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Maps from $M_n(\mathbb{F})$ to $\mathbb{F}$ that are multiplicative with respect to the Jordan triple product

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Abstract. Let $\mathbb{F}$ be the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. Denote by $M_n(\mathbb{F})$ the set of all $n \times n$ matrices over the field $\mathbb{F}$. We show that if Φ is a map from $M_n(\mathbb{F})$ to $\mathbb{F}$ that is multiplicative with respect to Jordan triple product, that is, a map: $\Phi : M_n(\mathbb{F}) \to \mathbb{F}$ satisfying

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A), \quad A, B \in M_n(\mathbb{F})$$

then there exists a multiplicative function $\varphi : \mathbb{F} \to \mathbb{F}$ such that either $\Phi(A) = \varphi(\det A)$ for all $A \in M_n(\mathbb{F})$ or $\Phi(A) = -\varphi(\det A)$ for all $A \in M_n(\mathbb{F})$.

There is a lot of papers on so called preserver problems. Reader can find many facts and references in an excellent survey paper written by Šemrl [8]. In the paper many open problems are mentioned. One of the problems is to characterize maps that are multiplicative with respect to the Jordan triple product, that is, maps Φ from $M_n(\mathbb{F})$ to $M_m(\mathbb{F})$ satisfying

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A) \tag{1}$$

for all $A,B \in M_n(\mathbb{F})$. Molnár [6] found the general form of such bijective mappings on $M_n(\mathbb{C})$ for $n \geq 3$, Lu [5] presented a purely algebraic proof that worked also in the dimension 2, Lešnjak and Sze [4] characterized all such injective maps, Kuzma [3] characterized nondegenerate maps that preserve the

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Jordan triple product on $M_n(\mathbb{F})$, $n \geq 3$. Continuous maps from the set of all $n \times n$ complex positive definite, positive semi-definite and Hermitian matrices to real numbers were studied by MOLNÁR [7]. A characterization of the determinant on the above sets of matrices has been obtained. In our paper the maps $M_n(\mathbb{F})$ to $\mathbb{F}$ that are multiplicative with respect to the Jordan triple product are studied. We shall see that also in this case the multiplicativity with respect to the Jordan triple product characterizes the determinant on the set of all (real or complex) matrices rather firmly.

The identity matrix is denoted by I, and a matrix with square equal to I is called an involution.

For a map satisfying (1) by $\Phi(I) = \Phi(I^3) = (\Phi(I))^3$, it follows that $\Phi(I)$ is equal 1, -1 or 0, and since $\Phi(0) = \Phi(0^3) = (\Phi(0))^3$, $\Phi(0)$ is 1, -1 or 0.

At first we consider the case $\Phi(I) = 1$, $\Phi(0) = 0$.

Lemma 1. *If we suppose that $\Phi(I) = 1$, then $\Phi(A^2) = (\Phi(A))^2$ for all $A \in M_n(\mathbb{F})$.*

PROOF.
$$\Phi(A^2) = \Phi(AIA) = \Phi(A)\Phi(I)\Phi(A) = (\Phi(A))^2 \tag{2}$$

□

Lemma 2. *If $\Phi(I) = 1$, and a matrix A is invertible then $\Phi(A) \neq 0$, and $\Phi(A^{-1}) = (\Phi(A))^{-1}$.*

PROOF.
$$1 = \Phi(I) = \Phi(AA^{-2}A) = \Phi(A)\Phi(A^{-2})\Phi(A)$$
Therefore $\Phi(A) \neq 0$.

From $\Phi(A) = \Phi(AA^{-1}A) = \Phi(A)\Phi(A^{-1})\Phi(A)$ we get $\Phi(A^{-1}) = (\Phi(A))^{-1}$. □

Lemma 3. *If $\Phi(I) = 1$, and matrices A and B are similar then $\Phi(A) = \Phi(B)$.*

PROOF. If matrices A are B similar then there exists an invertible matrix P such that $A = P^{-1}BP$. We can suppose that $\det P = \pm 1$. In [1] it is shown that an $n \times n$ matrix over a field is a product of four involutions if and only if $\det A = \pm 1$. Thus $P = V_1V_2V_3V_4$ with $V_1^2 = V_2^2 = V_3^2 = V_4^2 = I$. Using this result we obtain

$$\begin{aligned}\Phi(A) &= \Phi(V_4V_3V_2V_1BV_1V_2V_3V_4) = \Phi(V_4)\Phi(V_3V_2V_1BV_1V_2V_3)\Phi(V_4) = \cdots \\ &= \Phi(V_4)\Phi(V_3)\Phi(V_2)\Phi(V_1)\Phi(B)\Phi(V_1)\Phi(V_2)\Phi(V_3)\Phi(V_4).\end{aligned}$$

