

(ii) *σ-compact* if there exists a sequence $\{K_n : n \in \omega\}$ of compact subsets such that $R = \cup_{n\in\omega} K_n$,

(iii) *ω-bounded* if for every neighborhood V of 0 there exists a countable subset $S \subset R$ such that $R = S + V$,

(iv) *totally bounded* if for every neighborhood V of 0 there exists a finite subset S such that $R = S + V$,

(v) *finitely generated modulo open sets* if for every nonempty open set $U \subset R$, there exists a finite subset $F \subset R$ such that $R = \langle F \cup U \rangle$,

(vi) *almost metrizable* provided there exist a nonempty compact subset $K \subset R$ and a sequence $\{U_n\}_{n\in\omega}$ of open subsets containing K such that every open subset of R containing K contains some U_n. The family $\{U_n\}_{n\in\omega}$ is called a *neighborhood basis* of K.

Theorem 3.1. *If a topological ring R contains a compactly generated dense subring then its additive group is ω-bounded and R is finitely generated modulo open sets.*

PROOF. Let V be an arbitrary neighborhood of 0, and let K be a compact subset of R for which $\langle K \rangle$ is a dense subring. We may assume without loss in generality that $0 \in K$ and K is symmetric. Obviously, there exist compact subsets K_n, $0 \in K_n$, $n \in \omega$ such that $\langle K \rangle = \cup_{n\in\omega} K_n$. For every $n \in \omega$ there exists a finite subset F_n of K_n such that $K_n \subset F_n + V$; and hence $R = (\cup_{n\in\omega} F_n) + V + V$.

We have to prove that R is finitely generated modulo open sets. Let U be any nonempty subset of R. If $u \in U$, then $-u + U$ is a 0-neighborhood of R. There exists a finite subset T of K such that $K \subset T - u + U$. Put $F = T \cup \{-u\}$; then $\langle F \cup U \rangle = R$. □

Lemma 3.2 ([FS]). *Let X be a topological space and let $K \subset X$ be a compact set with a neighborhood basis $\{U_n\}_{n\in\omega}$. Suppose that we have compact sets $C_n \subset \cup_{k\leq n} U_k$ for all $n \in \omega$. Then the set $C = K \cup \left(\bigcup_{n\in\omega} C_n\right)$ is also compact.*

Theorem 3.3. *An almost metrizable topological ring R contains a dense compactly generated subring if and only if it is ω-bounded and finitely generated modulo open sets.*

PROOF. The implication ($\Rightarrow$) is obviously true, so it suffices to prove the implication ($\Leftarrow$). Let K be a compact subset of R with a countable neighborhood basis $\{U_n\}_{n\in\omega}$. Since the additive group of R is ω-bounded, for each $n \in \omega$ there exists a countable subset $S_n \subset R$ such that $R = S_n + U_n$. The set $S = \cup_{n\in\omega} S_n$ is countable, so we can fix its enumeration $S = \{s_n\}_{n\in\omega}$. Let $r \in R$ and let V be an arbitrary neighborhood of 0. Then $K - V$ is a neighborhood of K, and so there exists $n \in \omega$ such that $K \subset U_n \subset K - V$. Since $S_n + U_n = R$, there is $s \in S_n$ such that $r \in s + U_n \subset s + K - V$. Let $r = s + k - v$ with $k \in K$ and $v \in V$. Then

$$r + v = s + k \in (r + V) \cap (S + K) \neq \varnothing.$$

It follows that $S + K$ is dense in R. By the conditions for each $n \in \omega$ there is a finite subset F_n such that $R = \langle F_n \cup U_n\rangle$. Set $E_0 = F_0 \cup \{s_0\}$; it follows that $R = \langle F_0 \cup U_0\rangle$. So there is a finite subset $E_1 \subset U_0$ such that

$$F_1 \cup \{s_1\} \subset \langle E_0 \cup E_1\rangle,$$

which implies that $\langle E_0 \cup E_1 \cup U_1\rangle = R$. Hence there is a finite set $E_2 \subset U_1$ such that $F_2 \cup \{s_2\} \subset \langle E_0 \cup E_1 \cup E_2\rangle$. We obtain in this way finite subsets $E_{n+1} \subset U_n, n \in \omega$ such that

$$F_{n+1} \cup \{s_{n+1}\} \subset \langle E_0 \cup E_1 \cup \cdots \cup E_{n+1}\rangle.$$

