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Distribution of additive and q-additive functions under some conditions II.

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Abstract. Distribution of additive function over the set of integers having fixed number of prime divisors, and the distribution of q-additive functions over the set of integers for which the value of the sum of divisors function is fixed are investigated.

§1. Introduction

1.1. Notation. $\mathbb{N}, \mathbb{R}, \mathbb{C}$ as usual denote the set of natural, real and complex numbers, respectively, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathcal{P}$ be the set of the primes, p with or without suffixes always denote prime numbers. The letters $c, c_1, c_2, \ldots$ denote constants not necessary the same at every occurence. Let $\Phi(y)$ be the Gaussian distribution function, $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2} du$.

1.2. q-additive and q-multiplicative functions. Let $q \geq 2$ be an integer, the q-ary expansion of $n \in \mathbb{N}_0$ is defined as

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \tag{1.1}$$

where the digits $\varepsilon_j(n)$ are taken from $\mathbb{A}_q := \{0, 1, \ldots, q-1\}$. Let $\mathcal{A}_q$ be the set of q-additive functions, and $\overline{\mathcal{M}}_q$ be the set of q-multiplicative functions of modulus

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1: $f : \mathbb{N}_0 \to \mathbb{R}$ belongs to $\mathcal{A}_q$ if $f(0) = 0$ and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j) \quad (n \in \mathbb{N}_0). \tag{1.2}$$

We say that $g : \mathbb{N}_0 \to \mathbb{C}$ belongs to $\overline{\mathcal{M}}_q$, if $g(0) = 1$,

$$g(n) = \prod_{j=0}^{\infty} g(\varepsilon_j(n)q^j) \quad (n \in \mathbb{N}_0) \tag{1.3}$$

and $|g(n)| = 1$ $(n \in \mathbb{N}_0)$.

Let $\alpha(n)$, $\beta_h(n)$ be defined as

$$\alpha(n) = \sum_{j=0}^{\infty} \varepsilon_j(n); \quad \beta_h(n) = \sum_{\varepsilon_j(n)=h} 1 \quad (h = 1, \dots, q-1). \tag{1.4}$$

It is clear that $\alpha, \beta_h \in \mathcal{A}_q$. H. Delange [1] proved that for every $g \in \overline{\mathcal{M}}_q$ the limit

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} g(n) = M(g) \tag{1.5}$$

exists and $M(g) \neq 0$, if

$$m_j := \frac{1}{q} \sum_{c \in \mathbb{A}_q} g(cq^j) \neq 0 \quad (j = 0, 1, 2, \dots) \tag{1.6}$$

and

$$\sum (1 - m_j) \tag{1.7}$$

is convergent. If these conditions hold, then

$$M(g) = \prod_{j=0}^{\infty} m_j. \tag{1.8}$$

Hence he deduced that for $f \in \mathcal{A}_q$ the values $f(n)$ possess a limit distribution if and only if both of the series

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}_q} f(bq^j), \tag{1.9}$$

$$\sum_{j=0}^{\infty} \sum_{b \in \mathbb{A}_q} f^2(bq^j) \tag{1.10}$$

are convergent.

Let $f \in \mathcal{A}_q$. Assume that it has the limit distribution

$$F(y) := \lim_{x\to\infty} \frac{1}{x} \#\{n < x \mid f(n) < y\}. \tag{1.11}$$

Delange proved that $F(y) = P(\xi < y)$, where ξ is the sum of the independent random variables $\xi_0, \xi_1, \dots,$ where $P(\xi_j = f(aq^j)) = 1/q$ $(a \in \mathbb{A}_q)$. Thus the characteristic function $\varphi(\tau)$ of $F(y)$ can be written as

$$\varphi(\tau) = \prod_{j=0}^{\infty} \left\{ \frac{1}{q} \sum_{a=0}^{q-1} e^{i\tau f(aq^j)} \right\}. \tag{1.12}$$

Let $r_1, r_2, \dots, r_{q-1}$ be nonnegative integers, $\underline{r} = (r_1, \dots, r_{q-1})$ and $S_N(\underline{r}) = \{n < q^N \mid \beta_j(n) = r_j,\ j = 1, \dots, q-1\}$. Let $r_0 := N - (r_1 + \ldots + r_{q-1})$. $S_N(\underline{r})$ is empty if $r_0 < 0$. Let $M(N \mid \underline{r}) := \#S_N(\underline{r})$.

In [5] we proved the following Theorems A, B, C.

Theorem A. *Let $f \in \mathcal{A}_q$, and the series* (1.9), (1.10) *be convergent. Let $\underline{r}^{(N)} = \big(r_1^{(N)}, \dots, r_{q-1}^{(N)}\big)$ be such a sequence for which*

$$\left| \frac{q r_j^{(N)}}{N} - 1 \right| < \delta_N \quad (j = 1, \dots, q-1) \tag{1.13}$$

where $\delta_N \to 0 \quad (N \to \infty)$.

Then

$$\lim_{N\to\infty} \frac{1}{M(N \mid \underline{r})} \#\{n \in S_N(\underline{r}^{(N)}) \mid\ f(n) < y\} = F(y), \tag{1.14}$$

where $F(y) = P(\xi < y)$.

Theorem B. *Let $g \in \overline{\mathcal{M}}_q$ be such a function for which* (1.6) *holds and* (1.7) *is convergent. Let $\underline{r}^{(N)}$ be a sequence satisfying* (1.13). *Then*

$$\lim_{N\to\infty} \frac{1}{M(N \mid \underline{r})} \sum_{n \in S_N(\underline{r}^N)} g(n) = M(g). \tag{1.15}$$

Theorem C. *Let $q = 2$, $f \in \mathcal{A}_2$, $f(2^j) = O(1)$ $(j \in \mathbb{N})$,*

$$\eta_N = \frac{1}{N} \sum_{j=0}^{N-1} f(2^j), \quad B_N^2 := \frac{1}{4} \sum_{j=0}^{N-1} (f(2^j) - \eta_N)^2.$$

Assume that $B_N \to \infty$.

Let $\rho_N \to 0$, and $k = k^{(N)}$ be such a sequence of integers for which

$$\left| \frac{k}{N} - 1/2 \right| < \rho_N \tag{1.16}$$

holds.

Then

$$\lim_{N\to\infty} \frac{1}{\binom{N}{k}} \# \left\{ n < 2^N \;\middle|\; \frac{f(n) - k\eta_N}{B_N} < y,\ \alpha(n) = k \right\} = \Phi(y), \tag{1.17}$$

the convergence is uniform in y.

In [6] we continued our work and proved the following Theorems D, E.

Let

$$\eta_{N,k} := \frac{k}{N}, \quad \mathcal{E}_{N,k} = \{n < 2^N \mid \alpha(n) = k\}. \tag{1.18}$$

Theorem D. *Let $g \in \overline{\mathcal{M}}_2$ be such a function for which*

$$\sum_{j=0}^{\infty} (1 - g(2^j)) \tag{1.19}$$

is convergent. Let

$$M_\xi := \prod_{j=0}^{\infty} ((1-\xi) + g(2^j)\xi) \quad (0 < \xi < 1). \tag{1.20}$$

Let $\delta > 0$ be a constant. Then

$$\lim_{N\to\infty} \max_{\delta \le \frac{k}{N} \le 1-\delta} \left| \frac{1}{\binom{N}{k}} \sum_{\substack{n \in \mathcal{E}_{N,k} \\ n \le q^N}} g(n) - M_{\eta_{N,k}} \right| = 0. \tag{1.21}$$

Theorem E. *Let $f \in \mathcal{A}_2$, such that $\sum f(2^j)$, $\sum f^2(2^j)$ are convergent. Let $\xi_0, \xi_1, \dots$ be independent random variables, $P(\xi_\nu = 0) = 1 - \eta$, $P(\xi_\nu = f(2^\nu)) = \eta$, $\Theta = \sum_{j=0}^{\infty} \xi_j$,*

$$F_\eta(y) := P(\Theta < y). \tag{1.22}$$

Then

$$\lim_{N\to\infty} \max_{\delta \le \frac{k}{N} \le 1-\delta} \sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k}} \#\{n \in \mathcal{E}_{N,k},\ f(n) < y\} - F_{\frac{k}{N}}(y) \right| = 0. \tag{1.23}$$

Here $\delta > 0$ is an arbitrary small constant.

In [6] we mentioned that we would be able to prove

Theorem F. *Let* $f \in \mathcal{A}_2$, $f(2^j) = O(1)$. *Let* $h_N \in \mathcal{A}_2$ *be defined by* $h_N(2^j) := f(2^j) - \frac{1}{N}A_N$, $A_N = \sum_{j=0}^{N-1} f(2^j)$, $\sigma_N^2(\eta) := (1-\eta)\eta \sum_{j=0}^{N-1} h_N^2(2^j)$.
Assume that $\lim_{N\to\infty} \sigma_N(1/2) = \infty$.
Let $0 < \delta < 1/2$ *be a constant. Then*

$$\lim_{N\to\infty} \sup_{\frac{k}{N}\in[\delta,1-\delta]} \sup_{y\in\mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \left\{ n \in \mathcal{E}_{N,k} \mid \frac{f(n) - \frac{k}{N}A_N}{\sigma_N\left(\frac{k}{N}\right)} < y \right\} - \Phi(y) \right| = 0.$$

Here we shall prove that for the fulfilment of (1.23) the convergence of $\sum f(2^j)$, and of $\sum f^2(2^j)$ is necessary. Namely we shall prove the following

Theorem 1. *Let* $f \in \mathcal{A}_2$. *Assume that there exists a sequence of integers* k_N, $\frac{k_N}{N} \to \xi$ $(N \to \infty)$, $0 < \xi < 1$ *such that*

$$\lim_{N\to\infty} \sup_{y\in\mathbb{R}} \left| \frac{1}{\binom{N}{k_N}} \# \left\{ n \in \mathcal{E}_{N,k_N},\ f(n) < y \right\} - G(y) \right| = 0$$

with a suitable distribution function $G(y)$. *Then both of the series* (1.9), (1.10) *are convergent and* $G(y) = F_\xi(y)$, $F_\xi(y)$ *is defined in Theorem E.*

1.3. Additive functions. We say that $f : \mathbb{N} \to \mathbb{R}$ is additive if $f(mn) = f(m) + f(n)$ holds for every coprime pairs of integers. We say that $g : \mathbb{N} \to \mathbb{C}$ is multiplicative, if $g(1) = 1$, and $g(mn) = g(m) \cdot g(n)$, whenever $(m,n) = 1$. Let $\mathcal{A}, \mathcal{M}$ be the sets of additive, and multiplicative functions, let $\overline{\mathcal{M}} = \{g \in \mathcal{M} \mid |g(n)| = 1\ (n \in \mathbb{N})\}$. For the sake of brevity we shall write $x_1 = \log x$, $x_2 = \log x_1, \dots$.