Since the field is commutative, and $(\Phi(V_k))^2 = \Phi(V_k^2) = \Phi(I) = 1,\ k = 1, 2, 3, 4,$ we finally get $\Phi(A) = \Phi(B)$. □

Denote by $D(i, a)$, $i = 1, 2, \cdots n$ the diagonal $n \times n$ matrix with a at the (i, i) entry, elsewhere 1.

Lemma 4.

$$\Phi(D(1, a)) = \Phi(D(k, a)), \quad k = 1, 2, \ldots n. \tag{3}$$

Proof. For any $k = 1, 2, \ldots, n$ the matrices $D(1, a)$ and $D(k, a)$ are similar. Thus $\Phi(D(1, a)) = \Phi(D(k, a)), k = 1, 2, \ldots n$ by Lemma 3. □

Now, let us define

$$\varphi(a) = \Phi(D(1, a)) = \Phi\left(\begin{bmatrix} a & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \ddots & \cdot \\ 0 & 0 & \cdots & 1 \end{bmatrix}\right). \tag{4}$$

Lemma 5. *The function $\varphi : \mathbb{F} \to \mathbb{F}$ is multiplicative, $\varphi(1) = 1$, $\varphi(0) = 0$.*

Proof. First, let us show that $\varphi(0) = 0$. The zero matrix can be written as

$$0 = D(1, 0)D(2, 0)\ldots D(n, 0)ID(n, 0)D(n - 1, 0)\ldots D(1, 0).$$

By Lemma 4 we have $\Phi(D(1, 0)) = \Phi(D(k, 0))$ for $k = 1, 2, \ldots n$. Therefore $0 = (\varphi(0))^{2n}$, so $\varphi(0) = 0$. By Lemma 1 we get $\varphi(a^2) = (\varphi(a))^2$ for all $a \in \mathbb{F}$.

For complex case we show that φ is multiplicative by

$$\begin{aligned} \varphi(ab) = \varphi(\sqrt{a}b\sqrt{a}) &= \Phi\left(\begin{bmatrix} \sqrt{a}b\sqrt{a} & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 1 \end{bmatrix}\right) \\ &= \Phi\left(\begin{bmatrix} \sqrt{a} & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 1 \end{bmatrix}\begin{bmatrix} b & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 1 \end{bmatrix}\begin{bmatrix} \sqrt{a} & \cdot & 0 \\ \cdot & \cdot & \cdot \\ 0 & \cdot & 1 \end{bmatrix}\right) \\ &= \varphi(\sqrt{a}\,)\varphi(b)\varphi(\sqrt{a}\,) = \varphi(a)\varphi(b). \end{aligned}$$

The upper proof is valid also in the real case if at least one of the numbers a or b is positive.

It remains to prove the multiplicativity for $\mathbb{F} = \mathbb{R}$, in the case $a < 0$ and $b < 0$.

In this case $ab > 0$, thus $\varphi((ab)a) = \varphi(ab)\varphi(a)$. From (1) we get $\varphi(aba) = \varphi(a)\varphi(b)\varphi(a)$. Thus

$$\varphi(ab)\varphi(a) = \varphi(a)\varphi(b)\varphi(a).$$

Since $\varphi(a) \neq 0$ by invertibility of $D(1,a)$, we get

$$\varphi(ab) = \varphi(a)\varphi(b).$$

From $\Phi(I) = 1$ we get $\varphi(1) = 1$. □

Lemma 6. *If $\mathbb{F} = \mathbb{C}$ and $\Phi(I) = 1$, then*

$$\Phi(D) = \varphi(\det D) \tag{5}$$

for any diagonal matrix D. In the real case the same is valid if no more than one entry of the diagonal matrix is negative.

