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Local solutions of an alternative Cauchy equation

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1. Introduction

In a previous paper [8] we studied the alternative Cauchy equation

$$g(xy) \neq g(x)g(y) \quad \text{implies} \quad f(xy) = f(x)f(y)\,, \tag{1}$$

where f, g are unknown functions from a group $(X, \cdot)$ into a group $(S, \cdot)$ (For the motivation of (1) and some related problems see [4]–[6], [10]–[14]). Among the results there is a complete description of the solutions of (1) when $(X, \cdot) = (\mathbb{R}^n, +)$ and one of the two functions, say g, satisfies a suitable topological condition (weaker than continuity).

It is well known (see [1]–[3], [7]) that each solution of the local Cauchy equation

$$f(x+y) = f(x)f(y), \qquad (x,y) \in T$$

where $T := \{(x,y) \in \mathbb{R}^2 : x, y, x+y \in I\}$, $I = (0,1)$ and $f : I \to S$, has a unique extension to an additive function on the whole $\mathbb{R}$. Hence it is natural to ask if this is also true for the local version of (1), i.e. if each pair of functions $f, g : I \to S$, solution of the local alternative equation

$$g(x+y) \neq g(x)g(y) \quad \text{implies} \quad f(x+y) = f(x)f(y) \qquad \text{for all} \quad (x,y) \in T\,, \tag{2}$$

can be extended to a pair of functions $\hat{f}, \hat{g} : \mathbb{R} \to S$ satisfying the alternative equation

$$\hat{g}(x+y) \neq \hat{g}(x)\hat{g}(y) \quad \text{implies} \quad \hat{f}(x+y) = \hat{f}(x)\hat{f}(y) \qquad \text{for all} \quad (x,y) \in \mathbb{R}^2\,. \tag{2'}$$

In the present paper we prove that under suitable hypotheses on one of the two functions f and g the answer is affirmative.

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2. Notations and preliminary results

Denote by $\mathbb{Z}$ and $\mathbb{N}_0$ the classes of the integers and the non-negative integers respectively, and by $p_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, the maps given by :

$$p_1(x, y) = x, \quad p_2(x, y) = y, \quad p_3(x, y) = x + y\,.$$

Given an open interval $E \subset \mathbb{R}$ and a function $\varphi : E \to S$, we define

$$\Omega_\varphi := \{(x, y) \in (E \times E) \cap p_3^{-1}(E) : \varphi(x + y) \neq \varphi(x)\varphi(y)\} \tag{3}$$

and

$$A_\varphi := \{(x, y) \in (E \times E) \cap p_3^{-1}(E) : \varphi(x + y) = \varphi(x)\varphi(y)\}.$$

A_φ° and Ω_φ° denote the interior of A_φ and Ω_φ respectively.

A function $\varphi : E \to S$ is said *locally affine* in $x \in E$ if there exists $a \in \mathrm{Hom}(\mathbb{R}, S)$ such that $\varphi(x+u) = \varphi(x)a(u)$ for all u in an open interval $U \ni 0$. (Note that the homomorphism a may depend on the point x). A function $\varphi : E \to S$ is said *locally affine* in an interval $V \subset E$ if it is locally affine in each point of V.

We shall use the following simple properties:

Lemma 1. i) *If $(x_0, y_0) \in A_\varphi^\circ$ then φ is locally affine in $x_0, y_0, x_0 + y_0$.*

ii) *If $V \subset \mathbb{R}$ is an open interval and φ is locally affine in each point of V, then there exist $a \in \mathrm{Hom}(\mathbb{R}, S)$ and $\alpha \in S$ such that*

$$\varphi(x) = \alpha a(x), \quad x \in V.$$

iii) *Let J, K, L be open intervals and*

$$\varphi(x) = \begin{cases} \alpha a(x), & x \in J \\ \beta b(x), & x \in K \\ \gamma c(x), & x \in L \end{cases}, \qquad a, b, c \in \mathrm{Hom}(\mathbb{R}, S).$$

If there exists $(x_0, y_0) \in A_\varphi^\circ$ with $x_0 \in J,\ y_0 \in K,\ x_0 + y_0 \in L$, then

$$\gamma = \alpha\beta \qquad \text{and} \qquad b(x) = c(x) = \beta^{-1}a(x)\beta.$$

Proof. i) Take $U = (-\varepsilon, \varepsilon)$ such that $(x_0, y_0) + (U \times U) \subset A_\varphi^\circ$. If for all $u \in U$ we define

$$a(u) = (\varphi(x_0))^{-1}\varphi(x_0 + y_0 + u)(\varphi(y_0))^{-1} \quad \text{and}$$
$$b(u) = (\varphi(y_0))^{-1}a(u)\varphi(y_0)\,,$$

then by the property $\varphi(x_0+y_0+u)=\varphi(x_0+u)\varphi(y_0)=\varphi(x_0)\varphi(y_0+u)$ we get

$$\varphi(x_0+u)=\varphi(x_0)a(u), \qquad \varphi(y_0+u)=a(u)\varphi(y_0)=\varphi(y_0)b(u)$$
$$\varphi(x_0+y_0+u)=\varphi(x_0)\varphi(y_0)b(u)=\varphi(x_0+y_0)b(u), \qquad u\in U\,.$$

Furthermore, since

$$a(u+v)=(\varphi(x_0))^{-1}\varphi(x_0+y_0+u+v)(\varphi(y_0))^{-1}= \\ =(\varphi(x_0))^{-1}\varphi(x_0+u)\varphi(y_0+v)(\varphi(y_0))^{-1}=a(u)a(v)$$

for all $u,v\in U\times U$ with $u+v\in U$, a is the restriction of a homomorphism from $\mathbb{R}$ into S; the same is also true for b.