Let $\Omega(n) =$ number of distinct prime powers of n, $\mathcal{N}_k = \{n \mid \Omega(n) = k\}$,

$$N_k(x) := \#\{n \le x,\ n \in \mathcal{N}_k\}, \quad N_k(x \mid D) := \#\{n \le x \mid (n,D) = 1,\ n \in \mathcal{N}_k\}.$$

According to a classical theorem of Erdős and Wintner, if $f \in \mathcal{A}$ and the following three series

$$\sum_{|f(p)|<1} \frac{f(p)}{p}, \qquad \sum_{|f(p)|<1} \frac{f^2(p)}{p}, \qquad \sum_{|f(p)|\ge 1} 1/p \tag{1.24}$$

are convergent, then

$$\lim_x \frac{1}{x} \#\{n \le x \mid f(n) < y\} = F(y) \tag{1.25}$$

exists at every continuity points of F, where F is a distribution function. They proved also that the convergence of the series in (1.24) is necessary for the existence of satisfying (1.25).

In [6] we proved the following two theorems.

Theorem G. *Assume that $f \in \mathcal{A}$, the series (1.24) are convergent and $k = k(x)$ satisfies the inequality*

$$\left| \frac{k}{x_2} - 1 \right| < \delta_x, \tag{1.26}$$

where $\delta_x \downarrow 0$. Then

$$\lim_{x\to\infty} \frac{1}{N_k(x)} \#\{n \le x,\ n \in \mathcal{N}_k,\ f(n) < y\} = F(y), \tag{1.27}$$

where $F(y)$ is defined by (1.25).

Theorem H. *Let $g \in \overline{\mathcal{M}}$, and assume that*

$$\sum_p \frac{1 - g(p)}{p} \tag{1.28}$$

is convergent. Let $k = k(x)$ be such a sequence for which (1.26) is satisfied. Then

$$\frac{1}{N_k(x)} \sum_{\substack{n\le x \\ n\in\mathcal{N}_k}} g(n) = (1 + o_x(1))M(g),$$

$$M(g) = \prod_p e_p, \quad e_p = (1 - 1/p)\left(1 + \frac{g(p)}{p} + \frac{g(p^2)}{p^2} + \dots\right).$$

Here we shall prove

Theorem 2. *Let g be as in Theorem H satisfying the conditions formulated there. Let $\delta > 0$ be a fixed constant, $\xi_{k,x} := \frac{k}{x_2}$. Let*

$$M_\eta(g) := \prod_p e_p(\eta), \quad e_p(\eta) = \left(1 - \frac{\eta}{p}\right)\left(1 + \frac{g(p)\eta}{p} + \frac{g(p^2)\eta^2}{p^2} + \dots\right).$$

We have

$$\lim_{x\to\infty} \sup_{\delta\le\xi_{k,x}\le 2-\delta} \left| \frac{1}{N_k(x)} \sum_{\substack{n\le x \\ n\in\mathcal{N}_k}} g(n) - M_{\xi_{k,x}}(g) \right| = 0.$$

Theorem 3. *Let $f \in \mathcal{A}$, $f(p^\alpha) = O(1)$ if $p \in \mathcal{P}$, and $\alpha \in \mathbb{N}$. Let $A_x = \sum_{p\le x} \frac{f(p)}{p}$, $f^*(p^\alpha) = (p^\alpha) - \frac{\alpha}{x_2} A_x$, $B_x^2 = \sum_{p\le x} \frac{1}{p}(f^*(p))^2$. Assume that f^* is extended to $\mathbb{N}$ so that $f^* \in \mathcal{A}$. Let $B_x \to \infty$. Let $\xi_{k,x} := \frac{k}{x_2}$, $\delta \in (0, 1/2)$ be a constant. Then*

$$\lim_{x\to\infty} \max{}_{k}{}_{\xi_{k,x}\in[\delta,2-\delta]} \max_{y\in\mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \le x \;\middle|\; \frac{f^*(n)}{B_x\sqrt{\xi_{k,x}}} < y,\ n \in \mathcal{N}_k \right\} - \Phi(y) \right| = 0.$$

Theorem 4. *Assume that the conditions of Theorem 3 hold true. Let δ, A be positive constants, so that $0 < \delta < 1/2$, $A > 2 + \delta$. Then*

$$\lim_{x\to\infty} \max\nolimits_k_{\xi_{k,x}\in[2+\delta,A]} \max_{y\in\mathbb{R}} \left| \frac{1}{N_k(x)} \# \left\{ n \le x \;\middle|\; \frac{f^*(n)}{B_x\sqrt{2}} < y \right\} - \Phi(y) \right| = 0.$$

Theorem 5. *Let $f \in \mathcal{A}$, and assume that the 3 series in (1.24) are convergent. For some $\eta \in (0,2)$ and $p \in \mathcal{P}$ let $\xi_p = \xi_p(\eta)$ be the random variable distributed by $P(\xi_p = f(p^\alpha)) = \left(1 - \frac{\eta}{p}\right)\left(\frac{\eta}{p}\right)^\alpha$ $(\alpha = 0,1,2,\dots)$. Assume that $\xi_p (p \in \mathcal{P})$ are completely independent, $\Theta(\eta) := \sum \xi_p(\eta)$.*

Let $F_\eta(y) := P(\Theta(\eta) < y)$. Let furthermore

$$F_{k,x}(y) := \frac{1}{N_k(x)} \#\{n \le x,\ n \in \mathcal{N}_k,\ f(n) < y\}.$$

Let $0 < \delta < 1/2$.

Then

$$\lim_{x\to\infty} \max_{\xi_{k,x}\in[\delta,2-\delta]} \sup_{y\in\mathbb{R}} |F_{k,x}(y) - F_{\xi_{k,x}}(y)| = 0.$$

Theorem 6. *Let $g \in \overline{\mathcal{M}}$, (1.28) is convergent. Assume furthermore that $g(2^\alpha) = 1$ $(\alpha = 1,2,\dots)$. Let $A > 2 + \delta$ be constants. In the notations of Theorem 4 we have*

$$\lim_{x\to\infty} \sup_{2+\delta\le\xi_{k,x}\le A} \left| \frac{1}{N_k(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}_k}} g(n) - M_2^*(g) \right| = 0,$$

where

$$M_2^*(g) = \prod_{p>2} e_p(2).$$

Theorem 7. *Let $f \in \mathcal{A}$ be as in Theorem 7. Assume furthermore that $f(2^\alpha) = 0$ $(\alpha = 1,2,\dots)$. Then*

$$\lim_{x\to\infty} \max_{2+\delta\le\xi_{k,x}\le A} |F_{k,x}(y) - F_2^*(y)| = 0,$$

where

$$F_2^*(y) = P\left(\sum_{p>2} \xi_p(2) < y \right).$$

Here $\xi_{k,x} = \frac{k}{x_2}$.

Remark. In Theorems 6 and 7 we have to assume something on the values $g(2^\alpha)$ and on $f(2^\alpha)$, since for the function $\nu(n)$ defined by $2^{\nu(n)}\|n$,

$$\lim_{x\to\infty} \frac{1}{N_k(x)} \#\{n \le x,\ \nu(n) < c,\ n \in \mathcal{N}_k\} = 0$$

for every fixed c.

In the proof of some of the theorems we use the following analogue of the Turán–Kubilius inequality.

Theorem 8. *Let $f \in \mathcal{A}$, $A_x = \sum_{p\le x} \frac{f(p)}{p}$, $\tilde{B}_x^2(\eta) := \sum_{p^\alpha \le \sqrt{x}} \frac{f^2(p^\alpha)\eta^{2\alpha}}{p^\alpha}$. Assume that $f(p^\alpha) = 0$ if $p^\alpha > x^{1/4}$ or if $p \in \mathcal{P}$ and $\alpha > \sqrt{x_2}$.*

Let $\delta > 0$ be a constant, $\xi_{k,x} := \frac{k}{x_2}$. Then

$$\frac{1}{N_k(x)} \sum_{\substack{n\le x \\ n\in\mathcal{N}_k}} (f(n) - \xi_{k,x}A_x)^2 \le c\tilde{B}_x^2(\xi_{k,x}), \tag{1.29}$$

if $\xi_{k,x} \in [\delta, 2-\delta]$. Here c is an absolute constant.

Theorem 9. *Let f be as in Theorem 8. Assume that $f(2^\alpha) = 0$ $(\alpha = 1, 2, \dots)$. Let δ and $A > 2 + \delta$ be constants. Then, for $(2+\delta)x_2 \le k \le Ax_2$,*

$$\frac{1}{N_k(x)} \sum_{\substack{n\le x \\ n\in\mathcal{N}_k}} (f(n) - 2A_x)^2 \le c\tilde{B}_x^2(2), \tag{1.30}$$

where c is a constant that may depend on δ and A.