By Lemma 3.2 the set $C = K \cup (\bigcup_{n\in\omega} E_n)$ is compact. The subring $\langle C\rangle$ is dense since it contains $S + K$. It follows that R contains a compactly generated dense subring. $\square$

Corollary 3.4. *A topological ring with the first axiom of countability contains a dense compactly generated subring if and only if it is separable and finitely generated modulo open sets.*

Lemma 3.5. *If a σ-compact almost metrizable ring R contains a dense compactly generated subring, then R itself is compactly generated.*

Proof. Suppose that $R = \cup_{n\in\omega} L_n$, where L_n are compact, and let $H = \langle L_0 \rangle$ be dense in R. Let $K \subset R$ be a compact set with a neighborhood basis $\{U_n\}_{n\in\omega}$; we may assume that $\overline{U_{n+1}} \subset U_n$ for every $n \in \omega$. For $n \in \omega$ and $l \in L_n$, then $(l - U_{n+1}) \cap H \neq \varnothing$. Then there exist $u_l \in U_{n+1}$ and $h_l \in H$ such that $l - u_l = h_l$, hence $l = h_l + u_l \in h_l + U_{n+1}$. By compactness of L_n, there exists a finite subset F_n of H such that $L_n \subset U_{n+1} + F_n$. Evidently, $C_n = \overline{(L_n - F_n) \cap U_{n+1}}$ is a compact subset and is contained in U_n. We claim that $L_n \subset C_n + F_n$. Indeed, if $l \in L_n$, then $l = u + f$, where $u \in U_{n+1}$ and $f \in F_n$. It follows that $u = l - f \in (L_n - F_n) \cap U_{n+1} \subset C_n$ which implies that $l \in C_n + F_n$, and so $L_n \subset C_n + F_n \subset \langle C_n \cup L_0 \rangle$. Therefore, by setting, $C = L_0 \cup K \cup (\bigcup_{n\in\omega} C_n)$, we obtain $\langle C \rangle = R$. It follows from Lemma 3.2 that C is compact. □

Evidently, a topological ring without proper open subrings is finitely generated modulo open sets.

Corollary 3.6. *Let R be an almost metrizable σ-compact ring. If R is either without proper open subrings or is a connected ring or is totally bounded, then it is compactly generated.*

Theorem 3.7. *For a locally compact ring R the following conditions are equivalent:*

(i) *R has a dense compactly generated subring,*

(ii) *R is compactly generated,*

(iii) *R is finitely generated modulo open sets.*

Proof. (i) ⇒ (ii) Evidently.

(ii) ⇒ (iii) Follows from Theorem 3.1.

(iii) ⇒ (i) There exists a finite subset F such that $\langle F \cup U \rangle = R$. Then $R = \langle F \cup \overline{U} \rangle$ is compactly generated. □

4. Topologically finitely generated modulo open sets linearly compact rings

In ([U1], Theorem 2), we have proved that a linearly compact topological ring having a set of topological generators of cardinality less than 2^ω is strictly linearly compact. Here, we will extend this result.

Recall that a complete linearly topologized module M is called *strictly linearly compact* provided for each open submodule V the factor module M/V is artinian. A topological ring R is called *strictly linearly compact* provided the left topological R-module ${}_RR$ is strictly linearly compact.

Theorem 4.1. *Let M a linearly compact module such that for every open submodule V the factor module M/V has cardinality less than 2^ω. Then M is strictly linearly compact.*

PROOF. The module M is topologically isomorphic to the inverse limit $\varprojlim M/V$, where V runs over all open submodules of M. By ([U1], Theorem 2), each M/V is strictly linearly compact. Then M is strictly linearly compact as the class of strictly linearly compact modules is closed under taking of inverse limits by ([B2], Chapter III, Exercise 19). □

Corollary 4.2. *Every σ-compact linearly compact R-module is strictly linearly compact. Every linearly compact ring finitely generated modulo open sets R having a fundamental system of neighborhoods of 0 consisting of ideals is strictly linearly compact.*

Open Question. Is it true that every linearly compact ring R finitely generated modulo open sets is strictly linearly compact?

