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Restricted stability and shadowing

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Abstract. We prove a shadowing result for non-surjective mappings. As a corollary we obtain stability of the Cauchy functional equation in the case when in the target space the multiplication by 2 is only an injection.

Although there is no stability in the classical sense, we prove that there is stability on the respective subset of the domain.

1. Introduction

The essence of Hyers method consists in defining the unknown exact solution a to the given equation with the use of the approximate solution f by the formula

$$a(x) := \frac{f(2^n x)}{2^n}.$$

As this idea is crucial in many stability results, authors have long considered the question what happens in the case when there is no global unique divisibility by 2 in the target space. It occurs that it is enough to assume that the division by 2 is locally uniquely performable [2], [3], [8].

We have observed in [8] that the abstract Hyers method is strictly related to the dynamical systems stability notion called shadowing. This abstract way of looking at Hyers method enables to generalize some well-known classical stability results.

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In the present paper we apply the similar approach to investigation of the stability of the Cauchy equation in the case when in the target space we have no local 2-divisibility. As illustrates the following example, this usually causes lack of stability.

Example 1.1. Let $F : [0, \infty) \to [1, \infty)$ be defined by the formula

$$F(x) = \frac{1}{3}x + 1.$$

Then

$$|F(x+y) - F(x) - F(y)| \le 1 \quad \text{for } x, y \in [0, \infty),$$

but clearly there is no additive function $A : [0, \infty) \to [1, \infty)$ satisfying the inequality

$$\sup_{x \in [0,\infty)} |F(x) - A(x)| < \infty.$$

The same holds for the function $G : \mathbb{Z} \to \mathbb{Z}$ defined by $G(k) := \lfloor k/2 \rfloor$.

However, F and G can be approximated by additive functions on subsemigroups of the domain: $|F(x) - \frac{1}{3}x| \le 1$ for $x \in [3, \infty)$ and $|G(k) - \frac{1}{2}k| = 0$ for $k \in 2\mathbb{Z}$.

Our aim in this paper is to show that the situation described above is typical.

2. Shadowing

In this section we assume that (X, d) is a complete metric space. By $\mathbb{N}$ we denote the set of nonnegative integers. One of the basic well-known shadowing results can be formulated as follows [5]:

Theorem. *Let $\delta > 0$, $M > 1$ be arbitrary and let $\phi : X \to X$ be a continuous surjection such that*

$$d(\phi(x), \phi(y)) \ge M d(x, y) \quad \textit{for } x, y \in X.$$

Then for every sequence $(x_k)_{k \in \mathbb{N}} \subset X$ satisfying the condition

$$d(x_{k+1}, \phi(x_k)) \le \delta \quad \textit{for } k \in \mathbb{N},$$

there exists a unique $u \in X$ such that

$$d(x_k, \phi^k(u)) \le \frac{\delta}{M-1} \quad \textit{for } k \in \mathbb{N}.$$

As shows the following example, the assumption that ϕ is surjective is essential.

Example 2.1. Let $X = [1, \infty)$ and let $\phi(x) = 2x$, $M = 2$. We put

$$x_k = \max(1, 2^{k-1}) \quad \text{for } k \in \mathbb{N}.$$

Then $d(x_{k+1}, \phi(x_k)) \leq 1$ for $k \in \mathbb{N}$, but clearly there is no $u \in [1, \infty)$ such that

$$\sup_{k \in \mathbb{N}} d(x_k, \phi^k(u)) < \infty.$$

If such u would exist, then $0 = \lim_{k\to\infty} \frac{1}{2^k}|2^k u - x_k| = |u - \frac{1}{2}|$, but $\frac{1}{2} \notin [1, \infty)$.

Our main result (which we later use as a basic tool) generalizes the Theorem quoted at the beginning of this section and shows that under some additional assumptions surjectivity condition may be omitted. The essential (necessary) additional assumption, (2), means that the given pseudoorbit is close to the images of X under the respective iterates of ϕ.