Remark. In Theorem 9

$$\tilde{B}_x^2(2) = \sum_{\substack{p>2 \\ p^\alpha\le\sqrt{x}}} \frac{f^2(p^\alpha)}{p^\alpha}.$$

§2. Some lemmas and proof of Theorem 1

Let $f \in \mathcal{A}_2$, and

$$Q_{k,N}(D) := \sup_{y\in\mathbb{R}} \#\{n \in \mathcal{E}_{N,k},\ f(n) \in [y, y+D]\}.$$

Lemma 1. *Let $D > 0$ be fixed. If* $\limsup_j |f(2^j)| = \infty$, *then*

$$\max_{\delta \le k/N \le 1-\delta} \frac{Q_{k,N}(D)}{\binom{N}{k}} \to 0 \quad (N \to \infty).$$

Proof. By changing the sign of f, if needed, we may assume that $\limsup f(2^j) = \infty$.

Let $l_1 < l_2 < \dots$ be such a sequence of integers for which: $2D \le f(2^{l_1})$, $f(2^{l_{h+1}}) \ge 2f(2^{l_h})$.

Let N be a large integer, T be defined such that $l_T \le N-1 < l_{T+1}$. Let

$$U = \{l_1, l_2, \dots, l_T\}, \quad V = \{0, 1, \dots, N-1\} \backslash U.$$

Let

$$\alpha_1(n) = \sum_{s \in V} \varepsilon_s(n), \quad \alpha_2(n) = \sum_{t \in U} \varepsilon_t(n),$$

$$\mathcal{E}_h := \{n \in \mathcal{E}_{k,N},\ \alpha_2(n) = h\}, \quad h = 0, 1, \dots, T.$$

Then

$$\mathcal{E}_{N,k} = \bigcup_{h=0}^{T} \mathcal{E}_h.$$

Assume that $h \ge 1$. Then

$$\mathcal{E}_h = \bigcup_{a_1, a_2, \dots, a_h} \mathcal{E}_h^{(a_1, \dots, a_h)},$$

where $a_1, a_2, \dots, a_h$ run over all strictly monotonic sequences of length h from the set U,

$$\mathcal{E}_h^{(a_1,\dots,a_h)} := \{n \in \mathcal{E}_{k,N};\ \varepsilon_{a_\nu}(n) = 1 \text{ if } \nu = 1, \dots, h;\ \varepsilon_b(n) = 0 \text{ if } b \in U \backslash \{a_1, \dots, a_h\}\}.$$

If $n \in \mathcal{E}_h^{(a_1,\dots,a_h)}$, then $n = m + \rho_h$, where

$$\rho_h = \sum_{\nu=1}^{h} 2^{a_\nu}, \quad m = \sum_{\substack{j=0 \\ j \in V}}^{N-1} \delta_j \cdot 2^j, \qquad (\delta_j \in \{0,1\}).$$

It is clear that $\left|f\left(\rho_h^{(1)}\right) - f\left(\rho_h^{(2)}\right)\right| > D$ if $\rho_h^{(1)} \ne \rho_h^{(2)}$.

Let y and h be fixed. Then, for a fixed m, no more than one ρ_h may exist for which $f(\rho_h + m) \in [y, y+D]$.

Thus

$$\#\{n \in \mathcal{E}_h \mid f(n) \in [y, y+D]\} \le \binom{N-T}{k-h}.$$

This inequality holds for $h = 0$ as well.

We have

$$\frac{\binom{N-T}{k-h}}{\binom{N}{k}} = \frac{(N-T)!k!(N-k)!}{N!(k-h)!(N-T-(k-h))!}.$$

It is clear that, if $\{l_\nu\}$ satisfies the conditions stated above, then these conditions hold for every infinite subsequence of it. Therefore we may assume that $T^2/N \to 0$ as $N \to \infty$, whence we can deduce that

$$\frac{\binom{N-T}{k-h}}{\binom{N}{k}} = (1 + o_N(1)) \frac{k^h \cdot (N-k)^{T-h}}{N^T},$$

and so

$$\begin{aligned}\frac{Q_{k,N}(D)}{\binom{N}{k}} &\le (1 + o_N(1)) \sum_{h=0}^{T} \left(\frac{k}{N}\right)^h \left(1 - \frac{k}{N}\right)^{T-h} \\ &\le cT \max\left\{\left(1 - \frac{k}{N}\right)^T, \left(\frac{k}{N}\right)^T\right\} \le cT(1-\delta)^T \to 0 \quad \text{as } T \to \infty.\end{aligned}$$

The proof of Lemma 1 is complete. □

Lemma 2. *Let* $f \in \mathcal{A}_2$, $f(2^j) = O(1)$, $h_N \in \mathcal{A}_2$,

$$h_N(2^j) := f(2^j) - \frac{1}{N} A_N, \qquad A_N = \sum_{j=0}^{N-1} f(2^j), \qquad B_N^2 = \sum_{j=0}^{N-1} h_N^2(2^j).$$

Assume that $\limsup_{N\to\infty} B_N^2 = \infty$. *Then*

$$\lim_{N\to\infty} \max_{\frac{k}{N} \in [\delta, 2-\delta]} \frac{Q_{k,N}(D)}{\binom{N}{k}} = 0.$$

Proof. The assertion is clear from Theorem F. □

Proof of Theorem 1. Assume that the conditions hold. Then $Q_{k_N,N}(D) > c\binom{N}{k_N}$ with $c > 0$, if $\frac{k_N}{N} \in (\delta, 1-\delta)$. Thus $f(2^j) = O(1)$, and B_N^2 is bounded. One can prove simply that

$$\frac{1}{\binom{N}{k_N}} \sum_{\substack{n < 2^N \\ n \in \mathcal{E}_{N,k_N}}} h_N^2(n) = \frac{k_N}{N} \cdot \frac{(N-k_N)}{(N-1)} B_N^2, \tag{2.1}$$

whence

$$\frac{1}{\binom{N}{k_N}} \#\{n \in \mathcal{E}_{N,k_N} \mid |h_N(n)| > \Delta\} < \frac{c(\delta)}{\Delta^2}, \tag{2.2}$$

where $c(\delta)$ is a constant, and Δ is an arbitrary positive number. If f has a limit distribution on $\mathcal{E}_{N,k_N}$, then

$$\limsup_{N\to\infty} \frac{1}{\binom{N}{k_N}} \#\{n \in \mathcal{E}_{N,k_N} \mid |f(n)| > \Delta\} \le \varepsilon(\Delta), \tag{2.3}$$

where $\varepsilon(\Delta) \to 0$ as $\Delta \to \infty$.

From (2.2), (2.3) we obtain that $|h_N(n) - f(n)| \le 2\Delta$ holds for at least $\left(1 - 2\varepsilon(\Delta) - \frac{c(\Delta)}{\Delta^2}\right)\binom{N}{k_N}$ integers $n \in \mathcal{E}_{N,k_N}$, whence we obtain that $A_N = O(1)$. Thus $\sum f^2(2^j) < \infty$ holds.

Let $M < N$, $A_{M,N} = A_N - A_M$.

Let $0 < \eta < 1$, $\xi_i(\eta)$ be independent random variables,

$$P(\xi_i(\eta)) = -\eta f(2^j)) = 1 - \eta, \quad P(\xi_i(\eta) = (1-\eta) f(2^j)) = \eta,$$

$$\Theta_M(\eta) := \xi_0(\eta) + \xi_1(\eta) + \ldots + \xi_{M-1}(\eta).$$

Since $\sum f^2(2^i) < \infty$, therefore $P(\Theta_M(\eta) < z)$ converges weakly to a distribution function as $M \to \infty$.

Let

$$G_{M,\eta}(y) = P(\Theta_M(\eta) < y) \to G_\eta(y) = P(\Theta_\infty(\eta) < y).$$

Let $\tau \in \mathbb{R}$, $g(n) = e^{i\tau f(n)}$, $g_M(n) = \prod_{j=0}^{M-1} g(\varepsilon_j(n) 2^j)$,

$$h(n) = \tau f(n), \quad h_M^*(n) = \sum_{j=M}^{N-1} h(\varepsilon_j(n) \cdot 2^j),$$

$$u_M(n) := \sum_{j=M}^{N-1} h(\varepsilon_j(n) \cdot 2^j).$$

Repeating the simple computation used in [5], we can deduce that

$$\frac{1}{\binom{N}{k_N}} \sum_{n \in \mathcal{E}_{N,k_N}} \left(h_M^*(n) - \eta\tau A_{M,N}\right)^2 \le c_1(\delta) \sum_{j=M}^{N-1} h^2(2^j)$$
$$+ \frac{c_2(\delta)}{N} \sum_{i,j=M}^{N-1} |h(2^i)| \cdot |h(2^j)| \le c_3(\delta) \sum_{j=M}^{N-1} h^2(2^j),$$

with suitable constants $c_j(\delta)$, $j=1,2,3$.

We have

$$g(n)=g_M(n)e^{ih_M^*(n)}=g_M(n)e^{i\tau\eta A_{M,N}}+g_M(n)\left(e^{ih_M^*(n)}-e^{i\eta\tau A_{M,N}}\right),$$

whence $|g(n)-g_M(n)e^{i\eta\tau A_{M,N}}|\le|h_M^*(n)-\eta\tau A_{M,N}|$, and in the notations

$$M_{N,\frac{k}{N}}(\tau):=\frac{1}{\binom{N}{k}}\sum_{\substack{n<2^N\\ n\in\mathcal{E}_{N,k}}}g(n),$$

$$\varphi_{M,\eta}(\tau)=\prod_{l=0}^{M-1}\left(\eta e^{i\tau(1-\eta)f(2^l)}+(1-\eta)e^{-i\tau\eta f(2^l)}\right),$$

we obtain that

$$\left|M_{N,k_N/N}(\tau)-e^{i\frac{k_N}{N}\tau A_{M,N}}\cdot\frac{1}{\binom{N}{k_N}}\sum_{n<2^N}g_M(n)\right|\le c_4(\delta)|\tau|\sqrt{\sum_{j\ge M}f^2(2^j)}.$$

Arguing as in [5], we can deduce that

$$\frac{1}{\binom{N}{k}}\sum_{n\in\mathcal{E}_{N,k}}g_M(n)=(1+o_N(1))\prod_{j=0}^{M-1}\left(\left(1-\frac{k}{N}\right)+\frac{k}{N}\cdot g(2^j)\right)$$
$$=(1+o_N(1))e^{i\tau\frac{k}{N}A_M}\varphi_{M,\frac{k}{N}(\tau)},$$

thus

$$\left|M_{N,k_N/N}(\tau)-e^{i\frac{k_N}{N}\tau A_N}\varphi_{M,\frac{k_N}{N}}(\tau)\right|\le o_N(1)+c_5(\delta)\varepsilon_M|\tau|,$$

where

$$\varepsilon_M^2=\sum_{j=M}^{\infty}f^2(2^j),\ \varepsilon_M\to 0\quad\text{as}\quad M\to\infty.$$

Let $\psi_\eta(\tau)=\lim_{N\to\infty}M_{N,\frac{k_N}{N}}(\tau)$. From the condition we know that ψ_η exists. Furthermore $\lim_{N\to\infty}\varphi_{M,\frac{k_N}{N}}(\tau)=\varphi_{M,\eta}(\tau)$ obviously holds (due to $\frac{k_N}{N}\to\eta$). Finally, we shall prove that $\lim A_N$ exists.