5. Free compact rings

Recall that a topological space is called *separable* if it contains a dense subset of cardinality $\leq \omega$, see ([E], 1.1.3).

Example 5.1. A separable compact associative Hausdorff ring which cannot be embedded in a compact ring with a finite number of topological generators.

Fix a prime number p and let $\mathbb{F}_p$ be the field of p elements. According to Theorem of Hewitt–Marczewski–Pondiczery ([E],Theorem 2.3.15), the compact ring $R = \mathbb{F}_p^c$, $c = 2^\omega$, is separable. We claim that R cannot be embedded in a compact associative ring with a finite number of topological generators. Assume on the contrary that there exists a compact ring S with a finite number of topological generators that contains R. Denote by S_0 the connected component of S, i.e., the maximal connected subgroup of the additive group of S. Since $S_0 \cap R = 0$, the ring R is topologically isomorphic to a subring of S/S_0. Since S/S_0 is topologically finitely generated, it is metrizable ([U2], Theorem 27.30), and hence R is metrizable, a contradiction.

Lemma 5.1. *Let p be a prime and let $R_n = M(1,\mathbb{F}_p) \times \cdots \times M(n,\mathbb{F}_p)$. For each $i \in [1,n]$, fix a pair of generators x_i, y_i of $M(i,\mathbb{F}_p)$; then $R_n = \langle x,y\rangle$, where $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$.*

Proof. Obviously, $pr_i(\langle x,y\rangle) = M(i,\mathbb{F}_p)$, where pr_i means the projection of R_n on $M(i,\mathbb{F}_p)$, $\quad i \in [1,n]$. Furthermore, denote

$$I_i = \ker(pr_i\langle x,y\rangle), \qquad i \in [1,n].$$

Since $I_1 \cap I_2 \cap \cdots \cap I_n = 0$, then $\langle x,y\rangle$ is a subdirect product of $M(i,\mathbb{F}_p)$. Therefore $\langle x,y\rangle$ is a finite semisimple in the sense of Jacobson ring; and hence has identity. If $i,j \in [1,n]$, $i \neq j$, then

$$\langle x,y\rangle/I_i \cong M(i,\mathbb{F}_p), \quad \text{and} \quad \langle x,y\rangle/I_j \cong M(j,\mathbb{F}_p)$$

which implies that $I_i \neq I_j$. Thus $I_i + I_j = \langle x,y\rangle$ for $i,j \in [1,n]$, $i \neq j$ since both I_i and I_j are maximal ideals. By Chinese Remainder Theorem ([AF], p. 103), $\langle x,y\rangle$ is isomorphic to the direct product

$$(\langle x,y\rangle/I_1) \times \cdots \times (\langle x,y\rangle/I_n) \cong M(1,\mathbb{F}_p) \times \cdots \times M(n,\mathbb{F}_p);$$

and it follows that $\langle x,y\rangle = R_n$. $\square$

Lemma 5.2. *For any prime p, the ring $R = \prod_{n=1}^{\infty} M(n,\mathbb{F}_p)$ is compact second countable and has two topological generators.*

Proof. For each $n \in \mathbb{N}$, fix a pair x_n and y_n of generators of $M(n,\mathbb{F}_p)$ and set $x = (x_n)$, $y = (y_n)$ in R. Let q_n denotes the projection of R on $R_n = M(1,\mathbb{F}_p) \times \cdots \times M(n,\mathbb{F}_p)$. By Lemma 5.1, $q_n(\langle x,y\rangle) = R_n$ which means that $\langle x,y\rangle$ is dense in R. $\square$

Let R be a finite associative ring with identity 1 of prime characteristic p. Then R is a subring of $EndR$, where R considered as a vector $\mathbb{F}_p$ -space. Indeed, the mapping $R \to EndR$: $r \mapsto L_r$ $(x \mapsto rx)$ is an embedding of R in $EndR$.

The following Remark is well known, see [C]:

Remark 5.1. Every compact ring R of prime characteristic p is an open ideal of a compact ring S of characteristic p.

Indeed, considering the group $S = \mathbb{F}_p \times R$ with the product topology and multiplication

$$(\alpha,x)(\beta,y) = (\alpha\beta, \alpha y + \beta x + xy),$$

the ring R is topologically isomorphic to the open ideal $\{0\} \times R$.