ii) Fix $x_0\in V$; then there is $a_0\in \mathrm{Hom}(\mathbb{R},S)$ such that

$$\varphi(x_0+u)=\varphi(x_0)a_0(u)=\varphi(x_0)a_0(-x_0)a_0(x_0+u)=\alpha_0a_0(x_0+u)$$

for all u in a suitable neighbourhood U_{x_0} of the origin. Denote by F_0 the set of all $x\in V$ for which there exists a neighbourhood U_x of the origin such that

$$\varphi(x+u)=\alpha_0a_0(x+u), \quad u\in U_x.$$

Let $x_1\in x_0+U_{x_0}$ and let V_{x_1} be a neighbourhood of the origin such that $x_1+V_{x_1}\subset x_0+V_{x_0}$. We have

$$\varphi(x_1+v)=\alpha_0a_0(x_0+(x_1-x_0)+v)=\alpha_0a_0(x_1+v), \quad v\in V_{x_1}\,;$$

thus the set F_0 is open. Since φ is locally affine in each point of V, also the set $V\setminus F_0$ is open. The connectedness of V implies $F_0=V$.

iii) Let $(x_0,y_0)\in A_\varphi^\circ$ with $x_0\in J,\ y_0\in K,\ x_0+y_0\in L$; then

$$\gamma c(x_0)c(y_0)=\varphi(x_0+y_0)=\varphi(x_0)\varphi(y_0)=\alpha a(x_0)\beta b(y_0)$$

and so, for all $u\in\mathbb{R}$ such that $y_0+u\in K$ and $x_0+y_0+u\in L$,

$$\gamma c(x_0)c(y_0)c(u)=\gamma c(x_0)c(y_0+u)=\varphi(x_0+y_0+u)=\varphi(x_0)\varphi(y_0+u)= \\ =\alpha a(x_0)\beta b(y_0)b(u)=\gamma c(x_0)c(y_0)b(u).$$

It follows $b=c$ and $\gamma c(x_0)=\alpha a(x_0)\beta$.

Take now $u\in\mathbb{R}$ such that $x_0+u\in J$ and $x_0+y_0+u\in L$. Then

$$\gamma c(x_0)c(u)c(y_0)=\varphi(x_0+u+y_0)=\varphi(x_0+u)\varphi(y_0)= \\ =\alpha a(x_0)a(u)\beta c(y_0)=\gamma c(x_0)\beta^{-1}a(u)\beta c(y_0)\,.$$

So we deduce $c(x)=\beta^{-1}a(x)\beta$ and, since $(x_0,y_0)\in A_\varphi,\ \gamma=\alpha\beta.$ □

3. Local solutions

A pair (f,g) is called a *trivial solution* of (2) if either f or g is the restriction of a homomorphism of $\mathbb{R}$ into S. In the following we find the non-trivial solutions of (2) under the assumption that one of the two functions, say g, satisfies the following property:

$$p_i(\Omega_g) = p_i(\Omega_g^\circ), \qquad i = 1, 2\,. \tag{4}$$

Remark 1. a) The hypothesis (4) is the same condition under which in [8] we solved the functional equation (2′).

b) Note that condition (4) is obviously satisfied when S is a topological group and g is continuous. Furthermore there are noncontinuous functions satisfying (4): a "typical example" (see [5], [6], [10]) is the real function $g(x) = [x]$ (integral part of x). It can be easily proved that if $(S,\cdot) = (\mathbb{R},+)$ then condition (4) is fulfilled by every function $g : I \to \mathbb{R}$ satisfying the following properties:

i) the set D of the points of discontinuity of g is at most countable;
ii) for each $x_0 \in D$ there exists $\lim_{x\to x_0^-} g(x)$ and g is right-continuous;
iii) for each $x_0 \in D$ either $g(x_0+y) - g(x_0) - g(y) = 0$ for all $y \in I$ or $g(x_0+y) - g(x_0) - g(y)$ assumes at least two distinct non-zero values.

Define

$$W := I \setminus \big(p_1(\Omega_g) \cup p_2(\Omega_g)\big). \tag{5}$$

By (4) the set W is closed in I and is characterized by the property

$$W = \{t \in I : \forall x \in (0, 1-t), g(x+t) = g(x)g(t) = g(t)g(x)\}. \tag{6}$$

Note that, since $\Omega_g \subset A_f$, by (4) and Lemma 1-i) f is locally affine in each point of $I \setminus W$.

Theorem 1. *All the solutions of* (2) *with* $W = \emptyset$ *or* $W = I$ *are trivial.*

Proof. If $W = \emptyset$, the function f is locally affine in I and, by Lemma 1-ii), $f(x) = \alpha a(x)$. Since $\emptyset \neq \Omega_g \subset A_f$ we have $\alpha = e$ (the unit element of $(S,\cdot)$) and so f is the restriction of a homomorphism.