Assume indirectly that $\alpha=\liminf A_N$, $\beta=\limsup A_N$, $\alpha\ne\beta$, $N_\nu\nearrow\infty$, $R_\mu\to\infty$, $A_{N_\nu}\to\alpha$ $(\nu\to\infty)$, $R_\mu\to\beta$ $(\mu\to\infty)$. Then

$$\left|M_{N_\nu,\frac{k_{N_\nu}}{N_\nu}}(\tau)-M_{R_\mu,\frac{k_{R_\mu}}{R_\mu}}(\tau)-e^{i\frac{k_{N_\nu}}{N_\nu}\tau A_{N_\nu}}\varphi_{M,\frac{k_{N_\nu}}{N_\nu}}(\tau)-e^{i\frac{k_{R_\nu}}{R_\mu}\tau A_{R_\mu}}\varphi_{M,\frac{k_{R_\mu}}{R_\mu}}(\tau)\right|$$
$$\le o_{\min(N_\nu,R_\mu)}(1)+c_6(\delta)\varepsilon_M|\tau|.$$

It is clear that $\varphi_{M,\lambda}$ is continuous uniformly in $\lambda \in [\delta, 1-\delta]$, and $\lim_{M\to\infty} \varphi_{M,\lambda}(\tau)$ is continuous as well. Hence we obtain that $|e^{i\alpha\tau} - e^{i\beta\tau}| = 0$.

This holds only if $\alpha = \beta$.

The proof is completed. □

§3. Some useful lemmas

The following two lemmas can be found in [7], pages 59 and 60.

Lemma 3 (Wintner, Fréchet–Shohat)**.** *Let $F_n(z)$ $(n = 1, 2, \dots)$ be a sequence of distribution functions. For each non-negative integer k let*

$$\alpha_k = \lim_{n\to\infty} \int_{-\infty}^{\infty} z^k dF_n(z)$$

exist.

Then there is a subsequence $F_{n_j}(z)$, $(n_1 < n_2 < \dots)$, which converges weakly to a limiting distribution $F(z)$ for which

$$\alpha_k = \int_{-\infty}^{\infty} z^k dF(z) \quad (k = 0, 1, 2, \dots).$$

Moreover, if the set of moments α_k determine $F(z)$ uniquely, then as $n \to \infty$ the distributions $F_n(z)$ converge weakly to $F(z)$.

Lemma 4. *In the notations of Lemma 3 let the series*

$$\phi(t) = \sum_{l=0}^{\infty} \alpha_l \frac{(it)^l}{l!}$$

converge absolutely in a disc of complex t-values $|t| < \tau$, $\tau > 0$.

Then the α_k determine the distribution function $F(u)$ uniquely. Moreover, the characteristic function $\phi(t)$ of this distribution had the above representation in the disc $|t| < \tau$, and can be analytically continued into the strip $|Im(t)| < \tau$.

Remark. The proof of Lemma 3 can be found in [3], while the proof of Lemma 4 is given in [7], (Vol. I, page 60).

Remark. The characteristic function $\varphi(t) = e^{-t^2/2}$ of the standard normal distribution can be written as

$$\varphi(t) = \sum_{l=0}^{\infty} \frac{\mu_{2l}(it)^{2l}}{2l!}, \qquad \mu_{2l} = \frac{(2l)!}{2^l \cdot l!}$$

$(l = 0, 1, 2, \dots)$. The expansion is absolute convergent on the whole complex plane.

Lemma 5 (Newton–Girard formulas)**.** *Let $\mathcal{B}$ be a finite set of primes, $M = \#\mathcal{B}$, $\psi : \mathcal{B} \to \mathbb{R}$,*

$$E_l = (-1)^l \sum_{\substack{p_1 < \dots < p_l \\ p_\nu \in \mathcal{B}}} \psi(p_1) \dots \psi(p_l), \quad s_h = \sum_{p \in \mathcal{B}} \psi^h(p).$$

Then

$$E_1 + s_1 = 0$$

$$2E_2 + E_1 s_1 + s_2 = 0$$

$$\vdots$$

$$rE_r + E_{r-1}s_1 + \ldots + E_1 s_{r-1} + s_r = 0 \quad (r = 1, 2, \ldots M).$$

We shall use some of the results from the book of Tenenbaum [4] (Part II., Chapter II. 6).

Let

$$\nu(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 - \frac{z}{p}\right)^{-1} (1 - 1/p)^z$$

be defined in $|z| < 2$. Since $\nu(z)$ is analytic in the open set $|z| < 2$, therefore

$$\nu(z) = \sum_{m=0}^{\infty} \frac{\nu^{(m)}(0)}{m!} z^m, \quad \left|\frac{\nu^{(m)}(0)}{m!}\right| \leq \frac{c}{(2 - \delta/2)^m}$$

with any $\delta > 0$ and a suitable constant $c = c(\delta)$.

Let

$$b_m := \frac{\nu^{(m)}(0)}{m!}$$

$$Q_{0,k}(y) = \sum_{l=0}^{k-1} \frac{1}{l!} b_{k-1-l} y^l.$$

For some polynomial $P(x) \in \mathbb{R}[x]$, $P(x) = \sum u_l x^l$, let $\|P\|(x) = \sum |u_l| x^l$.

We have

$$Q_{0,k}(y + \lambda) - Q_{0,k}(y) = \sum_{\mu=1}^{k-1} \frac{1}{\mu!} Q_{0,k}^{(\mu)}(y) \cdot \lambda^\mu,$$

and so

$$\sum_{\mu=1}^{k-1} \frac{1}{\mu!} \|Q_{0,k}^{(\mu)}\|(y) = \sum_{\mu=1}^{k-1} \frac{1}{\mu!} \sum_{k-1 \geq l \geq \mu} \frac{1}{(l-\mu)!} |b_{k-1-l}| y^{l-\mu}$$

$$= \sum_{t=0}^{k-2} \frac{1}{t!} \left(\sum_{l=t+1}^{k-1} \frac{1}{(l-t)!} |b_{k-1-l}| \right) y^t = \sum_{t=0}^{k-2} d_t \cdot y^t.$$

It is clear that $d_t \le \frac{c}{t!}$ with a suitable constant c.
We formulate the above assertion as

Lemma 6. *We have*

$$\sum_{\mu=1}^{k-1} \frac{1}{\mu!} \|Q_{0,k}^{(\mu)}\|(y) = \sum_{t=0}^{k-2} d_t y^t, \ d_t < \frac{c}{t!}$$

with a suitable constant c.

Let

$$N_k^*(x) = \frac{x}{x_1} Q_{0,k}(x_2).$$

Lemma 7. *Let δ satisfy $0 < \delta < 1$. Then, for $x \ge 3$, $1 \le k \le (2-\delta)x_2$*

$$N_k(x) = N_k^*(x) + O_\delta\left(\frac{x_2}{k} N_k^*(x) \cdot \frac{1}{x_1}\right).$$

(See Tenenbaum *[4] Theorem 5 in p. 205.)*

Let $1 \le D \le x^{\varepsilon_x}$, where $0 < \varepsilon_x < 0,1$. Let $\eta_D := \frac{\log D}{x_1}$, $\Theta_D := \log(1-\eta_D)$,

$$\psi_{k,D}(y) := \frac{1}{1-\eta_D} \left\{ 1 + \Theta_D \cdot \frac{Q'_{0,k}(y)}{Q_{0,k}(y)} + \ldots + \Theta_D^{k-1} \frac{Q_{0,k}^{(k-1)}(y)}{Q_{0,k}(y)} \right\}. \tag{3.1}$$

After easy computation we have

$$N_k^*\left(\frac{x}{D}\right) = \frac{N_k^*(x)}{D} \psi_{k,D}(x_2). \tag{3.2}$$

§4. Proof of Theorem 3

Assume that the conditions of Theorem 3 are satisfied.
Let h be completely additive, $J_x = [K_x, x^{\varepsilon_x}]$,

$$h(p) = \begin{cases} \dfrac{f^*(p)}{B_x} & \text{if } p \in J_x \\ 0 & \text{if } p \notin J_x, \end{cases}$$

where $\varepsilon_x \downarrow 0$, $K_x \uparrow \infty$ so slowly that

$$\lim_{x\to\infty} \sup_{\frac{k}{x_2}\in[\delta,2-\delta]} \frac{1}{N_k(x)} \# \left\{ n \in \mathcal{N}_k,\ n \le x,\ \left| h(n) - \frac{f^*(n)}{B_x} \right| > \varepsilon \right\} = 0 \tag{4.1}$$

for each $\varepsilon > 0$.

$$\lim_{x\to\infty} \sup_{\frac{k}{x_2}\in[\delta,2-\delta]} \frac{1}{N_k(x)} \#\{n \in \mathcal{N}_k \mid n \le x,\ \exists p > K_x,\ p^2 \mid n\} = 0. \tag{4.2}$$

Let $p < K_x$, count those $n \in \mathcal{N}_k$ for which $p^\alpha \mid n$. The size of those n is no more than

$$N_{k-\alpha}\left(\frac{x}{p^\alpha}\right) < \frac{cx}{p^\alpha x_1} \frac{x_2^{k-\alpha-1}}{(k-1-\alpha)!} \le \frac{c_1(2-\delta/2)^{\alpha+1}}{p^\alpha} N_k(x), \tag{4.3}$$

assuming e.g. that $p^\alpha \le x_1$. Hence we obtain that

$$\sup_{\frac{k}{x_2}\in[\delta,2-\delta]} \frac{1}{N_k(x)} \# \left\{ n \in \mathcal{N}_k,\ \left| \sum_{\substack{p^\alpha \| n \\ p < K_x}} \frac{f^*(p^\alpha)}{B_x} \right| > \varepsilon \right\} \to 0 \ (x \to \infty)$$

if $K_x \uparrow \infty$ sufficiently slowly. Since the number of prime divisors p in $(x^{\varepsilon_x}, x]$ of n is less than $\frac{1}{\varepsilon_x}$ therefore (4.1) clearly holds.