Theorem 5.3. *Every associative compact second countable ring S of prime characteristic p is topologically isomorphic to a subring of $R = \prod_{n\in\mathbb{N}} M(n, \mathbb{F}_p)$.*

Proof. We may consider, according to Remark 5.1, that S has an identity. Fix a fundamental system $\{S_i\}_{i\in\mathbb{N}}$ of 0_S consisting of two-sided ideals and associate to each $i \in \mathbb{N}$ a number $k_i \in \mathbb{N}$ for which S/S_i is embedded in $M(i, \mathbb{F}_p)$. We can assume without loss of generality that $k_i < k_{i+1}, i \in \mathbb{N}$; then S is embedded in $\prod_{i\in\mathbb{N}} M(k_i, \mathbb{F}_p)$ which is embedded in R. □

Let X be a set and p be a prime. Considering the ring $\mathbb{F}_p \langle X\rangle$ of all non-commutative polynomials over X with coefficients from $\mathbb{F}_p$, let $\mathfrak{B}$ be the set of all cofinite ideals V of $\mathbb{F}_p \langle X\rangle$ with the property that $X\backslash V$ is finite. Then $\mathfrak{B}$ is a filter basis, $\cap\mathfrak{B} = 0$, and hence $\mathfrak{B}$ gives a totally bounded ring topology $\mathcal{T}$.

The completion $(c\mathbb{F}_p \langle X\rangle, c\mathcal{T})$ is a compact ring of characteristic p and has the following universal property: every mapping $f : X \to S$ in a compact ring S with 1 and $\mathrm{char}S = p$ with the property that $X \setminus f^{-1}(V)$ is finite for every 0-neighborhood V of S has an extension $\widehat{f} : c\mathbb{F}_p \langle X\rangle \to S$, $\widehat{f}(1) = 1$. The ring $(c\mathbb{F}_p \langle X\rangle, c\mathcal{T})$ is called the *free compact ring of characteristic p over X*. We will denote the ring $(c\mathbb{F}_p \langle X\rangle, c\mathcal{T})$ briefly by $c\mathbb{F}_p \langle X\rangle$. When $X = \{x_1, x_2, \ldots, x_n\}$ is finite, we will write $c\mathbb{F}_p \langle x_1, x_2, \ldots, x_n\rangle$.

Note that the ring $c\mathbb{F}_p \langle X\rangle$ is analogous to the free pro-p-group, see ([Se], I.1.5).

Lemma 5.4. *For any set X, $w(c\mathbb{F}_p \langle X\rangle) = \max\{|X|, \omega\}$.*

Proof. It is well known that every compact totally disconnected ring with a finite number of topological generators is metrizable ([U2], Theorem 27.30). Therefore, the assertion is true when X is finite. Assume that X is infinite, then the ring $c\mathbb{F}_p \langle X\rangle$ is mapped on $\mathbb{F}_p^X$, and hence $w(c\mathbb{F}_p \langle X\rangle) \geq |X|$. Every open ideal of $(\mathbb{F}_p \langle X\rangle, \mathcal{T})$ contains a cofinite ideal of the form $(X\backslash F)$, where F is finite. Since the cardinality of the family of sets of the form $X\backslash F$, where F runs over finite subsets of X, is $|X|$, we obtain that $w(\mathbb{F}_p \langle X\rangle, \mathcal{T}) \leq |X|$. Therefore $w(c\mathbb{F}_p \langle X\rangle) \leq |X|$ which implies that $w(c\mathbb{F}_p \langle X\rangle) = |X|$. □

Theorem 5.5. *Let p be a prime and $X = \{x, y\}$. The ring $c\mathbb{F}_p \langle x, y\rangle$ contains an isomorphic copy of every compact second countable associative ring of characteristic p.*

Proof. Let $R = c\mathbb{F}_p \langle x, y\rangle$. Consider the Wedderburn-Mal'cev decomposition $R = S \oplus J(R)$ (a topological direct sum of subgroups S and $J(R)$) of R, where S is a compact semisimple ring. The existence of this decomposition for compact rings was proved in [Ze], see also ([U2], p. 170). Evidently, $1 \in S$.