Theorem 2.1. *Let $\phi : X \to X$ be a mapping with a closed graph. We assume that we are given $M > 0$ such that*

$$d(\phi(x), \phi(y)) \geq M d(x, y) \quad \text{for } x, y \in G.$$

Let $(x_k)_{k\in\mathbb{N}} \subset X$ be a sequence satisfying

$$\sum_{k=0}^{\infty} \frac{1}{M^k} d(x_{k+1}, \phi(x_k)) < \infty. \tag{1}$$

If

$$\liminf_{k\to\infty} \frac{1}{M^k} d(x_k, \phi^k(X)) = 0, \tag{2}$$

then there exists a unique element $u \in X$ such that

$$\lim_{k\to\infty} \frac{1}{M^k} d(x_k, \phi^k(u)) = 0. \tag{3}$$

Morever, then

$$d(x_k, \phi^k(u)) \leq \frac{1}{M^{-k+1}} \sum_{i=k}^{\infty} \frac{1}{M^i} d(x_{i+1}, \phi(x_i)) \quad \text{for } k \in \mathbb{N}. \tag{4}$$

Proof. Let $(x_k)_{k\in\mathbb{N}} \subset X$ be a sequence satisfying (1) and (2).

We first prove that

$$d(\phi^k(v), x_k) \leq \frac{1}{M^{l-k}} d(\phi^l(v), x_l) + \sum_{i=k}^{l-1} \frac{1}{M^{i-k+1}} d(x_{i+1}, \phi(x_i)) \tag{5}$$

for $v \in X$ and $l, k \in \mathbb{N}$, $l > k$.

Clearly

$$d(\phi^k(v), x_k) \leq \frac{1}{M} d(\phi^{k+1}(v), \phi(x_k)) \leq \frac{1}{M} d(\phi^{k+1}(v), x_{k+1}) + \frac{1}{M} d(x_{k+1}, \phi(x_k))$$

for $v \in X$, $k \in \mathbb{N}$. Applying the above inequality inductively we obtain (5).

Due to (2) we can find a sequence $(u_k)_{k\in\mathbb{N}} \subset X$ and an increasing sequence of positive integers $(N_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} \frac{1}{M^{N_k}} d(\phi^{N_k}(u_k), x_{N_k}) = 0. \tag{6}$$

Now we prove that $(\phi^n(u_k))_{k\in\mathbb{N}}$ is a Cauchy sequence for arbitrarily fixed $n \in \mathbb{N}$. For $k, l \in \mathbb{N}$, $l > k > n$ we obtain by (5)

$$\begin{aligned}
&d(\phi^n(u_k), \phi^n(u_l)) \leq \frac{1}{M^{N_k-n}} d(\phi^{N_k}(u_k), \phi^{N_k}(u_l)) \\
&\leq \frac{1}{M^{N_k-n}} \left[d(\phi^{N_k}(u_k), x_{N_k}) + d(x_{N_k}, \phi^{N_k}(u_l)) \right] \leq \frac{1}{M^{N_k-n}} d(\phi^{N_k}(u_k), x_{N_k}) \\
&+ \frac{1}{M^{N_k-n}} \left[\frac{1}{M^{N_l-N_k}} \phi^{N_l}(u_l), x_{N_l}) + \sum_{i=N_k}^{N_l-1} \frac{1}{M^{i-N_k+1}} d(x_{i+1}, \phi(x_i)) \right] \\
&\leq \frac{1}{M^{-n}} \left[\frac{1}{M^{N_k}} d(\phi^{N_k}(u_k), x_{N_k}) + \frac{1}{M^{N_l}} d(\phi^{N_l}(u_l), x_{N_l}) \right] \\
&+ \frac{1}{M^{1-n}} \sum_{i=N_k}^{\infty} \frac{1}{M^i} d(x_{i+1}, \phi(x_i)).
\end{aligned}$$

By (6) and (1) we obtain that the above sum tends to zero as $k, l \to \infty$. Thus $(\phi^n(u_k))_{k\in\mathbb{N}}$ is a Cauchy sequence (for any n), and consequently it is convergent. We put

$$\tilde{u}_n := \lim_{k\to\infty} \phi^n(u_k), \qquad u := \tilde{u}_0.$$

Since ϕ has a closed graph, we obtain that

$$\tilde{u}_{n+1} = \lim_{k\to\infty} \phi^{n+1}(u_k) = \phi(\tilde{u}_n) \quad \text{for } n \in \mathbb{N}.$$

Hence $\tilde{u}_n = \phi^n(u)$.