(4.2) can be proved easily. We use (4.3) if $K_x \le p \le x_1$ with $\alpha = 2$, and for $p > x_1$ we use the obvious

$$\#\{n \in \mathcal{N}_k,\ n \le x \mid \exists p^2 \mid n,\ p > x_1\} \le \sum_{p > x_1} \frac{x}{p^2} \le \frac{x}{x_1}$$

inequality.

Thus (4.2) is true.

We have

$$\frac{1}{B_x^2} \sum_{p < K_x} \frac{f^{*2}(p)}{p} \ll \frac{\log\log K_x}{B_x^2}, \quad \frac{1}{B_x^2} \sum_{x^{\varepsilon_x} < p < x} \frac{f^{*2}(p)}{p} \ll \frac{\log 1/\varepsilon_x}{B_x^2},$$

and so

$$\sum \frac{h^2(p)}{p} = 1 + H_x, \quad |H_x| \ll \frac{\log\log K_x + \log 1/\varepsilon_x}{B_x^2}. \tag{4.4}$$

Assuming that K_x and $1/\varepsilon_x$ are increasing sufficiently slowly, we can and will assume that $H_x \to 0$ $(x \to \infty)$.

To prove the theorem it is enough to show that for every $r = 1, 2, \dots$,

$$\sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \left| \frac{1}{N_k(x)} \sum_{\substack{n \le x \\ n \in \mathcal{N}_k}} \frac{h^r(n)}{\beta_k^r} - \mu_r \right| \to 0 \quad \text{as } x \to \infty,$$

and then apply the Frechet–Shohat theorem.

Let us consider the sum

$$U_{k,r}(x) := \frac{1}{N_k(x)} \sum_{\substack{n \le x \\ n \in \mathcal{N}_k}} h^r(n). \tag{4.5}$$

Since h is completely additive, therefore

$$U_{k,r}(x) = \sum_{s=1}^{r} \sum_{l_1 + \dots + l_s = r} \frac{c(r; l_1, \dots, l_s)}{N_k(x)} \sum_{p_1, p_2, \dots, p_s}^{*} h^{l_1}(p_1) \dots h^{l_s}(p_s) N_{k-s}\left(\frac{x}{p_1 \dots p_s}\right) \tag{4.6}$$

where star indicates that we sum over all those s tuples $p_1, \dots, p_s$ of primes for which $p_i \ne p_j$, if $i \ne j$. Here $c(r; l_1, \dots, l_s) = \frac{r!}{l_1! \dots l_s!}$.

Let

$$V_{k,r}(x \mid l_1, \dots, l_s) = \frac{1}{N_{k-s}^*(x)} \sum_{p_1, \dots, p_s}^{*} h^{l_1}(p_1) \dots h^{l_s}(p_s) N_{k-s}^*\left(\frac{x}{p_1 \dots p_s}\right), \tag{4.7}$$

$$\tilde{U}_{k,r}(x) = \sum_{s=1}^{r} \frac{c(r; l_1, \dots, l_s) \cdot N_{k-s}^*(x)}{N_k(x)} V_{k,r}(x \mid l_1, \dots, l_s). \tag{4.8}$$

From Lemma 7 we can deduce simply that $U_{k,r}(x) - \tilde{U}_{k,r}(x) \to 0$ $(x \to \infty)$ uniformly as $\frac{k}{x} \in [\delta, 2-\delta]$. We estimate $V_{k,r}(x \mid l_1, \dots, l_s)$ by using (3.1), (3.2) with $D = p_1 \dots p_s$. We can write $\psi_{k,D}(y)$ as a convergent power series of η_D.

We try to estimate

$$E(l_1, t_1; l_2, t_2; \dots; l_s t_s) := \sum_{p_1, \dots, p_s}^{*} \frac{h^{l_1}(p_1)(\log p_1)^{t_1}}{p_1 x_1^{t_1}} \dots \frac{h^{l_s}(p_s) \cdot (\log p_s)^{t_s}}{p_s x_1^{t_s}}. \tag{4.9}$$

Let

$$\kappa(l, t) := \frac{1}{x_1^t} \sum_{p \in J_x} \frac{h^l(p)(\log p)^t}{p} \quad (l = 1, 2, \dots;\ t = 0, 1, \dots). \tag{4.10}$$

From (4.4) we have

$$\kappa(2,0) = 1 + H_x, \ |H_x| < \frac{c \log\log K_x + \log 1/\varepsilon_x}{B_x^2}. \tag{4.11}$$

We have

$$\begin{aligned}\kappa(1,0) &= \frac{1}{B_x} \sum_{p\in J_x} \frac{1}{p}\left(f(p) - \frac{A_x}{x_2}\right) = \frac{1}{B_x}\sum_{p<x_2}\frac{1}{p}\left(f(p) - \frac{A_x}{x_2}\right) \\ &- \frac{1}{B_x}\sum_{p<B_x}\frac{1}{p}\left(f(p) - \frac{A_x}{x_2}\right) - \frac{1}{B_x}\sum_{x^{\varepsilon_x}\le p<x}\frac{1}{p}\left(f(p) - \frac{A_x}{x_2}\right) \\ &= \sum\nolimits_1 - \sum\nolimits_x - \sum\nolimits_3.\end{aligned}$$

Since $x_2 - \sum_{p<x_2} 1/p = O(1)$, therefore

$$\sum\nolimits_1 = \frac{1}{B_x}\left(A_x - \frac{A_x}{x_2}\sum_{p<x_2} 1/p\right) = \frac{A_x}{x_2 B_x}\left(x_2 - \sum_{p<x_2} 1/p\right) = O\left(\frac{1}{B_x}\right).$$

Furthermore

$$\sum\nolimits_1 = O\left(\frac{\log\log K_x}{B_x}\right), \qquad \sum\nolimits_2 = O\left(\frac{\log 1/\varepsilon_x}{B_x}\right).$$

Consequently

$$|\kappa(1,0)| \le \frac{c(\log\log K_x + \log 1/\varepsilon_x)}{B_x}$$

with a suitable constant c.

It is known that

$$\sum_{p<y}\frac{(\log p)^s}{p} < c\frac{(\log y)^s}{s}$$

for $s \ge 1$.

Let Λ_x be defined by

$$\Lambda_x := \frac{c(\log\log K_x + \log 1/\varepsilon_x)}{B_x} + \frac{r}{K_x} \ge |\kappa(1,0)| + \frac{r}{K_x}. \tag{4.12}$$

It is known that

$$\sum_{p<y}\frac{(\log p)^s}{p} < c\frac{(\log y)^s}{s},$$

for $s \ge 1$.

Hence, by using the Cauchy–Schwarz inequality,

$$\kappa(1,t) \le \left(\sum_{p\in J_x} \frac{h^2(p)}{p}\right)^{1/2} \left(\frac{1}{x_1^{2t}} \sum_{p\le x^{\varepsilon_x}} \frac{(\log p)^{2t}}{p}\right) \le \frac{c\varepsilon_x^t}{\sqrt{t}}. \tag{4.13}$$

For $l \ge 2$, $t \ge 1$

$$|\kappa(l,t)| \le c\varepsilon_x^t \kappa(l,0), \tag{4.14}$$

$$|\kappa(l,0)| \le c\left(\frac{1}{B_x}\right)^{l-2}. \tag{4.15}$$

Assume first that there exists at least one $(l_j,t_j) = (1,0)$. Assume that $(l_j,t_j) = (1,0)$ if $j = 1,\dots,h$ and $(l_j,t_j) \ne (1,0)$ if $j > h$. We have

$$E(l_1,t_1;\dots;l_s,t_s)$$
$$= \sum_{p_{h+1},\dots,p_s}^{*} \left(\prod_{\nu=h+1}^{s} \frac{h^{l_\nu}(p_\nu)}{p_\nu} \cdot \frac{(\log p_\nu)^{t_\nu}}{x_1^{t_\nu}}\right) \left\{\sum_{p_1,\dots,p_h}^{**} \frac{h(p_1)}{p_1} \cdots \frac{h(p_h)}{p_h}\right\}$$

where $*$ means that $p_{h+1},\dots,p_s$ are distinct primes, and $**$ means that $p_1,\dots,p_h$ are distinct primes, none of them belongs to the set $\{p_{h+1},\dots,p_s\}$. First we estimate the inner sum. Let us apply Lemma 5 with $\mathcal{B} = \{p \mid p < x^{\varepsilon_x}\} \setminus \{p_{h+1},\dots,p_s\}$, $\psi(p) = \frac{h(p)}{p}$. In the notation of Lemma 5 $\sum_{p_1,\dots,p_h}^{**} \frac{h(p_1)}{p_1} \cdots \frac{h(p_h)}{p_h} = (-1)^h h! E_h$. Since $|E_1| = \left|\sum_{p\in\mathcal{B}} \frac{h(p)}{p}\right| \le \Lambda_x$ (see (4.12)), from the Newton–Girard formulas (by using induction on h e.g.) we obtain that $|E_h| \le c\Lambda_x$, where c is a constant that may depend on r at most.