By KAPLANSKY's Theorem ([K], [U2]), we have that

$$S \cong_{\text{top}} \prod_{i\in\omega} M\left(n_i, F_i\right).$$

Identifying S with $\prod_{i\in\omega} M\left(n_i, F_i\right)$, we claim that the ring $M\left(n, \mathbb{F}_p\right)$ appears in the decomposition of S for each $n \in \omega$. As the ring $M\left(n, \mathbb{F}_p\right)$ can be generated by two elements, there exists a surjective homomorphism $h : c\mathbb{F}_p\left\langle x, y\right\rangle \to M\left(n, \mathbb{F}_p\right)$. Evidently, $h\left(S\right) = M\left(n, \mathbb{F}_p\right)$, and hence S contains an ideal isomorphic to $M\left(n, \mathbb{F}_p\right)$. It follows that S contains an ideal isomorphic to $\prod_{n=1}^{\infty} M\left(n, \mathbb{F}_p\right)$ which, by Theorem 5.3, contains an isomorphic copy of every compact second countable associative ring of characteristic p. □

Theorem 5.6. *If $|X| = \omega$, then $c\mathbb{F}_p\left\langle X\right\rangle$ has the property that every compact second countable associative ring with 1 of prime characteristic p is a continuous homomorphic image of $c\mathbb{F}_p\left\langle X\right\rangle$ and every compact second countable associative ring of characteristic p is a subring of $c\mathbb{F}_p\left\langle X\right\rangle$.*

PROOF. The ring $c\mathbb{F}_p\left\langle X\right\rangle$ contains a copy of the topological ring $c\mathbb{F}_p\left\langle x, y\right\rangle$. By Theorem 5.5, $c\mathbb{F}_p\left\langle X\right\rangle$ contains an isomorphic copy of every compact second countable associative ring of prime characteristic p.

Let R be any compact second countable associative ring of characteristic p. We may assume without loss of generality that R is infinite, then $R(+) \cong_{top} (\mathbb{Z}/p\mathbb{Z})^{\omega}$. It follows from the structure of $(\mathbb{Z}/p\mathbb{Z})^{\omega}$ that R contains a countable subset Y such that $R(+)$ is topologically generated by Y and for every open ideal V of Y, the set $Y \setminus V$ is finite. Fix any bijection $f : \omega \to Y$. There exists a continuous homomorphism $h : c\mathbb{F}_p\left\langle X\right\rangle \to R$ extending f such that $h(1) = 1$. □

The notion of a free compact topologically nilpotent associative ring of prime characteristic p over a set X is similar to the notion of a free compact ring. Namely, let X be a set and p be a prime number. Consider the free ring $\mathbb{F}_p(X)$ of all noncommutative polynomials over X with coefficients from $\mathbb{F}_p$ without terms of degree zero. Let $\mathfrak{B}$ be the set of all cofinite ideals V of $\mathbb{F}_p(X)$ with the property that $X\backslash V$ is finite and $\mathbb{F}_p(X)/V$ is nilpotent. Then $\mathfrak{B}$ is a filter basis, $\cap\mathfrak{B} = 0$, and hence $\mathfrak{B}$ gives a totally bounded ring topology $\mathcal{T}$.

The completion $(c\mathbb{F}_p(X), c\mathcal{T})$ is a compact ring of characteristic p and has the following universal property: every mapping $f : X \to S$ in a compact ring topologically nilpotent ring S of characteristic p with the property that $X\backslash f^{-1}(V)$ is finite for every 0-neighborhood V of S has an extension $\widehat{f} : c\mathbb{F}_p(X) \to S$. The ring $(c\mathbb{F}_p(X), c\mathcal{T})$ is called the *free compact topologically nilpotent ring of characteristic p over X*.

Remark 5.2. We will need some properties of inverse limits of topological rings. Let $\{R_\alpha : f_{\alpha\beta}, \Omega\}$ be an inverse system of topological rings, $R = \varprojlim R_\alpha$ and π_α denotes the canonical projection of R in R_α. Then:

(i) The family $\{\pi_\alpha^{-1}(V_\alpha)\}_{\alpha\in\Omega}$ where V_α is a neighborhood of 0_{R_α} is a fundamental system of 0_R.
This implies immediately the following fact:

(ii) If S is a subring of $R = \varprojlim R_\alpha$ and $\pi_\alpha(S)$ is dense in R_α for every $\alpha \in \Omega$, then S is a dense subring of R.