If $W = I$, then g is obviously the restriction of a homomorphism. □

Therefore from now on we assume that (f,g) is a solution of (2) with

$$\emptyset \neq W \neq I\,.$$

Lemma 2. *Let $\bar{t} \in W$ and $(x,y) \in T$. If $(x+n\bar{t}, y+m\bar{t}) \in T$ for some $m,n \in \mathbb{Z}$, then*

$$(x,y) \in \Omega_g \iff (x+n\bar{t}, y+m\bar{t}) \in \Omega_g.$$

Proof. Obviously it is enough to consider the case $m,n \geq 0$. By (6) we have

$$g(x+n\bar{t}+y+m\bar{t}) = g(x+y+(m+n)\bar{t}) = g(x+y)g(\bar{t})^{m+n}$$
$$g(x+n\bar{t}) = g(x)g(\bar{t})^n, \qquad g(y+m\bar{t}) = g(y)g(\bar{t})^m .$$

Therefore, since $g(\bar{t})$ commutes with $g(y)$ for all $y \in (0,1-\bar{t})$, we obtain

$$g(x+y+(m+n)\bar{t})\big[g(x+n\bar{t})g(y+m\bar{t})\big]^{-1} =$$
$$= g(x+y)g(\bar{t})^{n+m}g(\bar{t})^{-m}g(y)^{-1}g(\bar{t})^{-n}g(x)^{-1} =$$
$$= g(x+y)g(y)^{-1}g(x)^{-1} = g(x+y)\big[g(x)g(y)\big]^{-1}. \qquad \square$$

Lemma 3. *Let $\bar{t} \in W$ and let $\tilde{g} : \mathbb{R} \to S$ be defined as follows:*

$$\tilde{g}(x) = g(x-n\bar{t})g(\bar{t})^n \quad \text{if} \quad n\bar{t} < x \leq (n+1)\bar{t}, \quad n \in \mathbb{Z}. \tag{7}$$

Then g is the restriction of $\tilde{g}$ on I and the set

$$H_{\tilde{g}} := \{t \in \mathbb{R} : \forall x \in \mathbb{R}, \tilde{g}(t+x) = \tilde{g}(t)\tilde{g}(x) = \tilde{g}(x)\tilde{g}(t)\}$$

is a subgroup of $\mathbb{R}$ with $\bar{t} \in H_{\tilde{g}}$.

Proof. By (6) the function g is the restriction of $\tilde{g}$ on I. We now prove (as in [8]) that $H_{\tilde{g}}$ is a subgroup of $\mathbb{R}$.

Since $\tilde{g}(0) = e$ we have $0 \in H_{\tilde{g}}$. Let $t \in H_{\tilde{g}}$; then

$$e = \tilde{g}(0) = \tilde{g}(t-t) = \tilde{g}(t)\tilde{g}(-t)$$

and so $\tilde{g}(-t) = [\tilde{g}(t)]^{-1}$. Moreover, for every $x \in \mathbb{R}$ we have

$$\tilde{g}(x) = \tilde{g}(t-t+x) = \tilde{g}(t)\tilde{g}(x-t) = \tilde{g}(x-t)\tilde{g}(t)$$

and so $\tilde{g}(x-t) = \tilde{g}(-t)\tilde{g}(x) = \tilde{g}(x)\tilde{g}(-t)$, i.e. $-t \in H_{\tilde{g}}$.
Finally, let $t_1, t_2 \in H_{\tilde{g}}$; for every $x \in \mathbb{R}$ we get

$$\tilde{g}(t_1+t_2+x) = \begin{cases} \tilde{g}(t_1)\tilde{g}(t_2+x) = \tilde{g}(t_1)\tilde{g}(t_2)\tilde{g}(x) = \tilde{g}(t_1+t_2)\tilde{g}(x) \\ \tilde{g}(t_2+x)\tilde{g}(t_1) = \tilde{g}(x)\tilde{g}(t_2)\tilde{g}(t_1) = \tilde{g}(x)\tilde{g}(t_1+t_2) \end{cases}$$

i.e. $t_1+t_2 \in H_{\tilde{g}}$.

Let $x \in \mathbb{R}$ and let $n \in \mathbb{Z}$ such that $n\bar{t} < x \leq (n+1)\bar{t}$; from (7) we have

$$\tilde{g}(\bar{t}+x) = g(\bar{t}+x-(n+1)\bar{t})g(\bar{t})^{n+1}, \quad \tilde{g}(x) = g(x-n\bar{t})g(\bar{t})^n$$

and so $\bar{t} \in H_{\tilde{g}}$. $\square$

Lemma 4. *Assume $\emptyset \neq W \neq I$. The set W has a minimum $\tau\,(>0)$.*

Proof. Since W is closed in I, if $W \cap (0,1/2) = \emptyset$ then $\tau := \inf W \in W$. Otherwise let $\bar t \in W \cap (0,1/2)$ and assume it is not the minimum of W. Let $\tilde g$ be the function defined by (7). Since $\bar t < 1/2$, the open square $(0,\bar t\,)^2$ is contained in T and so

$$\Omega_g \cap (0,\bar t\,)^2 = \Omega_{\tilde g} \cap (0,\bar t\,)^2 . \tag{8}$$

By Lemma 2 the set Ω_g satisfies the equalities

$$(0,\bar t\,) \setminus W = \Big(\bigcup_{i=1,2} p_i(\Omega_g) \Big) \cap (0,\bar t\,) = \bigcup_{i=1,2} p_i\Big(\Omega_g \cap (0,\bar t\,)^2\Big) .$$

Moreover, since $H_{\tilde g} = \mathbb{R} \setminus (p_1(\Omega_{\tilde g}) \cup p_2(\Omega_{\tilde g}))$, by construction the set $\Omega_{\tilde g}$ satisfies the similar equalities

$$(0,\bar t\,) \setminus H_{\tilde g} = \left(\bigcup_{i=1,2} p_i\left(\Omega_{\tilde g}\right) \right) \cap (0,\bar t\,) = \bigcup_{i=1,2} \left(p_i(\Omega_{\tilde g}) \cap (0,\bar t\,)^2\right) .$$

By (8) we get

$$(0,\bar t\,) \setminus W = (0,\bar t\,) \setminus H_{\tilde g} . \tag{8}$$

Since we have assumed $W \neq I$, by using again Lemma 2 we have that $(0,\bar t\,) \setminus W$ is a non-empty open set. Thus, from (9) and Lemma 3, $H_{\tilde g}$ is a proper closed subgroup of $\mathbb{R}$, i.e. $H_{\tilde g} = \tau\mathbb{Z}$ for some $\tau \in (0,\bar t\,)$. Since by (9) $(0,\bar t\,) \cap W = (0,\bar t\,) \cap H_{\tilde g}$, we get $\tau = \min W$. □

We can now state the main result (for the proof see Section 4).