Thus

$$E(l_1,t_1;\dots;l_s,t_s) \le c\Lambda_x \kappa(l_{h+1},t_{h+1}) \dots \kappa(l_s,t_s).$$

By the inequalities (4.13), (4.14), (4.15) we have

$$E(l_1,t_1;\dots;l_s,t_s) \le c_1 \Lambda_x\, \varepsilon_x^{t_1+\dots+t_s} \prod_{l_j\ge 2} \left(\frac{1}{B_x}\right)^{l_j-2}. \tag{4.16}$$

c_1 is a constant which may depend on r.

Similarly, if $(l_j,t_j) \ne (1,0)$ holds for every j, then

$$E(l_1,t_1,\dots,l_s,t_s) \le c_1 \varepsilon_x^{t_1+\dots+t_s} \prod_{l_j\ge 2} \left(\frac{1}{B_x}\right)^{l_j-2}. \tag{4.17}$$

We can observe that the right hand side of (4.16), (4.17) tends to zero except the case, when for every j, $(l_j, t_j) = (2, 0)$. This can be happen only if $r = 2R$ is even. Observe that

$$E(2,0;\dots,2,0) = \sum_{p_1,\dots,p_R}^{*} \frac{h^2(p_1)}{p_1}\dots\frac{h^2(p_R)}{p_R},$$

and hence we can deduce easily that

$$E(2,0;\dots;2,0) = \kappa(2,0)^R + o_x(1) = 1 + o_x(1). \tag{4.18}$$

Let us go back to (4.7). See furthermore (3.1):

$$V_{k,r}(x \mid l_1,\dots,l_s) = \sum^{*} \frac{h^{l_1}(p_1)\dots h^{l_s}(p_s)}{p_1\dots p_s} T_{k-s}(\eta_{p_1\dots p_s}),$$

where

$$\begin{aligned} T_{k-s}(W) &= \frac{1}{1-W}\Big\{1 + \log(1-W)\cdot S_1 + \log^2(1-W)S_2 + \dots \\ &\qquad + \log^{k-s-1}(1-W)\cdot S_{k-s-1}\Big\} \\ S_j &:= \frac{Q^{(j)}_{0,k-s-1}(x_2)}{Q_{0,k-s-1}(x_2)}. \end{aligned}$$

Let

$$V^{(T)}_{k,r}(x \mid l_1,\dots,l_s) = \sum_{p_1,\dots,p_s}^{*} \frac{h^{l_1}(p_1)\dots h(p_s)^{l_s}}{p_1\dots p_s}\left(\frac{\log p_1\dots p_s}{x_1}\right)^T.$$

Then

$$V^{(T)}_{k,r}(x \mid l_1,\dots,l_s) = \sum_{t_1+\dots+t_s=T} \frac{T!}{t_1!\dots t_s!} E(l_1,t_1;\dots,l_s,t_s).$$

In the case $T = 0$ it was already proved that $V^{(0)}_{k,r}(x \mid l_1,\dots,l_s) = o_x(1)$, except the case when $l_1 = l_2 = \dots = l_s = 2$, $s = R$, $r = 2R$, when $V^{(0)}_{k,2R}(x \mid 2, \dots, 2) = 1 + o_x(1)$.

Let now $T \ge 1$. From (4.16), (4.17) we obtain that

$$V^{(T)}_{k,r}(x \mid l_1,\dots,l_s) \le c\varepsilon_x^T. \tag{4.19}$$

Let $u(w) = p_0 + p_1 w + \dots$ be a power series with nonnegative coefficients, and assume that it converges in the disc $|w| < 1$.

Since

$$\sum_{p_1,\dots,p_s}^{*} \frac{h^{l_1}(p_1)\dots h^{l_s}(p_s)}{p_1\dots p_s} u\left(\frac{\log p_1\dots p_s}{x_1}\right) = \sum_{T=0}^{\infty} p_T V_{k,r}^{(T)}(x \mid l_1,\dots,l_s),$$

from (4.19) we obtain that the left hand side of (4.20) is less than

$$\le \sum_{T=0}^{\infty} p_T c\varepsilon_x^T = cu(\varepsilon_x).$$

Since the coefficients of the Taylor expansion of $\frac{w}{1-w}$ and of $(-1)^j \frac{(\log(1-w))^j}{1-w}$ is positive, and they converge for $|w| < 1$, therefore

$$\left|\sum^{*} \frac{h^{l_1}(p_1)\dots h(p_s)^{l_s}}{p_1\dots p_s} u\left(\frac{\log p_1\dots p_s}{x_1}\right)\right| \le cu(\varepsilon_x)$$

holds, if

$$u(w) = \frac{(-1)^j \log^j(1-w)}{1-w}, \quad j = 1,\dots,k-s-1,$$

and if

$$u(w) = \frac{w}{1-w},$$

$l_1,\dots,l_s$ arbitrary, and in the case $u(w) = 1$, $(l_1,\dots,l_s) \ne (2,\dots,2)$ the left hand side tends to 0.

Consequently, by Lemma 6 $V_{k,r}(x \mid l_1,\dots,l_s) \to 0$ $(x \to \infty)$ if $(l_1,\dots,l_s) \ne (2,\dots,2)$, while for $s = R$, $r = 2R$,

$$V_{k,2R}(x \mid 2,\dots,2) = 1 + o_x(1).$$

We are almost ready. We have to observe only that

$$\frac{N_{k-s}^*(x_2)}{N_k^*(x_2)} = \frac{k(k-1)\dots k-(s-1)}{x_2^s} = (1+o_x(1))\xi_{k,x_2}^s.$$

The proof is complete.

§5. Proof of Theorem 4

Theorem 10. *Let $0 < \delta$, $A > 2+\delta$ be constants. Then for all $k \in [(2+\delta)x_2,\ Ax_2]$ we have*

$$N_k(x) = \frac{cxx_1}{2^k}\left\{1 + O_A\left(x_1^{-\delta^2/5}\right)\right\}.$$

See [4].

To prove the theorem we can use the argument of the proof of Theorem 5. Instead of (3.1), (3.2) we can use the formula

$$N_k^*(x) = \frac{cxx_1}{2^k},\ N_k^*\left(\frac{x}{D}\right) = \frac{1}{D}\ N_k^*(x)\left(1 - \frac{\log D}{x_1}\right).$$

We omit the details.

§6. Proof of Theorem 8

If (1.29) holds for $f_1, f_2 \in \mathcal{A}$, then it holds for $f = f_1 + f_2$. Let $\gamma < 1/4$ be a small positive constant, $f_1(p^\alpha) = f(p^\alpha)$ if $p^\alpha < x^\gamma$, and $f_1(p^\alpha) = 0$ if $p^\alpha \geq x^\gamma$, and let $f_2(p^\alpha) = f(p^\alpha) - f_1(p^\alpha)$.

We have

$$\begin{aligned} S := \sum_{\substack{n\in\mathcal{N}_k \\ n\le x}} f_2^2(n) &\leq \sum_{p_1\neq p_2} |f_2(p_1^{\alpha_1})| \cdot |f_2(p_2^{\alpha_2})| \cdot N_{k-\alpha_1-\alpha_2}\left(\frac{x}{p_1^{\alpha_1}p_2^{\alpha_2}}\right) \\ &+ \sum_{p^\alpha} |f_2(p^\alpha)| \cdot N_{k-\alpha}\left(\frac{x}{p^\alpha}\right). \end{aligned}$$

From the conditions of the theorem $f_2(p_i^{\alpha_i}) = f(p_i^{\alpha_i}) = 0$ if $p_i^{\alpha_i} > x^{1/4}$, or if $\alpha_i > \sqrt{x_2}$.

Assume that $p_i^{\alpha_i} \leq x^{1/4}$ and $\alpha_i \leq \sqrt{x_2}$. Then

$$N_{k-\alpha_1-\alpha_2}\left(\frac{x}{p_1^{\alpha_1}p_2^{\alpha_2}}\right) \leq \frac{cN_k(x)}{p_1^{\alpha_1}p_2^{\alpha_2}}\xi_{k,x}^{\alpha_1+\alpha_2}, \tag{6.1}$$

(c is an absolute constant) and we deduce that

$$\frac{1}{N_k(x)}S \leq c\left(\sum_{x^\gamma<p^\alpha\leq x^{1/4}} \frac{|f_2(p^\alpha)|}{p^\alpha}\xi_{k,x}^\alpha\right)^2 + c\tilde{B}_x^2(\xi_{k,x}).$$

Since

$$\sum \frac{|f_2(p^\alpha)|\xi_{k,x}^\alpha}{p^\alpha} \leq \left(\sum \frac{\xi_{k,x}^\alpha}{p^\alpha}\right)^{1/2} \tilde{B}_x(\xi_{k,x}),$$

where in the right hand side $x^\gamma < p^\alpha < x^{1/4}$, $\alpha \leq \sqrt{x_2}$, thus $p = 2$ cannot occur.

Therefore

$$\sum_{p^\alpha, p\geq 2} \frac{\xi_{k,x}^\alpha}{p^\alpha} \text{ is bounded by an absolute constant, and so}$$

$$\frac{1}{N_k(x)} S \leq c\tilde{B}_x^2(\xi_{k,x}).$$

Let

$$U_x = \sum \frac{f_2(p)}{p} = \sum_{x^\gamma \leq p < x^{1/4}} \frac{f(p)}{p}.$$

Since

$$\frac{1}{N_k(x)} \sum_{\substack{n\leq x\\ n\in\mathcal{N}_k}} (f_2(n) - U_x)^2 \leq \frac{2}{N_k(x)} S + 2|U_x|^2,$$

and

$$|U_x|^2 \leq \left\{ \sum_{x^\gamma < p < x^{1/4}} \frac{1}{p} \right\} \tilde{B}_x^2 \left(\frac{k}{x_2}\right),$$

therefore (1.29) holds for f_2.