Proof. If $\mathbb{F} = \mathbb{C}$ a diagonal matrix

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ . & . & \ddots & . \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

can be written as a product

$$D = D(1, \sqrt{d_1}\,) \dots D(n-1, \sqrt{d_{n-1}}\,) D(n, d_n) D(n-1, \sqrt{d_{n-1}}\,) \dots D(1, \sqrt{d_1}\,).$$

By Lemma 4 we obtain $\Phi(D(k, \sqrt{d_k})) = \Phi(D(1, \sqrt{d_k})) = \varphi(\sqrt{d_k})$. Thus by (1) we get

$$\begin{aligned}\Phi(D) &= \varphi(\sqrt{d_1}\,)\varphi(\sqrt{d_2}\,) \dots \varphi(\sqrt{d_{n-1}}\,)\varphi(d_n)\varphi(\sqrt{d_{n-1}}\,) \dots \varphi(\sqrt{d_2}\,)\varphi(\sqrt{d_1}\,) \\ &= \varphi(d_1 d_2 \dots d_n) = \varphi(\det(D)).\end{aligned}$$

We have to compute square roots on $n-1$ entries, so in the real case one entry can be negative. □

Theorem 7. *Let $\mathbb{F}$ be $\mathbb{C}$ or $\mathbb{R}$, and $\Phi : M_n(\mathbb{F}) \to \mathbb{F}$ a map satisfying*

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A), \quad A, B \in M_n(\mathbb{F}). \tag{6}$$

Then there exists a multiplicative function $\varphi : \mathbb{F} \to \mathbb{F}$ such that either

$$\Phi(A) = \varphi(\det A) \quad \textit{for all } A \in M_n(\mathbb{F})$$

or

$$\Phi(A) = -\varphi(\det A) \quad \textit{for all } A \in M_n(\mathbb{F}).$$

Proof. Since $\Phi(I) = \Phi(I^3) = (\Phi(I))^3$ it follows that $\Phi(I) = 1, -1$ or 0. Also by $\Phi(0) = \Phi(0^3) = (\Phi(0))^3$ we have $\Phi(0) = 1, -1$ or 0.

The cases $\Phi(I) = 0$, $\Phi(0) \neq 0$ are contradictory because we get $\pm 1 = \Phi(0) = \Phi(I0I) = \Phi(I)\Phi(0)\Phi(I) = 0$.

In the case $\Phi(I) = 1$, $\Phi(0) = 1$ we get $\Phi(A) = 1$ for all $A \in M_n(\mathbb{F})$.

In the case $\Phi(I) = 0$, $\Phi(0) = 0$ we get $\Phi(A) = 0$ for all $A \in M_n(\mathbb{F})$.

Thus in upper two cases the theorem is true.

If a map Φ satisfies (6) the same is true for the map $-\Phi$.

Therefore only the case $\Phi(I) = 1$ and $\Phi(0) = 0$ remains to be proved.

From now on we will assume that Φ maps the zero matrix to 0 and the identity matrix to 1.

Proof for $\mathbb{C}$. Let $A \in M_n$. Denote by J the Jordan canonical form of the matrix A. By Lemma 3 $\Phi(A) = \Phi(J)$. If A is not invertible move a Jordan block corresponding to eigenvalue 0 (one of them) in upper left corner of the matrix J. Then there exists $m \in \mathbb{N}$ such that the first column and the first row of the matrix J^m are 0. From

$$\begin{aligned}\Phi(J^m) &= \Phi((I - E_{11})J^m(I - E_{11})) = \Phi(I - E_{11})\Phi(J^m)\Phi(I - E_{11}) \\ &= \varphi(0)\Phi(J^m)\varphi(0) = 0\end{aligned}$$

we conclude $\Phi(A) = 0$. From now on suppose that A is invertible.

For $k > 0$, $\lambda \neq 0$ and $a \neq 0$ denote by

$$Y_k(\lambda, a) = \begin{bmatrix} \lambda & a & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & a \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \tag{7}$$

a matrix of size $k \times k$. Note that

$$Y_k^{-1}(1, a) = \begin{bmatrix} 1 & -a & a^2 & \cdots & * \\ 0 & 1 & -a & \cdots & * \\ 0 & 0 & 1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad Y_k^2(1, a) = \begin{bmatrix} 1 & 2a & a^2 & \cdots & 0 \\ 0 & 1 & 2a & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{8}$$

The only eigenvalue of the matrices $Y_k(1, a)$, $Y_k^{-1}(1, a)$ and $Y_k^2(1, a)$ is 1, and since $a \neq 0$, each of them has only one eigenvector. Therefore they have the same Jordan form, i.e., they are similar.