Theorem 2. *Assume (f,g) to be a non-trivial solution of (2) with g satisfying condition (4). Then the set W has a minimum $\tau\,(>0)$ and*

$$f(x) = f_0(x)a(x), \qquad g(x) = g_0(x)c(x) \tag{10}$$

where :

A) a and c are homomorphisms from $\mathbb{R}$ into S which commute with f_0 and g_0 respectively ;

B) the pair (f_0,g_0) has one of the following forms :

$$\begin{cases} f_0(x) = \alpha^{i+1} \\ g_0(x) = \gamma^i \end{cases} \quad \text{if } x \in \big[i\tau,(i+1)\tau\big) \cap I, \quad \alpha,\gamma \neq e, \quad i \in \mathbb{N}_0, \tag{11}$$

$$(12) \qquad \begin{cases} f_0(x) = \alpha^i \\ g_0(x) = \gamma^{i+1} \end{cases} \quad \textit{if } x \in \big(i\tau, (i+1)\tau\big] \cap I, \quad \alpha, \gamma \neq e, \quad i \in \mathbb{N}_0,$$

$$(13) \qquad \begin{cases} f_0(x) = e \quad \textit{if} \quad x \in I \backslash E, \qquad f_0(x) \neq e \quad \textit{if} \quad x \in E \\ \textit{where } \emptyset \neq E \subset \tau\mathbb{N}_0 \cap I \\ \qquad\qquad \textit{and } g_0 \textit{ satisfies the conditions} \\ g_0(x+\tau) = g_0(x)g_0(\tau) = g_0(\tau)g_0(x), \quad x \in (0, 1-\tau) \\ g_0(\tau) = g_0(x)g_0(\tau - x), \qquad x \in (0,\tau)\,, \end{cases}$$

$$(14) \qquad \begin{cases} f_0(x) = e \quad \textit{if } x \in I\backslash\{\xi\}, \quad f_0(\xi) \neq e \\ \textit{with } \xi \in W\backslash\tau\mathbb{N}_0, \qquad \max\{\tau, 1-\tau\} < \xi < 1 \\ \qquad\qquad \textit{and } g_0 \textit{ satisfies the conditions} \\ g_0(x+\tau) = g_0(x)g_0(\tau) = g_0(\tau)g_0(x), \quad x \in (0, 1-\tau) \\ g_0(x+\xi) = g_0(x)g_0(\xi) = g_0(\xi)g_0(x), \quad x \in (0, 1-\xi) \\ g_0(\xi) = g_0(x)g_0(\xi - x), \qquad x \in (0,\xi)\,. \end{cases}$$

Moreover all pairs (f,g) of the above mentioned forms are nontrivial solutions of (2).

Corollary 1. *Each solution (f,g) of* (2) *satisfying* (4) *is the restriction on I of a solution $(\hat{f},\hat{g})$ of the alternative equation* (2′).

Proof. In a previous paper ([8], Theorem 5) we have described the solutions of (2′) satisfying (4), where the set E in the definition of Ω_φ is the whole $\mathbb{R}$. We prove that each solution of (2) is extendible to a solution of (2′) of one of the forms described in Theorem 5 of [8]. The solutions of the form (11) and (12) are extendible in an obvious way to the solutions of the form iii) of Theorem 5 in [8]. The extension in the remaining cases (13) and (14) is given by (7) of Lemma 3 where the role of $\bar{t}$ is now assumed by τ or ξ respectively. In such a way we get solutions of (2′) which are of the form i) of Theorem 5 in [8]. □

Remark 2. The extension of the solutions of the form (14) is based on the properties of ξ, and the equation

$$g_0(x+\tau) = g_0(x)g_0(\tau) = g_0(\tau)g_0(x)$$

doesn't play any role. So, starting from the solutions of the form (13) or (14) we get solutions on $\mathbb{R}$ of the same form. Nevertheless in the triangle

T the properties of Ω_g yield in a natural way the value τ (and not ξ). So these two kinds of solutions are essentially different. Therefore it is natural to ask whether solutions of the form (14) exist. Clearly this depends on the parameters τ and ξ and on the group S. In [9] this problem has been completely solved.

Since property (4) is satisfied if the function g_0 and the homomorphism c are continuous, starting from the results of [9] in the case $S = \mathbb{R}$, we may list under which conditions on τ and ξ there exist continuous functions $g_0 : I \to \mathbb{R}$ satisfying the equations in (14) with $\tau = \min W$.