Let now f_3 be defined on prime powers p^β such that $f_3(p^\beta) = f_1(p^\alpha) - f_1(p^{\alpha-1})$ $(\alpha = 1, 2, \dots)$. Then, with the classical meaning of summation,

$$f_1(n) = \sum_{p^\beta | n} f_3(p^\beta).$$

Let

$$f_4(n) = \sum_{p|n} f_3(p), \qquad f_5(n) = \sum_{\substack{p^\beta | n\\ \beta\geq 2}} f_3(p^\beta).$$

Let us estimate first

$$S_1 := \sum_{\substack{n\leq x\\ n\in\mathcal{N}_k}} f_5(n)^2 = \sum_{\substack{p_1^{\alpha_1}; p_2^{\alpha_2}\\ p_1\neq p_2}} f_3(p_1^{\alpha_1}) f_3(p_2^{\alpha_2}) N_{k-\alpha_1-\alpha_2}\left(\frac{x}{p_1^{\alpha_1} p_2^{\alpha_2}}\right)$$

$$+ \sum_{\substack{p\\ \alpha_1,\alpha_2}} f_3(p^{\alpha_1}) f_3(p^{\alpha_2}) N_{k-\max(\alpha_1,\alpha_2)}\left(\frac{x}{p^{\max(\alpha_1,\alpha_2)}}\right) = S_2 + S_3.$$

Since $f_3(p_i^{\alpha_i}) = 0$, if $p_k^{\alpha_i} > x^\gamma$, or if $\alpha_i = 1$, or $\alpha_i > \sqrt{x_2}$, from (6.1) we deduce that

$$\frac{S_2}{N_k(x)} \leq \left(\sum_{\alpha\geq 2} \frac{|f_3(p^\alpha)|}{p^\alpha} \xi_{k,x}\right)^2 + 2\sum_{\alpha_1=2}^{\infty} \sum_{\alpha_2=2}^{\alpha_1} \sum_p \frac{|f_3(p^{\alpha_1}) f_3(p^{\alpha_2})|}{p^{\alpha_1}} \xi_{k,x}^{\alpha_1}.$$

The first sum on the right hand side is less than $c\tilde{B}_x^2(\xi_{k,x})$. To estimate the second sum we start from

$$\left|\xi_{k,x}^{\alpha_1} f_3(p^{\alpha_1}) f_3(p^{\alpha_2})\right| \le 2 f_3^2(p_1^{\alpha}) \xi_{k,x}^{2\alpha_1} + f_3^2(p_2^{\alpha})$$

and deduce that it is less than

$$4\tilde{B}_x(\xi_{k,x}) + 4 \sum_{\alpha_2=2}^{\infty} \sum_p \frac{|f_3(p_2^{\alpha})|^2}{p^{\alpha_2}} \sum \frac{1}{1-1/p} \le 4\tilde{B}_x^2(\xi_{k,x}) + 8\tilde{B}_x^2(1).$$

Thus

$$\frac{S_2}{N_k(x)} \le c_1 \tilde{B}_x^2(\xi_{k,x}) + 8\tilde{B}_x^2(1).$$

Finally we prove that

$$T := \sum_{\substack{n \le x \\ n \in \mathcal{N}_k}} \left(f_4(n) - \xi_{k,x_2} A_x^*\right)^2 \le c B_x^2 N_k(x),$$

$$A_x^* = \sum_{p < x^\gamma} \frac{f_4(p)}{p}. \tag{6.2}$$

Let $\rho_x := \sum_{p<x^\gamma} 1/p$.

Let $\tilde{f}_4(p) = f_4(p) - \frac{A_x^*}{\rho_x}$, $\tilde{f}_4(n) = \sum_{p|n} \tilde{f}_4(p)$. Then $\tilde{f}_4(n) = f_4(n) - \frac{k}{\rho_x} A_x^*$ if $n \in \mathcal{N}_k$.

Let

$$\tilde{T} = \sum_{\substack{n \le x \\ n \in \mathcal{N}_k}} \tilde{f}_4(n)^2. \tag{6.3}$$

We shall prove that

$$\tilde{T} \le c N_k(x) \sum_{p \le x} \frac{\tilde{f}_4^2(p)}{p}. \tag{6.4}$$

Hence (6.2) easily follows.

We have

$$\tilde{T} = \sum_{p_1 \ne p_2} \tilde{f}_4(p_1) \tilde{f}_4(p_2) N_{k-2}\left(\frac{x}{p_1 p_2}\right) + \sum_p \tilde{f}_4^2(p) N_{k-1}\left(\frac{x}{p}\right).$$

From Lemma 7 we obtain that

$$\tilde{T} = \tilde{T}_1 + \tilde{T}_2 + \text{error, where}$$

$$\tilde{T}_1 = \sum_{p_1,p_2} \tilde{f}_4(p_1)\tilde{f}_4(p_2)N^*_{k-2}\left(\frac{x}{p_1p_2}\right),$$

$$\tilde{T}_2 = \sum \tilde{f}_4^2(p)\left(N_{k-1}\left(\frac{x}{p}\right) - N_{k-2}\left(\frac{x}{p_2}\right)\right),$$

where the error is clearly less than $cN_{k-2}(x)\sum \frac{\tilde{f}_4^2(p)}{p}$.

Let

$$E_l := \sum_{p<x^\gamma} \frac{\tilde{f}_4(p)}{p}\frac{(\log p)^l}{x_1^l}.$$

It is clear that $E_0 = 0$, and

$$|E_l| \le \left(\sum \frac{\tilde{f}_4^2(p)}{p}\right)^{1/2}\left(\sum_{p<x^\gamma} \frac{(\log p)^{2l}}{px_1^{2l}}\right)^{1/2} \le 2\left(\sum \frac{\tilde{f}_4^2(p)}{p}\right)^{1/2}\frac{\gamma^l}{\sqrt{2l}},$$

if $x > x_0$, and $l \ge 0$. Thus

$$\left|\sum_{p_1,p_2} \frac{\tilde{f}_4(p_1)}{p_1}\frac{\tilde{f}_4(p_2)}{p_2}\frac{(\log p_1p_2)^\nu}{x_1^\nu}\right| = \left|\sum_{l=0}^{\nu-1} E_lE_{\nu-l}\cdot\binom{\nu}{l}\right| \le 4(2\gamma)^l. \tag{6.5}$$

We have

$$\frac{\tilde{T}_1}{N_{k-2}(x)} = \sum_{p_1,p_2} \frac{\tilde{f}_4(p_1)}{p_1}\frac{\tilde{f}_4(p_2)}{p_2}\psi_{k-2,p_1p_2}(x_2),$$

where ψ_{k-2,p_1p_2} is defined in (3.1). By using Lemma 6 and (6.5), furthermore that $\tilde{T}_2 \ll \xi_{k,x}\sum \frac{\tilde{f}_4^2(p)}{p}N_k(x)$, we get (6.4).

Since $\tilde{f}_4^2(p) \le 2f_4^2(p) + 2\frac{|A_x^*|^2}{\rho_x^2}$, therefore

$$\sum \frac{\tilde{f}_4^2(p)}{p} \le 2B_x^2 + 2\frac{|A_x^*|^2}{\rho_x},\ A_x^{*2} \le \sum \frac{1}{p}B_x^2,$$

and so

$$\sum \frac{\tilde{f}_4^2(p)}{p} \le cB_x^2.$$

Finally $f_4(n) - \xi_{k,x_2}\ A_x^* = \tilde{f}_4(n) + \left(\frac{k}{\rho_x} - \xi_{k,x_2}\right)A_x^*$, and so

$$T \le 2\tilde{T} + \left|\frac{k}{\rho_x} - \xi_{k,x_2}\right|^2 |A_x^*|^2N_k(x).$$

Furthermore $|A_x^*|^2 \le B_X^2\rho_x$, and so

$$\left|\frac{k}{\rho_x} - \frac{k}{x_2}\right|^2 |A_x^*|^2 \le \rho_x\left|\frac{k(x_2-\rho_x)}{\rho_xx_2}\right|^1 B_x^2 = o_x(1)B_x^2.$$

Thus (6.2) holds true.

The proof of the theorem is complete.

§7. Proof of Theorem 9

We can argue similarly as in §6. Since now $N_k(x) = N_k^*(x)\big(1+O_A\big(x_1^{-\delta^2/5}\big)\big)$ (Lemma 8), $N_k^*\big(\frac{x}{D}\big) = \frac{1D}{N}{}^*_k(x) - \frac{\log D}{Dx_1}N_k^*(x)$, we obtain our theorem easier than that of Theorem 10.

We omit the details.

§8. Proof of Theorem 5

Assume that the conditions of the theorem hold. Let $\mathcal{B}$ be such a sequence of primes for which $\sum_{p\in\mathcal{B}} 1/p < \infty$. Let $\rho(Y) := \sum_{\substack{bp>Y\\ p\in\mathcal{B}}} 1/p$. Then $\rho(Y) \to 0$ as $Y \to \infty$.

Count

$$S_Y := \#\{n \le x \mid n \in \mathcal{N}_k, p \mid n \text{ for some } p > Y,\ p \in \mathcal{B}\}.$$

Then

$$\begin{aligned} S_Y &\le \sum_{\substack{Y<p<x^{1-\delta_x}\\ p\in\mathcal{B}}} N_{k-1}\left(\frac{x}{p}\right) + \sum_{\substack{\nu<x^{\delta_x}\\ \nu\in\mathcal{N}_{k-1}}} \pi\left(\frac{x}{\nu}\right) \le \\ &\le N_{k-1}(x) \sum_{\substack{Y<p<x^{1-\delta_x}\\ p\in\mathcal{B}}} \frac{1}{p}\frac{\log x}{(\log x - \log p)} + \frac{3x}{x_1} \sum_{\substack{\nu<x^{\delta_x}\\ \nu\in\mathcal{N}_{k-1}}} 1/\nu \\ &\le N_{k-1}(x)\cdot \frac{1}{\delta_x}\rho(Y) + \frac{3x}{x_1}\cdot\frac{1}{(k-1)!}\left(\sum_{p<x^{\delta_x}} 1/p\right)^{k-1}, \end{aligned}$$

whence

$$\frac{S_Y}{N_k(x)} \le \frac{k}{x_2}\cdot\frac{1}{\delta_x}\rho(Y) + 3\left(\frac{x_2 - \log 1/\delta_x}{x_2}\right)^{k-1} \le \frac{k}{x_2}\frac{1}{\delta_x}\rho(Y) + 3e^{-\frac{(k-1)}{x_2}\log\frac{1}{\delta_x}}.$$

The second sum is small if δ_x is small, the first sum is small if $\frac{\rho(Y)}{\delta_x}$ is small, i.e. if Y is large.