First let us compute the value of Φ for matrices of the form

$$B_{lk}(\lambda, a) = \begin{bmatrix} I_l & 0 & 0 \\ 0 & Y_k(\lambda, a) & 0 \\ 0 & 0 & I_{n-k-l} \end{bmatrix} \tag{9}$$

where $0 \le l \le n-k$. Since $B_{lk}(1,a)$ is similar to $B_{lk}^{-1}(1,a)$, $\Phi(B_{lk}(1,a)) = \Phi(B_{lk}^{-1}(1,a))$ by Lemma 3. We get $\Phi(B_{lk}(1,a)) = \pm 1$. The matrix $B_{lk}(1,a)$ is also similar to $B_{lk}^2(1,a)$, thus by Lemma 1

$$\Phi(B_{lk}(1,a)) = \Phi(B_{lk}^2(1,a)) = \Phi(B_{lk}(1,a))^2.$$

We have shown that

$$\Phi(B_{lk}(1,a)) = 1. \tag{10}$$

Using the identity

$$B_{lk}(\lambda, 1) = \begin{bmatrix} I_l & 0 & 0 \\ 0 & \sqrt{\lambda} I_k & 0 \\ 0 & 0 & I_{n-k-l} \end{bmatrix} B_{lk}(1, 1/\lambda) \begin{bmatrix} I_l & 0 & 0 \\ 0 & \sqrt{\lambda} I_k & 0 \\ 0 & 0 & I_{n-k-l} \end{bmatrix} \tag{11}$$

we get

$$\Phi(B_{lk}(\lambda,1)) = \Phi\left(\begin{bmatrix} I_l & 0 & 0 \\ 0 & \sqrt{\lambda} I_k & 0 \\ 0 & 0 & I_{n-k-l} \end{bmatrix}\right) \cdot 1 \cdot \Phi\left(\begin{bmatrix} I_l & 0 & 0 \\ 0 & \sqrt{\lambda} I_k & 0 \\ 0 & 0 & I_{n-k-l} \end{bmatrix}\right) = \varphi(\lambda^k)$$

by (6), (10) and Lemma 6.

Each Jordan block corresponding to nonzero eigenvalue can be written as a square of an upper triangular matrix with $\sqrt{\lambda}$ on diagonal

$$\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \lambda \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda} & \frac{1}{2}\lambda^{-1/2} & \cdots & \binom{\frac{1}{2}}{k-1}\lambda^{\frac{3}{2}-k} \\ 0 & \sqrt{\lambda} & \cdots & \binom{\frac{1}{2}}{k-2}\lambda^{\frac{5}{2}-k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda} \end{bmatrix}^2 = \left(\sqrt{Y_k(\lambda, 1)}\right)^2,$$

and write

$$\sqrt{B_{lk}(\lambda, 1)} = \begin{bmatrix} I_l & 0 & 0 \\ 0 & \sqrt{Y_k(\lambda, 1)} & 0 \\ 0 & 0 & I_{n-k-l} \end{bmatrix}.$$

We will finish the proof by induction on the Jordan blocks. Suppose that the dimensions of Jordan blocks in the matrix J are k_1,k_2,$\dots k_s$ and the corresponding eigenvalues are λ_1, $\lambda_2, \dots, \lambda_s$. The values λ_k need not to be distinct. We already know that

$$\Phi(B_{0k_1}(\lambda_1, 1)) = \varphi(\lambda_1^{k_1}). \tag{12}$$

Suppose that we know the value

$$\Phi\left(\begin{bmatrix} C & 0 \\ 0 & I_{n-l} \end{bmatrix}\right),$$

where the matrix C is block diagonal with the first $m-1$ Jordan blocks of the matrix J on the diagonal. The dimension of the matrix C is $l = k_1 + k_2 + \cdots + k_{m-1}$.

Since we can write

$$\begin{bmatrix} C & 0 & 0 \\ 0 & Y_{k_m}(\lambda_m, 1) & 0 \\ 0 & 0 & I_{n-k_m-l} \end{bmatrix} = \sqrt{B_{lk_m}(\lambda_m, 1)} \begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix} \sqrt{B_{lk_m}(\lambda_m, 1)}$$

we get

$$\Phi\left(\begin{bmatrix} C & 0 & 0 \\ 0 & Y_{k_m}(\lambda_m, 1) & 0 \\ 0 & 0 & I_{n-k_m-l} \end{bmatrix}\right) = \varphi(\lambda_m^{k_m})\Phi\left(\begin{bmatrix} C & 0 \\ 0 & I \end{bmatrix}\right).$$