(i)
$$1 - \frac{\tau}{1-h\tau} < \frac{\xi - h\tau}{2(1-h\tau)}$$

(ii)
$$1 - \frac{\tau}{1-h\tau} = \frac{\xi - h\tau}{2(1-h\tau)},$$
$$\frac{\xi - h\tau}{\xi - (h+1)\tau} - 2\left[\frac{\xi - h\tau}{2\big(\xi - (h+1)\tau\big)}\right] \notin \{0,1\}$$

(iii)
$$\frac{\xi - h\tau}{2(1-h\tau)} < 1 - \frac{\tau}{1-h\tau} < \frac{\xi - h\tau}{2(1-h\tau)} + \frac{\xi - (h+1)\tau}{2(1-h\tau)}, \quad \frac{\tau}{\xi} \notin \mathbb{Q}$$

(iv)
$$\frac{\xi - h\tau}{2(1-h\tau)} < 1 - \frac{\tau}{1-h\tau} < \frac{\xi - h\tau}{2(1-h\tau)} + \frac{\xi - (h+1)\tau}{2(1-h\tau)}\left(1 - \frac{1}{q}\right),$$
$$\frac{\tau}{\xi} \in \mathbb{Q} \quad \text{and} \quad \frac{\xi - h\tau}{\xi - (h+1)\tau} - 2\left[\frac{\xi - h\tau}{2\big(\xi - (h+1)\tau\big)}\right] = \frac{p}{q}, \quad (p,q) = 1$$

where $[t]$ denotes the integral part of t and $h = \left[\frac{\xi}{\tau}\right] - 1$.

4. Proof of the main result

By Theorem 1 we have $\emptyset \neq W \neq I$ and, by Lemma 4, W has a minimum $\tau > 0$.

We need some other notations and lemmas.

$$\begin{aligned} J_k &:= \{x \in I : k\tau < x < (k+1)\tau\}, \quad k \in \mathbb{N}_0 \\ T^1_{i,j} &:= \{(x,y) \in T : x \in J_i,\ y \in J_j,\ x+y \in J_{i+j}\} \\ T^2_{i,j} &:= \{(x,y) \in T : x \in J_i,\ y \in J_j,\ x+y \in J_{i+j+1}\} \\ Q_{i,j} &:= T^1_{i,j} \cup T^2_{i,j}, \quad i,j \in \mathbb{N}_0, \\ \nu &:= \max\{k \in \mathbb{N}_0 : (k+1)\tau \le 1\}, \\ D_u &:= \{(x, u-x) : x \in (0,u)\}, \quad u \in (0,1)\,. \end{aligned}$$

Remark 3. Note that, since $\Omega_g \subset A_f$, by (4) and by Lemmas 1 and 2 we have:

i) if $\nu \ge 1$ then f is locally affine in the intervals $J_i,\ i \in \{0, \cdots, \nu-1\}$, and $(\nu\tau, 1-\tau)$;

ii) if $\nu = 0$ then f is locally affine in the interval J_0.

Condition (4) implies that $\Omega_g \cap (0,\tau)^2 \cap T \neq \emptyset$ if and only if $\Omega_g \cap Q_{0,0} \neq \emptyset$. In the following Lemmas 5,6 and 8 we consider separately the three possible cases:

I) $\Omega_g \cap Q_{0,0} \subset T^2_{0,0}$

II) $\Omega_g \cap Q_{0,0} \subset T^1_{0,0}$

III) $\Omega_g \cap T^i_{0,0} \neq \emptyset, \quad i = 1, 2.$

Lemma 5. *If $\Omega_g \cap Q_{0,0} \subset T^2_{0,0}$ then each non-trivial solution (f,g) of (2) is given by (10) with (f_0, g_0) of the form (11).*

Proof. By the hypothesis, $T^1_{0,0} \subset A_g$ and so $g(x) = c(x)$, $x \in (0,\tau)$, where $c \in \mathrm{Hom}(\mathbb{R}, S)$. By Remark 3 and Lemma 1, $f(x) = \alpha a(x)$, $x \in (0,\tau)$, where $a \in \mathrm{Hom}(\mathbb{R}, S)$ and $\alpha \in S$. If $x \in (\tau, 2\tau) \cap I$, since $\tau \in W$ we have

$$g(x) = g(\tau)g(x-\tau) = g(\tau)c(x-\tau) = g(\tau)c(\tau)^{-1}c(x) = \gamma c(x). \tag{15}$$

It follows, for $x \in (0,\tau)$,

$$\gamma c(\tau)c(x) = \gamma c(\tau + x) = g(\tau + x) = g(\tau)g(x) = g(\tau)c(x) \tag{16}$$

and so $g(\tau) = \gamma c(\tau)$. From (15) and (16) we have $g(x) = \gamma c(x)$, $x \in [\tau, 2\tau) \cap I$. By Lemma 2, $T^1_{0,1} \subset A_g$ and so, $c(x)\gamma c(y) = g(x)g(y) = g(x+y) = \gamma c(x)c(y)$, for all $(x,y) \in T^1_{0,1}$. Hence $c(x)\gamma = \gamma c(x)$, $x \in (0,\tau)$, that is the homomorphism c commutes with γ. Moreover $\Omega_g \cap T^2_{0,0} \neq \emptyset$ implies $\gamma \neq e$ and so $T^2_{0,0} \subset \Omega_g$. It follows $T^2_{0,0} \subset A_f$, i.e. for all $(x,y) \in T^2_{0,0}$

$$\alpha a(x)\alpha a(y) = f(x)f(y) = f(x+y) = f(y)f(x) = \alpha a(y)\alpha a(x). \tag{17}$$

From (17) we get $a(x)\alpha a(y) = a(y)\alpha a(x)$, i.e. $a(x-y)\alpha = \alpha a(x-y)$; so a commutes with α. Furthermore (17) gives $f(x+y) = \alpha^2 a(x+y)$, i.e. $f(x) = \alpha^2 a(x)$, $x \in (\tau, 2\tau) \cap I$. Since $\gamma \neq e$, the points $(x, \tau - x)$, $x \in (0, \tau)$, are not in A_g and so $f(\tau) = \alpha^2 a(\tau)$; it follows $f(x) = \alpha^2 a(x)$, $x \in [\tau, 2\tau) \cap I$. We can now repeat this procedure to get f and g on the whole interval I. Note that $\alpha \neq e$ since (f, g) is not trivial. □

Lemma 6. *If $\Omega_g \cap Q_{0,0} \subset T^1_{0,0}$ then each non-trivial solution (f, g) of* (2) *is given by* (10) *with (f_0, g_0) of the form* (12).