Now we have by (5) and (6)

$$\begin{aligned} d(x_k, \phi^k(u)) &= \lim_{l\to\infty} d(x_k, \phi^k(u_l)) \\ &\le \lim_{l\to\infty} \left(\frac{1}{M^{N_l-k}} d(x_{N_l}, \phi^{N_l}(u_l)) + \sum_{i=k}^{N_l-1} \frac{1}{M^{i-k+1}} d(x_{i+1}, \phi(x_i)) \right) \\ &\le \frac{1}{M^{-k+1}} \sum_{i=k}^{\infty} \frac{1}{M^i} d(x_{i+1}, \phi(x_i)), \end{aligned}$$

which implies (4), and consequently (3).

It remains to prove uniqueness of u. Suppose that v also satisfies (3). Then for $k \in \mathbb{N}$ we have

$$d(u,v) \le \frac{1}{M^k} d(\phi^k(u), \phi^k(v)) \le \frac{1}{M^k} d(\phi^k(u), x_k) + \frac{1}{M^k} d(x_k, \phi^k(v)).$$

Letting $k \to \infty$, we obtain that by (3) the right hand side tends to zero. Thus $u = v$. □

3. Stability of the Cauchy functional equation

As we have shown in [8], shadowing results can be usually easily "translated" to stability of the linear-type functional equations. In this section we show such an application for the stability of the Cauchy functional equation in metric groupoids.

For the convenience of the reader we recall some basic facts concerning groupoids [1], [6], [7].

Definition 3.1. A set G with a binary operation $\circ$ is called *square-symmetric groupoid* (shortly square groupoid) if

$$(x \circ y) \circ (x \circ y) = (x \circ x) \circ (y \circ y) \quad \text{for } x, y \in G.$$

Clearly every commutative semigroup is a square groupoid.

In a square groupoid we define inductively the powers x^{2^n} of x by

$$x^{2^0} = x,\ x^{2^{n+1}} = x^{2^n} \circ x^{2^n}.$$

One can easily check that

$$(x \circ y)^{2^n} = x^{2^n} \circ y^{2^n} \quad \text{for } x, y \in G, n \in \mathbb{N}. \tag{7}$$

We will use the following denotation

$$A^{2^n} = \{a^{2^n} : a \in A\} \quad \text{for } A \subset G, n \in \mathbb{N}.$$

For our considerations we will need the notion of metric groupoid.

Definition 3.2. Let $(X, \circ)$ be a groupoid and d a metric in X. We say that X is a *metric groupoid* if there exists $K > 0$ such that

$$d(x \circ y, u \circ v) \leq K(d(x, u) + d(y, v)) \quad \text{for } x, y, u, v \in X.$$

In other words a groupoid is metric if the function $X \times X \ni (x, y) \mapsto x \circ y \in X$ is Lipschitz (in $X \times X$ we take the product metric).

We conclude this section with a quite general example of a metric groupoid.

Example 3.1. Let X be a Banach space and let $A, B : X \to X$ be commuting Lipschitz affine functions. We define the binary operation in X by the formula

$$x \circ y = Ax + By \quad \text{for } x, y \in X.$$

Then one can easily verify that X is a complete metric square groupoid.

The following theorem is the main result of the paper.