(iii) If R_α are compact and the canonical projections $f_{\alpha\beta}$ are surjective, then π_α are surjective ([E], Corollary 3.2.15).

Theorem 5.7. *The free compact topologically nilpotent ring R of prime characteristic p with a countable number of topological generators is a domain.*

PROOF. Consider for each $n \in \mathbb{N}$ the ring R_n of formal noncommutative series on n variables $x_1, \dots, x_n$. Consider the ring $R = \varprojlim R_n$ where $f_n : R_{n+1} \to R_n$ be the continuous homomorphism which sends x_i to x_i, $i = 1, \dots, n$ and x_{n+1} to 0. It is well known that each R_n is a domain. Set $y_i = (0 \dots 0 x_i x_i \dots)$, ($i-1$ zeros).

Claim I. For each $n \in \mathbb{N}$ the subring $\langle y_1, \dots, y_n\rangle$ is a free associative ring in the class of rings of characteristic p with $y_1, \dots, y_n$ as free generators.

Claim II. The subring $P = \langle y_i : i \in \mathbb{N}\rangle$ is the free associative ring of characteristic p with $y_i, i \in \mathbb{N}$ as free generators.

Claim III. The subring P is dense in R. Indeed, $\pi_n\langle y_1, \dots, y_n\rangle$ is $\langle x_1, \dots, x_n\rangle$, which is dense in R_n (see, [B1], Chapter II, §5).

Claim IV. R is a domain. This follows immediately since the inverse limit of domains is a domain.

Claim V. The ring R is second countable.

Claim VI. The ring R is free in the class of all second countable compact topologically nilpotent rings of characteristic p.

Indeed, let S be a second countable compact topologically nilpotent ring of characteristic p. Let $S = \overline{\langle a_i : i \in \mathbb{N}\rangle}$ where $\{a_i : i \in \mathbb{N}\}$ has the property that $\lim_{i\to\infty} a_i = 0$. Let $f_1 : P \to S$, $y_i \mapsto a_i$.

We claim that f_1 is continuous. Indeed, let V' be an open ideal of S. Let $n \in \mathbb{N}$ such that $a_i \in V'$ for $i \geq n$. Let $m \in \mathbb{N}, S^m \subset V', m \geq n$. Consider the neighborhood $W = (\langle x_1\rangle^m \times \cdots \times \langle x_1, \dots, x_n\rangle^m \times \prod_{i\geq n+1}\langle x_1, \dots x_i\rangle) \cap P$ of 0_P. If $w \in W$, then $w = g_1(y_1, \dots, y_n) + g_2(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})$, where $g_2 = 0$ or each monomial of $g_2(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})$ contains at least one element from $\{y_{n+1}, \dots, y_{n+k}\}$. Then $pr_i(w) = g_1(pr_i(y_1), \dots, pr_i(y_n))$ for each $i \leq n$

is of degree $\geq m$ (here pr_i is the projection of $\prod_{n\in\mathbb{N}}$ on R_i). This implies that $f_1[g_1(y_1,\dots,y_n)] \in V'$. It is obviously that $f_1[g_2(y_1,\dots,y_n,y_{n+1},\dots,y_{n+k})] \in V'$ and so $f_1(w) \in V'$.

We extend by continuity f_1 to a continuous homomorphism $f : R \to S$; obviously, f is surjective. □

Corollary 5.8. *Every compact second countable topologically nilpotent ring of prime characteristic is a continuous homomorphic image of a compact domain*

Recall that an associative ring is called *reduced* provided it has no nonzero nilpotent elements.

Example 5.2. The ring $M(2,\mathbb{F}_2)$ is not a continuous image of a compact reduced ring.