Proof. By the hypothesis, $T^2_{0,0} \subset A_g$ and so by Remark 3 and Lemma 1 we have

$$f(x) = \beta a(x), \quad g(x) = \gamma c(x), \quad x \in (0, \tau); \quad a, c \in \mathrm{Hom}(\mathbb{R}, S)\,. \tag{18}$$

Note that $\gamma \neq e$, otherwise $\Omega_g \cap T^1_{0,0} = \emptyset$ and so $\Omega_g = \emptyset$. It follows $T^1_{0,0} \subset \Omega_g$, i.e. $T^1_{0,0} \subset A_f$ and this forces $\beta = e$. If $(x, y) \in T^2_{0,0} (\subset A_g)$, from (18) we have $\gamma c(x)\gamma c(y) = g(x)g(y) = g(x+y) = g(y)g(x) = \gamma c(y)\gamma c(x)$ and, as in Lemma 5, we conclude that c commutes with γ and $g(x) = \gamma^2 c(x)$, $x \in (\tau, 2\tau) \cap I$. By Lemma 2 $T^1_{0,1} \subset \Omega_g$, i.e. $T^1_{0,1} \subset A_f$ and, by Lemma 1, $f(x) = \alpha a(x)$, $x \in (\tau, 2\tau) \cap I$. As in Lemma 5 we prove that a commutes with α. Since $\tau \in W$, if $x \in (0, \tau)$ we have

$$\gamma c(\tau)\gamma c(x) = \gamma^2 c(\tau + x) = g(\tau + x) = g(\tau)g(x) = g(\tau)\gamma c(x),$$

and so $g(x) = \gamma c(x)$, $x \in (0, \tau]$. Since $\gamma \neq e$, the points $(x, \tau - x)$, $x \in (0, \tau)$ are not in A_g and so we must have $f(x) = a(x), x \in (0, \tau]$. By the same procedure we obtain f and g on the whole interval I. □

Remark 4. As a consequence of Lemmas 5 and 6, we obtain that f is locally affine on each interval $J_i \cap I$, $i \geq 0$.

Lemma 7. *Assume $\mu \in \mathbb{R}$ with $|\mu| < \tau$ and*

$$\sigma := (\nu + 1)\tau + \mu \in W\,. \tag{19}$$

If $D_\sigma \subset A_g$ then $\sigma > 1 - \tau$ and there does not exist any $\mu \in (\sigma, 1)$ such that $D_\mu \subset A_g$.

Proof. Assume there is $\bar{u} := (\nu+1)\tau + \bar{\mu} \in (\sigma, 1)$ such that $D_{\bar{u}} \subset A_g$. Since $D_\sigma \subset A_g$ and $D_{\bar{u}} \subset A_g$, for all $x \in (0, \sigma)$ we have simultaneously

$$g(x) = \begin{cases} g(\sigma)[g(\sigma - x)]^{-1} = [g(\sigma - x)]^{-1} g(\sigma) \\ g(\bar{u})[g(\bar{u} - x)]^{-1} = [g(\bar{u} - x)]^{-1} g(\bar{u})\,. \end{cases} \tag{20}$$

Since $\tau \in W$, we get

$$
\left\{ \begin{array}{l} g(\sigma) = g((\nu+1)\tau+\mu) = [g(\tau)]^{\nu} g(\tau+\mu) \\ g(\bar{u}) = g((\nu+1)\tau+\bar{\mu}) = [g(\tau)]^{\nu} g(\tau+\bar{\mu}) \end{array} \right. \tag{21}
$$

and so, by (20), for all $x \in (0,\sigma)$ we obtain

$$
\left\{ \begin{array}{l} g(\tau+\mu)[g(\sigma-x)]^{-1} = g(\tau+\bar{\mu})[g(\bar{u}-x)]^{-1} \\ [g(\sigma-x)]^{-1} g(\tau+\mu) = [g(\bar{u}-x)]^{-1} g(\tau+\bar{\mu})\,. \end{array} \right. \tag{22}
$$

By (19) the points $(\sigma, \bar{\mu}-\mu)$ and $(\bar{\mu}-\mu, \sigma)$ belong to A_g and so, by using (21), we obtain

$$
g(\tau+\bar{\mu}) = g(\tau+\mu) g(\bar{\mu}-\mu) = g(\bar{\mu}-\mu) g(\tau+\mu)\,. \tag{23}
$$

Substituting (23) in (22) we have, for all $x \in (0,\sigma)$,

$$
g(\bar{u}-x) = g(\sigma-x) g(\bar{\mu}-\mu) = g(\bar{\mu}-\mu) g(\sigma-x)\,. \tag{24}
$$

We prove that $\sigma > 1-\tau$. If not, then by the definition of ν, it is $\mu < 0$; since $\sigma \in W$ and $|\mu| < \tau$, by Lemma 2 we have $\sigma - \nu\tau = \tau + \mu \in W$ and $0 < \tau + \mu < \tau$: a contradiction. Then $\sigma, \bar{u} \in (1-\tau, 1)$ and so $\bar{u} - \sigma = \bar{\mu} - \mu < \tau$. Since $\sigma \in W$ we have $\sigma \geq \tau$. Lemma 2 now implies that (24) holds for all $x \in (0, 1-(\bar{\mu}-\mu))$, i.e. $\bar{\mu}-\mu \in W$: a contradiction. □