Thus by induction we get

$$\Phi(J) = \varphi(\lambda_1^{k_1}\lambda_2^{k_1}\dots\lambda_s^{k_s}) = \varphi(\det A).$$

Proof for $\mathbb{R}$. We will use the real Jordan canonical form of a matrix. Proof of the real Jordan form, and some observations can be find in [2]. Each real matrix $A \in M_n(\mathbb{R})$ is similar to a block diagonal matrix of the form

$$\begin{bmatrix} J(\lambda_1) & & & & & \\ & \ddots & & & & \\ & & J(\lambda_{n_1}) & & & \\ & & & R(a_1, b_1) & & \\ & & & & \ddots & \\ & & & & & R(a_{n_2}, b_{n_2}) \end{bmatrix} \tag{13}$$

where $J(\lambda_1), J(\lambda_2), \dots, J(\lambda_{n_1})$ are Jordan blocks corresponding to real eigenvalues of the matrix A, and $a_k + ib_k$, $a_k, b_k \in \mathbb{R}$, $k = 1, 2, \dots n_2$, are nonreal eigenvalues of the matrix A. Values λ_k and the values $a_k + ib_k$ need not be distinct. Each real block triangular matrix $R(a_k, b_k)$ is of the form

$$R(a,b) = \begin{bmatrix} C(a,b) & I & \cdots & 0 \\ 0 & C(a,b) & \cdots & 0 \\ . & . & \ddots & . \\ 0 & 0 & \cdots & C(a,b) \end{bmatrix}, \tag{14}$$

where

$$C(a,b) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \quad a, b \in \mathbb{R}.$$

The similarity matrix can be chosen to be real.

Suppose that the dimension of $R(a,b)$ is $2m \times 2m$, and compute

$$\Phi\left(\begin{bmatrix} I_l & 0 & 0 \\ 0 & R(a,b) & 0 \\ 0 & 0 & I_{n-2m-l} \end{bmatrix}\right),$$

and show that the matrix

$$\begin{bmatrix} I & 0 & 0 \\ 0 & R(a,b) & 0 \\ 0 & 0 & I_{n-2m-l} \end{bmatrix}$$

has a square root.

In what follows (in next lines), we work only with the blocks in the position $2, 2$.

Multiply the matrix $R(a,b)$ by $\frac{1}{\sqrt[4]{a^2+b^2}} I$ from the left and right, and denote by

$$R_\alpha = \begin{bmatrix} \dfrac{a}{\sqrt{a^2+b^2}} & \dfrac{b}{\sqrt{a^2+b^2}} \\ \dfrac{-b}{\sqrt{a^2+b^2}} & \dfrac{a}{\sqrt{a^2+b^2}} \end{bmatrix}.$$

The matrix R_α can be viewed as a clockwise rotation about the origin by an angle α. Since

$$T = \frac{1}{\sqrt[4]{a^2+b^2}} I \, R(a,b) \, \frac{1}{\sqrt[4]{a^2+b^2}} I = \begin{bmatrix} R_\alpha & \dfrac{1}{\sqrt{a^2+b^2}} I & \cdots & 0 \\ 0 & R_\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_\alpha \end{bmatrix}, \tag{15}$$

and

$$\Phi\left(\frac{1}{\sqrt[4]{a^2+b^2}}I\right)=\varphi\left((a^2+b^2)^{\frac{-m}{2}}\right), \tag{16}$$

we have

$$\Phi(R(a,b))=\varphi((a^2+b^2)^m)\Phi(T)=\varphi(\det(R(a,b)))\Phi(T). \tag{17}$$

The inverse of the matrix

$$C=\begin{bmatrix} R_\alpha & I & \cdots & 0 \\ 0 & R_\alpha & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_\alpha \end{bmatrix}$$

is

$$C^{-1}=\begin{bmatrix} R_{-\alpha} & -R_{-2\alpha} & \cdots & (-1)^{m-1}R_{-m\alpha} \\ 0 & R_{-\alpha} & \cdots & (-1)^{m-2}R_{-(m-1)\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{-\alpha} \end{bmatrix}.$$

If we look matrices T, C and C^{-1} as an endomorphism of the space $\mathbb{C}^n$ they have the same (and only one) conjugate pair of eigenvalues, $\lambda_1=\cos\alpha+i\sin\alpha$ and $\lambda_2=\cos\alpha-i\sin\alpha$. All three matrices has invertible matrices just above the blocks on the diagonal. Therefore the geometric multiplicities of the eigenvalues are 1. So complex Jordan forms of the matrices T, C, and C^{-1} are equal. Following the construction of the real Jordan form from the complex one we can see that also the real Jordan canonical forms of the matrices T, C, and C^{-1} are equal. Therefore the matrices T, C, and C^{-1} are similar.