Theorem 3.1. *Let $(G, \circ)$ be a square groupoid and $(X, \circ, d)$ a complete metric square groupoid. We assume that we are given $\Psi : G \times G \to \mathbb{R}_+$ and constants $0 < M_\Psi < M$ such that*

$$d(x^2, y^2) \geq M d(x, y) \qquad \text{for } x, y \in X, \tag{8}$$

$$\Psi(g^2, h^2) \leq M_\Psi \Psi(g, h) \qquad \text{for } g, h \in G. \tag{9}$$

Let $F : G \to X$ be such that

$$d(F(g \circ h), F(g) \circ F(h)) \leq \Psi(g, h) \quad \text{for } g, h \in G. \tag{10}$$

We denote

$$G_S := \{g \in G : \liminf_{k \to \infty} \frac{1}{M^k} d(F(g^{2^k}), X^{2^k}) = 0\}.$$

Then G_S is a subgroupoid of G and there exists a unique additive function $A : G_S \to X$ such that

$$d(F(g), A(g)) \leq \frac{1}{M - M_\Psi} \Psi(g, g) \quad \text{for } g \in G_S. \tag{11}$$

Proof. We define the mapping $\phi : X \to X$ by the formula

$$\phi(x) = x^2.$$

Then clearly by (8)

$$d(\phi(x), \phi(y)) \geq M d(x, y) \quad \text{for } x, y \in X.$$

Since the groupoid X is metric, the mapping ϕ is continuous, and consequently has a closed graph.

We are going to construct the desired additive mapping A with the use of Theorem 2.1. Let $g \in G_S$ be fixed and let us consider the sequence $(x_k(g))_{k \in \mathbb{N}} \subset X$ defined by the formula

$$x_k(g) := F(g^{2^k}) \quad \text{for } k \in \mathbb{N}.$$

By the definition of G_S we obtain that

$$\liminf_{k \to \infty} \frac{1}{M^k} d(x_k(g), \phi^k(X)) = 0.$$

On the other hand, by (9) and (10)

$$d(x_{k+1}(g), \phi(x_k(g))) = d(F(g^{2^{k+1}}), (F(g^{2^k}))^2) \leq \Psi(g^{2^k}, g^{2^k}) \leq M_\Psi^k \Psi(g, g).$$

Thus

$$\sum_{k=0}^{\infty} \frac{1}{M^k} d(x_{k+1}(g), \phi(x_k(g))) \leq \sum_{k=0}^{\infty} (\frac{M_\Psi}{M})^k \cdot \Psi(g, g) = \frac{1}{1 - \frac{M_\Psi}{M}} \cdot \Psi(g, g) < \infty.$$

This means that we can apply Theorem 2.1 and obtain existence of a unique element $u(g)$ such that

$$\lim_{k \to \infty} \frac{1}{M^k} d(x_k(g), \phi^k(u(g))) = 0. \tag{12}$$

Moreover, by (4)

$$d(x_0(g), u(g)) \leq \frac{1}{M} \frac{\Psi(g, g)}{1 - \frac{M_\Psi}{M}} = \frac{1}{M - M_\Psi} \Psi(g, g).$$

We define the function A by the formula $A(g) := u(g)$. By the above inequality we obtain that (11) holds. It remains to prove that A is additive. Let $g, h \in G$ be arbitrary. We use, without mentioning it explicitly, the equalities (7).

By (12) we obtain that

$$\lim_{k\to\infty}\frac{1}{M^k}d(F(g^{2^k}),A(g)^{2^k})=0,\qquad \lim_{k\to\infty}\frac{1}{M^k}d(F(h^{2^k}),A(h)^{2^k})=0,$$

$$\lim_{k\to\infty}\frac{1}{M^k}d(F((g\circ h)^{2^k}),A(g\circ h)^{2^k})=0.$$

Applying the first two of the equalities above and the fact that X is a metric groupoid (with constant $K>0$), we obtain that

$$\begin{aligned}
&\lim_{k\to\infty}\frac{1}{M^k}d(F((g\circ h)^{2^k}),(A(g)\circ A(h))^k)\\
&\quad\le\lim_{k\to\infty}\frac{1}{M^k}\Big(d(F(g^{2^k}\circ h^{2^k}),F(g^{2^k})\circ F(h^{2^k}))\\
&\qquad+d(F(g^{2^k})\circ F(h^{2^k}),(A(g)\circ A(h))^k)\Big)\\
&\quad\le\lim_{k\to\infty}\left(\frac{M_\Psi^k}{M^k}\Psi(g,h)+\frac{K}{M^k}(d(F(g^{2^k}),A(g^{2^k}))+d(F(h^{2^k}),A(h^{2^k}))\right)=0.
\end{aligned}$$

Summing up, we have obtained that

$$\lim_{k\to\infty}\frac{1}{M^k}d(x_k(g\circ h),(A(g\circ h))^k)=0,\quad \lim_{k\to\infty}\frac{1}{M^k}d(x_k(g\circ h),(A(g)\circ A(h))^k)=0.$$

By the uniqueness part of Theorem 2.1 we obtain that $A(g\circ h)=A(g)\circ A(h)$. □

Let us just mention that taking Φ as respective functions one can obtain generalization of some of the results of Th. Rassias [4, Th. 2.1, 2.2].