Lemma 8. *Assume $\Omega_g \cap T^1_{0,0} \neq \emptyset$ and $\tau \leq s_0 < s_1 < \cdots < s_N \leq 1$. If*

$$
f(x) = \left\{ \begin{array}{ll} a(x), & x \in (0,s_0) \setminus W \\ \alpha a(x), & x \in \bigcup_{i=0}^{N-1} (s_i, s_{i+1}) \end{array} \right.
$$

with $\alpha \neq e$ and $a \in \mathrm{Hom}(\mathbb{R}, S)$, then:

i) $\{(x,y) : 0 < x < s_0,\ 0 < y < s_0,\ s_0 < x+y < s_N\} \setminus \bigcup_{i=0}^{N} D_{s_i} \subset A_g$;

ii) $\{(x,y) : 0 < x < s_0,\ 0 < y < s_0,\ 0 < x+y < s_0\} \subset \Omega_g$;

iii) $s_0 = \tau$.

Proof. Property i) is obvious. By i) and Lemma 1 we have $g(x) = \gamma c(x),\ x \in (0,s_0)$, where $c \in \mathrm{Hom}(\mathbb{R}, S)$. Since $s_0 \geq \tau$ and $T^1_{0,0} \cap \Omega_g \neq \emptyset$, we have $\gamma \neq e$. It follows

$$
\{(x,y) : 0 < x < s_0, \quad 0 < y < s_0, \quad 0 < x+y < s_0\} \subset \Omega_g
$$

and moreover, by the definitions of W and τ, $s_0 = \tau$. □

Lemma 9. *If* $\Omega_g \cap T^i_{0,0} \neq \emptyset$, $i = 1, 2$, *then each non-trivial solution* (f, g) *of* (2) *is given by* (10) *with* (f_0, g_0) *either of the form* (13) *or of the form* (14).

Proof. By Lemma 2 with $\bar{t} = \tau$, $\Omega^\circ_g \neq \emptyset$ implies $\Omega^\circ_g \cap Q_{0,0} \neq \emptyset$ and so

$$\Omega^\circ_g \cap T^1_{0,0} \neq \emptyset \quad \text{or} \quad \Omega^\circ_g \cap T^2_{0,0} \neq \emptyset\,. \tag{25}$$

Consider fist the case $\tau \leq 1/2$ (i.e. $\nu \geq 1$).

By Remark 3 and ii) of Lemma 1 we may write

$$f(x) = \alpha_i a_i(x), \quad x \in J_i \quad \text{where} \quad a_i \in \mathrm{Hom}(\mathbb{R}, S), \quad i = 0, \ldots, \nu - 1\,.$$

By (25) and iii) of Lemma 1 all homomorphisms a_i equal a same homomorphism a. Moreover from $\Omega_g \cap T^1_{i,0} \neq \emptyset$ we get $\alpha_i = e$ for $i = 0, \cdots, \nu - 1$. So

$$f(x) = a(x) \quad , \quad x \in \bigcup_{i=0}^{\nu-1} J_i \qquad \text{where} \quad a \in \mathrm{Hom}(\mathbb{R}, S).$$

It remains to consider the interval $(\nu\tau, 1)$.

If $(\nu+1)\tau < 1$ then $(\nu\tau, 1-\tau) \neq \emptyset$ and by Remark 3 f is locally affine on $(\nu\tau, 1-\tau)$. If $L := \{(1-\tau, y) : 0 < y < \tau\} \subset A_g$, then, by Lemma 2, we get $1 - (\nu+1)\tau \in W$: a contradiction, since $1 - (\nu+1)\tau < \tau$. Thus $L \cap \Omega_g \neq \emptyset$ and, by (4), $L \cap \Omega^\circ_g \neq \emptyset$. This implies f locally affine in $(\nu\tau, s)$ with $s > 1 - \tau$. Define

$$\rho := \sup\{s > 1 - \tau : f \text{ is locally affine in } (\nu\tau, s)\}.$$

Then $\rho \in W$ and, by ii) of Lemma 1, $f(x) = \alpha\,\bar{a}(x)$, $x \in (\nu\tau, \rho)$; moreover, since $L \cap \Omega^\circ_g \neq \emptyset$, by Lemma 1 we deduce $\bar{a}(x) = a(x)$. If $\alpha \neq e$, by Lemma 8 with $N = 1$, $s_0 = \nu\tau$ and $s_1 = \rho$ we have that the triangle

$$\{(x, y) : 0 < x < \nu\tau, \quad 0 < y < \nu\tau, \quad 0 < x + y < \nu\tau\}$$

is a subset of Ω_g; but this is impossible since this triangle contains the segment $\{(\rho - \nu\tau, y) : 0 < y < 1 - \rho\}$ which, by Lemma 2, is in A_g. Thus we have

$$f(x) = a(x), \qquad x \in \left(\bigcup_{i=0}^{\nu-1} J_i\right) \bigcup (\nu\tau, \rho).$$

If $(\nu+1)\tau = 1$ we define $\rho := \nu\tau$.

In the case $\tau > 1/2$ we have immediately $f(x) = a(x)$, $x \in (0, \tau)$ and we define $\rho := \tau$.

Summarizing, in all cases we can guarantee that

$$f(x) = a(x), \quad x \in (0,\rho)\setminus E_0 \tag{26}$$

where $E_0 \subset \{n\tau : n = 1, \dots, \nu\} \subset W$ is a finite set and $\rho \geq 1/2$.