Thus, by choosing $\delta_x = \sqrt{\rho(Y)}$ for example, we obtain that

$$\frac{S_Y}{N_k(x)} = o_Y(1).$$

From the convergence of the three series it is obvious that there is a sequence $\rho_p \downarrow 0$ such that for the set $\mathcal{B}_1 = \{p \mid |f(p)| > \rho_p\}$, $\sum_{p\in\mathcal{B}_1} 1/p < \infty$. Let $\mathcal{B}_1$ be fixed. Let $\mathcal{B}_2 = \{p^\alpha \mid p \in \mathcal{P},\ \alpha \geq 2\}$, and let

$$S_Y^* := \#\{n \leq x \mid n \in \mathcal{N}_k,\ p^\alpha \mid n \text{ for some } p^\alpha \in \mathcal{B}_2,\ p^\alpha > Y\}.$$

This is clear:

$$S_Y \leq \sum_{\substack{Y < p^\alpha \leq \sqrt{x} \\ p^\alpha \in \mathcal{B}_2}} N_{k-\alpha}\left(\frac{x}{p^\alpha}\right) + \sum_{x \geq p^\alpha \geq \sqrt{x}} \frac{x}{p^\alpha} \leq cN_k(x) \sum_{p^\alpha \geq Y} \left(\frac{k}{x_2}\right)^\alpha \frac{1}{p^\alpha} + cx^{3/4},$$

and so

$$\frac{S_Y^*}{N_k(x)} \leq c \sum_{2^\alpha \geq Y} \left(\frac{k}{x_2 \cdot 2}\right)^\alpha + \frac{1}{Y^{1/10}} \sum_{\substack{\alpha \geq 2 \\ p \geq 3}} \left(\frac{k}{x_2 \cdot p^{9/10}}\right)^\alpha + cx^{3/4}.$$

The first sum on the right hand side is $\ll \frac{Y^{-\delta/2 \log 2}}{1 - \frac{k}{2x_2}}$, the second sum after $\frac{1}{Y^{1/10}}$ is bounded by an absolute constant.

Thus

$$\limsup_{x\to\infty} \sup_{\frac{k}{x_2} \in [\delta, 2-\delta]} \frac{S_Y^*}{N_k(x)} \leq \varepsilon(Y),$$

where $\varepsilon(Y) \to 0$ as $Y \to \infty$.

Let $Y = Y_x$ be tending to infinity slowly. For some $n \leq x$ let $n = A(n) \cdot B(n)$, where $A(n) = \prod_{\substack{p^\alpha \| n \\ p<Y}} p^\alpha$, and $B(n) = \frac{n}{A(n)}$. Consider the set of integers $n \in \mathcal{N}_k$ up to x. Let us drop those n for which $p \mid n$ for some $p \in \mathcal{B}_1$, $p > Y$ and those for which $p^\alpha \mid n$ for some $p^\alpha \in \mathcal{B}_2$, $p^\alpha > Y$. The number of the dropped elements is $\ll \varepsilon_1(Y) N_k(x)$, where $\varepsilon_1(Y) \to 0$ uniformly as $\frac{k}{x_2} \in [\delta, 2-\delta]$. Let $f^* \in \mathcal{A}$ defined on prime powers p^α as follows:

$$f^*(p^\alpha) = \begin{cases} 0 & \text{if } \alpha \geq 2 \\ 0 & \text{if } \alpha = 1,\ p \leq Y,\ \text{if } p \geq \sqrt{x},\ \text{or if } p \in \mathcal{B}_1 \\ f(p) & \text{if } \alpha = 1,\ p \in (Y, \sqrt{x}\,]. \end{cases}$$

From Theorem 8 we have

$$\frac{1}{N_k(x)} \sum \left(f^*(n) - \xi_{k,x} \sum \frac{f^*(p)}{p}\right)^2 \leq c \sum_{Y \leq p \leq \sqrt{x}} \frac{f^{*2}(p)}{p}. \tag{8.1}$$

$$\limsup_{x} \sup_{\xi_{k,x}\in[\delta,2-\delta]} \frac{1}{N_k(x)} \#\{n \le x \mid n \in \mathcal{N}_k,\ |f^*(n)| \ge \lambda\} \tag{8.2}$$
$$\le \varepsilon_2(Y),\ \varepsilon_2(Y) \to 0$$

valid for every $\lambda > 0$.

Let $\mathcal{M}_Y$ be the set of those m, the largest prime power factor of which is not larger than Y, and if $p^\alpha \| m$, $\alpha \ge 2$, then $p^\alpha \le Y$. From the estimation of S_Y^* we obtain that

$$\frac{1}{N_k(x)} \#\{n \le x \mid n \in \mathcal{N}_k,\ A(n) \notin \mathcal{M}_Y\}$$

$$\limsup_{x\to\infty} \sup_{\frac{k}{x_2}\in[\delta,2-\delta]} \frac{1}{N_k(x)} \#\{n \le x \mid n \in \mathcal{N}_k,\ A(n) \notin \mathcal{M}_Y\} \le \varepsilon_3(Y),$$

where $\varepsilon_3(Y) \to 0$ as $Y \to \infty$.

Let $\mathcal{D}_{m,k} := \{n \in \mathcal{N}_k,\ A(n) = m\}$ $(m \in \mathcal{M}_Y)$, and let $h(n) := f(A(n))$. Thus $h(n)$ is constant on $D_{m,k}$, and from (8.2) we obtain that

$$\limsup_{x\to\infty} \sup_{\frac{k}{x_2}\in[\delta,-2\delta]} \frac{1}{N_k(x)} \#\{n \le x \mid n \in \mathcal{N}_k |f(n) - f(A(n))| > \lambda\} \le \varepsilon_2(Y).$$

Now we compute the density of the set $D_{m,k}$.

Let $\mathcal{N}_k(D) = \{n \in \mathcal{N}_k \mid n \in D\}$. Starting from the generating function

$$\prod_{p\nmid D} \frac{1}{1-\frac{z}{p^s}} = \prod_{p|D} \left(1 - \frac{z}{p^s}\right) \cdot \sum \frac{z^{\Omega(n)}}{n^s},$$

for $N_k(x|D) = \sum_{\substack{n\le x \\ (n,D)=1 \\ n\in\mathcal{N}_k}} 1$ we have

$$N_k(x, D) = \sum_{d|D} \mu(d) N_{k-\Omega(d)} \left(\frac{x}{D}\right).$$

Let $K_Y = \prod_{p\le Y} p$.

From the convergence of the series in (1.24) we obtain that

$$\sum \frac{f^*(p)}{p} = \sum_{\substack{Y\le p<\sqrt{x} \\ |f(p)|<1}} \frac{f(p)}{p} = \sum_{\substack{Y\le p<\sqrt{x} \\ \rho_p<|f(p)|<1}} \frac{f(p)}{p}$$

tends to zero as $x \to \infty$. The right hand side of (8.1) tends to zero as well. Applying these relations, from (8.1) we obtain

Consequently

$$\#(D_m, k) = N_{k-\Omega(m)}\left(\frac{x}{m} \mid K_Y\right) = \sum_{d\mid K_Y} N_{k-\Omega(m)-\Omega(d)}\left(\frac{x}{md}\right)\mu(d),$$

and so

$$\frac{\#(D_{m,k})}{N_k(x)} = (1+o_x(1))\frac{\xi_{k,x_2}^{\Omega(m)}}{m}\prod_{p\mid K_Y}\left(1-\frac{\xi_{k,x_2}}{p}\right) \tag{8.3}$$

uniformly as $\frac{k}{x_2} \in [\delta, 1-\delta]$, $m \in \mathcal{M}_Y$ even if $Y = Y_x \to \infty$ slowly. Hence the assertion easily follows.

§9. Proof of Theorem 2

This can be carried over by a simple application of Theorem 7 and of (8.3).

Let $f(p^\alpha) = \arg g(p^\alpha) \in [-\pi, \pi]$, f be extended so that $f \in \mathcal{A}$. Then $g(n) = e^{if(n)}$.

From the convergence of $\sum \frac{1-g(p)}{p}$ we obtain that $\sum \frac{f(p)}{p}$, $\sum \frac{f^2(p)}{p}$ are convergent. For some $n \in \mathcal{N}_k$ define $g_Y(n) := g(A(n))$. First we observe that $\frac{1}{N_k(x)}\sum_{n\le x}|g(n)-g_Y(n)| \le \varepsilon_1(Y)$, uniformly in $\frac{k}{x_2} \in [\delta, 2-\delta]$, where $\varepsilon_1(Y) \to 0$ if $Y \to \infty$. Furthermore

$$\sum_{m\in\mathcal{M}_Y} g_Y(m)\frac{\#(D_{m,k})}{N_k(x)} = (1+o_x(1))\sum_{m\in\mathcal{M}_Y}\frac{g(m)\xi_{k,x_2}^{\Omega(m)}}{m}\prod_{p\mid K_Y}\left(1-\frac{\xi_{k,x_2}}{p}\right).$$

The right hand side clearly tends to $M_{\xi_{k,x_2}}(g)$ defined in Theorem 4.

Since

$$\limsup_x \sup_k \frac{1}{N_k(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}_k\\ A(n)\notin\mathcal{M}_Y}} |g(n)| \to 0 \text{ as } Y \to \infty,$$

our theorem immediately follows.

§10. Proof of Theorem 6 and 7

The proof is completely analogous to that of Theorem 2 and 5. So we omit it.

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