Remark 3.1. Theorem 3.1 has a constructive character. Let, as in Theorem 3.1, f be an approximately additive function. To construct $A(g)$, for $g\in G_S$ we proceed in the following way: we take an arbitrary sequence $(u_k(g))\subset X$ such that $\liminf_{k\to\infty}\frac{1}{M^k}d(f(x^{2^k}),\phi^k(u_k(g)))=0$ and put $A(g):=\lim_{k\to\infty}u_k(g)$.

The most commonly met situation is when Ψ is a constant function. Let us now reformulate our theorem in this case, which corresponds to the classical Hyers Theorem.

Corollary 3.1. *Let G, X be square groupoids. We assume that we have a complete metric d in X and $M>1$ such that X is a metric groupoid and*

$$d(x^2,y^2)\ge Md(x,y)\quad \text{for } x,y\in X.$$

Let $\varepsilon>0$ and $F:G\to X$ be such that

$$d(F(g\circ h),F(g)\circ F(h))\le\varepsilon\quad\text{for } g,h\in G.$$

Then $G_S := \{g \in G : \liminf_{k\to\infty} \frac{1}{M^k} d(F(g^{2^k}), X^{2^k}) = 0\}$ *is a subgroupoid of* G *and there exists a unique additive function* $A : G_S \to X$ *such that*

$$d(F(g), A(g)) \le \frac{\varepsilon}{M-1} \quad \textit{for } g \in G_S.$$

The crucial role in the above result plays the assumption that $M > 1$. So the question arises what happens if this is not the case. The following example shows that we have no stability if $M = 1$.

Example 3.2. Let $G = X = [0, 1]$ with the standard metric. In G and X we consider the operations

$$g_1 \oplus g_2 = \min(g_1 + g_2, 1) \qquad \text{for } g_1, g_2 \in G,$$

$$x_1 \vee x_2 = \max(x_1, x_2) \qquad \text{for } x_1, x_2 \in X.$$

Clearly, $(G, \oplus)$ and $(X, \vee)$ are commutative groupoids. Moreover, X is a metric groupoid. We have $x^2 = x \vee x = x$ for $x \in X$, which implies that X satisfies (8) with $M = 1$.

Given $n \in \mathbb{N}$ we define the function $F_n : G \to X$ by the formula

$$F_n(g) = g^{1/n} \quad \text{for } g \in G.$$

One can easily check that

$$d(F_n(g_1 \oplus g_2), F_n(g_1) \vee F_n(g_2)) \le 2^{1/n} - 1 \to 0.$$

Let $0 < \varepsilon < 1/2$ be fixed. We show that for an arbitrarily chosen $n \in N$ there is no homomorphism $A : G \to X$ such that

$$d(F_n(g), A(g)) \le \varepsilon \quad \text{for } g \in G.$$

For an indirect proof, suppose that such a homomorphism exists. Then

$$A(1) = A(1/k \oplus \cdots \oplus 1/k) = A(1/k) \vee \cdots \vee A(1/k) = A(1/k) \quad \text{for } k \in \mathbb{N}, k \ge 1.$$

Consequently

$$\begin{aligned} 1 - (1/k)^{1/n} = d(F_n(1/k), F_n(1)) &\le d(F_n(1/k), A(1/k)) \\ &\quad + d(A(1/k), A(1)) + d(A(1), F_n(1)) \le 2\varepsilon. \end{aligned}$$

Letting $k \to \infty$ we obtain that $2\varepsilon \ge 1$, a contradiction.

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