By Lemma (3), $\Phi(T)=\Phi(C)$, and $\Phi(C)=\Phi(C^{-1})$. Therefore $\phi(C)=\pm 1$. Since the matrix C has square root

$$\sqrt{C}=\begin{bmatrix} R_{\alpha/2} & \frac{1}{2}R_{-\alpha/2} & \cdots & \binom{\frac{1}{2}}{m-1}R_{(3/2-m)\alpha} \\ 0 & R_{\alpha/2} & \cdots & \binom{\frac{1}{2}}{m-2}R_{(5/2-m)\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{\alpha/2} \end{bmatrix},$$

we have $\Phi(C)=1$ by Lemma 1.

We proved that

$$\Phi(\begin{bmatrix} I_l & 0 & 0 \\ 0 & R(a,b) & 0 \\ 0 & 0 & I_{n-2m-l} \end{bmatrix}) = \varphi(\det(R(a,b))), \tag{18}$$

and since the matrix C has a square root also the (similar) matrix T has a square root. From (15) we see that also matrix $R(a,b)$ has a square root.

Let

$$S_{2s} = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & -1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & -1 & 0 \end{bmatrix}$$

have s 2×2 blocks. Since $S_{2s}^4 = I$ we have

$$\Phi\left(\begin{bmatrix} S_{2s} & 0 \\ 0 & I \end{bmatrix}\right) = \pm 1$$

by Lemma 1.

Denote the real Jordan form of A by J and arrange the Jordan block to get three blocks on the diagonal. The first one consisting of the Jordan blocks corresponding to the negative eigenvalues of the matrix A, the second one consisting of the Jordan blocks corresponding to the positive eigenvalues of the matrix A, and the third one consisting of the real Jordan blocks corresponding to the conjugate pairs of complex eigenvalues of the matrix A.

$$J = \begin{bmatrix} J_- & 0 & 0 \\ 0 & J_+ & 0 \\ 0 & 0 & R \end{bmatrix}. \tag{19}$$

We will have two cases.

Case 1: If the dimension of J_- is even, $2s$, then multiply J from the left and right by

$$P = \begin{bmatrix} S_{2s} & 0 \\ 0 & I \end{bmatrix}. \tag{20}$$

Since $S_{2s}^2 = -I$ all eigenvalues of the matrix $S_{2s}J_-S_{2s}$ are positive. Namely, if $\mu < 0$ is an eigenvalue of $S_{2s}J_-S_{2s}$, then for an eigenvector x associated with μ we have

$$S_{2s}J_-S_{2s}x = \mu x, \quad \mu < 0.$$

Multiplying by $-S_{2s}$ from left, we get

$$J_-S_{2s}x = -\mu S_{2s}x, \quad \mu < 0.$$

This is a contradiction, since all the eigenvalues of J_- are negative.

Since $\Phi(P) = \pm 1$, we have

$$\Phi(PJP) = \Phi(P)\Phi(J)\Phi(P) = \Phi(J) = \Phi(A).$$

Also

$$\det(PJP) = \det A.$$

In this case all real eigenvalues of the matrix $J' = PJP$ are positive.

Case 2: If the dimension of J_- is odd, $2s+1$, then multiply J from left and right by

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & S_{2s} & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{21}$$

The product QJQ has exactly one negative eigenvalue and the complex eigenvalues remain the same. To see this, consider the matrix J as a block matrix corresponding to the partition of the matrix Q. Therefore the Jordan form of the matrix QJQ is of the form

$$J'' = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & J'_+ & 0 \\ 0 & 0 & R \end{bmatrix}, \tag{22}$$

where $\lambda < 0$ and all the eigenvalues of the matrix J'_+ are positive. As before, we have $\det J'' = \det A$ and $\Phi(J'') = \Phi(A)$.

In the first case all blocks have square roots so we are able to end the proof by induction as in the complex case.

In the second case when the sum of the dimensions of the blocks corresponding to negative eigenvalues is even we start the induction with

$$\Phi\left(\begin{bmatrix} \lambda & 0 \\ 0 & I_{n-1} \end{bmatrix}\right) = \varphi(\lambda).$$

After that each block we add has a square root. We conclude $\Phi(A) = \varphi(\det A)$. $\square$

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