Assume the contrary: let R be a compact reduced ring and $f : R \to M(2,\mathbb{F}_2)$ be a continuous surjective homomorphism. Pick up $e_{11} \in M(2,\mathbb{F}_2)$. By ([U2], Theorem 4.20) there exists an idempotent $e \in R, f(e) = e_{11}$; evidently e is central, a contradiction. □

6. Compact Noetherian rings

It is well known that a compact Noetherian ring has a finite number of topological generators ([U2], Corollary 6.45, p. 135). We have modified an example of L. W. Small, see [Sm], in order to obtain a compact left and right Noetherian ring having no classical quotient ring in abstract sense.

For a prime p, let $\mathbb{Z}_p$ denotes the ring of p-adic numbers with the natural compact topology, and let T be the ring of 2×2 matrices of the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$, where $a \in \mathbb{Z}_p$ and $b, c \in \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p$, such that $\mathbb{Z}_p$ acts on $\mathbb{Z}_p/p\mathbb{Z}_p$ in the usual way.

Lemma 6.1. *T is left and right Noetherian.*

Proof. The ring T has identity and its additive group is topologically isomorphic to $\mathbb{Z}_p \times (\mathbb{Z}/p\mathbb{Z})^2$. This group satisfies the ACC on closed subgroups. The ring T with the natural compact topology satisfies the ACC on closed subgroups, so T satisfies the ACC on left finitely generated ideals. Therefore T is left Noetherian. Analogously, T is right Noetherian. □

Set $T_1 = \left\{ \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} : a \in \mathbb{Z}_p,\ c \in \mathbb{Z}_p/p\mathbb{Z}_p \right\}$ and let S be the ring of 2×2 matrices of the form $\begin{bmatrix} t & t' \\ 0 & t_1 \end{bmatrix}$, where $t, t' \in T$, $t_1 \in T_1$.

Lemma 6.2. *The ring S is left and right Noetherian.*

PROOF. The ring S has identity and its additive group is topologically isomorphic to $(\mathbb{Z}_p)^3 \times (\mathbb{Z}/p\mathbb{Z})^5$. This group satisfies the ACC condition on closed subgroups. The ring S with the natural compact topology satisfies the ACC on closed subgroups, so S satisfies the ACC on left finitely generated ideals. Therefore S is left Noetherian. Analogously, S is right Noetherian. □

Theorem 6.3. *The ring S does not satisfy the left Ore condition.*

The last Theorem shows that as S does not satisfy the left Ore condition, S does not have a left quotient ring.

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References

[AF] F. W. ANDERSON and K. R. FULLER, Rings and Categories of Modules, Second Edition, *Springer-Verlag*, 1992.

[B1] N. BOURBAKI, Groupes et Algebres de Lie, Chap. I-III (Fasc. XXVI, XXXVII), *Hermann, Paris*, VI 1971,1972.

[B2] N. BOURBAKI, Algebre Commutative, Chap. I-VII (Fasc. XXVII, XXVIII, XXX, XXXI), *Paris, Hermann.*

[Co] P. M. COHN, Free Rings and their Relations, *Academic Press*, *London, New York*, 1971.

[C] J. CUDE, Compact integral domains, *Pacific J. of Math.* **32**(3) (1970), 615–619.

[E] R. ENGELKING, General Topology, *Warszawa*, 1977.

[FS] H. FUJITA and D. SHAKHMATOV, Topological groups with dense compactly generated subgroups, *Applied General Topology* **3**(1) (2002), 85–89.

[K] I. KAPLANSKY, Topological rings, *Amer. J. Math.* **69** (1947), 153–183.

[Mv] A. I. MAL'CEV, About one representation of nonassociative rings, *Uspehi Matematiceskih Nauk* **7**(1) (1952), 181–185.

[Se] J. P. SERRE, Galois Cohomology, *Springer-Verlag*, 1997.

[Sm] L. SMALL, On some questions in noetherian rings, *Bulletin of American Mathematical Society* **72**(5) (1966), 853–857.

[U1] M. URSUL, The relations between the concepts of compactness, linear compactness, and hereditarily linear compactness in topological rings, *Matematiceskie Issled.* **53** (1979), 136–147.

[U2] M. Ursul, Topological Rings Satisfying Compactness Conditions, Vol. 549, *Kluwer, Mathematics and Applications*, 2002.

[W] S. Warner, Topological Rings, *North-Holland*, *New York*, 1993.

[Ze] D. Zelinsky, Raising idempotents, *Duke Math. J.* **5** (1954), 627–630.

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