Let now

$$T_\rho := \{(x,y) : 0<x<1-\rho, 0<y<1-\rho, 0<x+y<1-\rho\},\ T'_\rho := T_\rho + (\rho, 0)$$
$$I_\rho := \{x : 0 < x < 1-\rho\}, \quad I'_\rho := I_\rho + \rho\,.$$

By (4) $p_i(T'_\rho \cap \Omega_g) = p_i(T'_\rho \cap \Omega_g^\circ)$ and since $\rho \in W$, by Lemma 2,

$$(T_\rho \cap \Omega_g) + (\rho, 0) = T'_\rho \cap \Omega_g. \tag{27}$$

Thus $p_i(T_\rho \cap \Omega_g) = p_i(T_\rho \cap \Omega_g^\circ)$ and all the results obtained up to now for T hold for T_ρ as well.

Define $W_\rho := I_\rho \setminus \big(p_1(T_\rho \cap \Omega_g) \cup p_2(T_\rho \cap \Omega_g)\big)$, i.e. W_ρ is the analog for T_ρ of the set W. For W_ρ we have different possibilities.

i) $W_\rho = \emptyset$.

In this case, by (27), $p_1(T'_\rho \cap \Omega_g) \cup p_2(T'_\rho \cap \Omega_g) = I'_\rho$ and so f is locally affine on I'_ρ, i.e. $f(x) = \alpha\overline{a}(x)$, $x \in I'_\rho$. Since $\Omega_g^\circ \cap T'_\rho \neq \emptyset$, by Lemma 1 $\overline{a} = a$. Moreover $\alpha = e$; if not, by ii) and iii) of Lemma 8 with $N = 1$, $s_0 = \rho$, $s_1 = 1$ we have $\rho = \tau$ and

$$R := \{(x,y) : 0 < x < \rho,\ 0 < y < \rho,\ \rho < x + y < 1\} \subset A_g\,.$$

So $R = T^2_{0,0}$ and this is a contradiction since, by hypothesis, $T^2_{0,0} \cap \Omega_g \neq \emptyset$.

ii) $W_\rho = I_\rho$.

This implies $T_\rho \subset A_g$ and $T'_\rho \subset A_g$; thus $I'_\rho \cup \{\rho\} \subset W$. Assume $f(\xi) \neq a(\xi)$ for some $\xi \in I'_\rho \cup \{\rho\}$. In this case we immediately conclude that the whole diagonal $\{(x,y) \in T : x + y = \xi\}$ is in A_g. By Lemma 7 this cannot happen for any other $\xi_1 \in I'_\rho \cup \{\rho\}, \xi_1 \neq \xi$. Thus either

$$f(x) = a(x), \quad x \in I'_\rho \cup \{\rho\}$$

or there exists $\xi \in I'_\rho \cup \{\rho\}$ such that

$$f(x) = a(x), \quad x \in I'_\rho \setminus \{\xi\} \qquad \text{and} \qquad f(\xi) \neq a(\xi)\,.$$

iii) $\emptyset \neq W_\rho \neq I_\rho$.

By Lemma 4 W_ρ has a minimum $\tau_\rho (> 0)$. Since all results obtained for T are also true for T_ρ, the proof is divided in the above mentioned cases I)–III) (with the obvious changes of meaning of simbols).

In cases I) and II), by Remark 4, f is locally affine in each interval

$$K_j := (\rho + j\tau_\rho, \rho + (j+1)\tau_\rho) \cap I'_\rho, \quad j \in \mathbb{N}_0\,,$$

and, as usual, we have $f(x) = \alpha_j a(x), x \in K_j$. By (27) and Lemmas 5 and 6 we explicitely know the set $\Omega_g \cap T'_\rho$. If follows immediately that all α_j are equal, i.e. $f(x) = \alpha a(x),\ x \in \bigcup_{j\in\mathbb{N}_0} K_j$. We prove that $\alpha = e$. On the contrary, by Lemma 8 with $(s_i, s_{i+1}) = K_i$, we have

$$T_\rho \subset \{(x,y) : 0 < x < \rho,\ 0 < y < \rho,\ 0 < x+y < \rho\} \subset \Omega_g\,,$$

contrary to the description, coming from Lemmas 5 and 6, of the set $\Omega_g \cap T_\rho$.

In case III) by using (27) we may argue as in first part of the present proof up to relation (26).

Summarizing, in all cases I)–III) we obtain

$$f(x) = a(x), \quad x \in (0, \rho + \rho_1)\backslash(E_0 \cup E_1)$$

where $\rho_1 \geq (1-\rho)/2$ and $E_1 \subset W$ is a finite set.
By iteration of this procedure we get the final result

$$f(x) = a(x), \quad x \in (0,1)\backslash E$$

where $E := \bigcup_{n\geq 0} E_n \subset W$ and each E_n is finite.

Assume $E \neq \emptyset$. If there exists $k \in \{1, \cdots, (\nu+1)\}$ such that $k\tau \in E$, i.e. $f(k\tau) \neq a(k\tau)$, then $\{(x,y) \in T : x+y = k\tau\} \subset A_g$; so, by Lemma 2, all diagonals $\{(x,y) \in T : x+y = i\tau\}$, $i = 1, \cdots, \nu+1$, are in A_g and by Lemma 7, E cannot contain any other point $x \notin \tau\mathbb{N}_0$. Thus $E \subset \tau\mathbb{N}_0$.

If $E \cap \tau\mathbb{N}_0 = \emptyset$ then, again by Lemma 7, $E = \{\xi\}$, with $\rho \leq \xi < 1$ and obviously $\xi \in W$. Note that, by the definition of ρ, we always have $\max\{\tau, 1-\tau\} < \xi < 1$. □

Theorem 2 follows immediately from Lemmas 5, 6 and 9.

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