Publ. Math. Debrecen
73/1-2 (2008), 101–117

Pointwise approximation theorems for Meyer-König and Zeller–Durrmeyer operators

By QIULAN QI (Shijiazhuang) and JUAN LIU (Shijiazhuang)

Abstract. In this paper, we give the estimates of the second and fourth-order moments for the Meyer-König and Zeller–Durrmeyer type operators. Secondly, using the equivalence between the unified moduli of smoothness $\omega^2_{\varphi^\lambda}(f,t)$ and the Peetre's K-functional $K^2_{\varphi^\lambda}(f,t^2)(0 \le \lambda \le 1)$, we obtain the direct, inverse and equivalence theorems for these operators.

1. Introduction

The Meyer-König and Zeller operators are given by [4], [8], [11], [12]

$$M_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) m_{n,k}(x), \quad 0 \le x < 1,$$

$$M_n(f,1) = f(1), \quad m_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}.$$

There are several studies about these operators in approximation theory (cf. [1], [4], [7]–[9], [11], [12]). Ditzian [5] introduced the unified moduli of smoothness $\omega^2_{\varphi^\lambda}(f,t)$ $(0 \le \lambda \le 1)$ and gave an interesting direct result for the Bernstein operators which united the result with $\omega^2(f,t)$ and $\omega^2_{\varphi}(f,t)$. It is difficult to get the estimates of the moments for the Meyer-König and Zeller type operators (cf. [2], [3], [10]). The direct and inverse results for Durrmeyer-type modifications

Mathematics Subject Classification: 41A25, 41A35, 41A27.
Key words and phrases: Meyer-König and Zeller–Durrmeyer type operators, moduli of smoothness, K-functional, moment, Hölder inequality.
Research supported by the National Natural Science Foundation of China (10571040).

of the Meyer-König and Zeller operators are not perfect. In this paper, we will consider a new modification of Meyer-König and Zeller–Durrmeyer type operators $\widetilde{M}_n(f,x)$:

$$\widetilde{M}_n(f,x)=\sum_{k=0}^{\infty}\Phi_{n,k}(f)m_{n,k}(x),\quad f(x)\in C[0,1],$$

where

$$\Phi_{n,k}(f)=\begin{cases}C_{n-2,k-1}^{-1}\int_0^1 f(t)m_{n-2,k-1}(t)dt, & k>0,\\ f(0), & k=0,\end{cases}$$

$$C_{n,k}=\int_0^1 m_{n,k}(t)dt=\frac{n+1}{(n+k+1)(n+k+2)}.$$

Let

$$\|f\|=\sup_{x\in[0,1]}|f(x)|,\quad \omega_{\varphi^\lambda}^2(f,t)=\sup_{0<h\le t}\|\Delta_{h\varphi^\lambda}^2\|,$$

where $\varphi(x)=\sqrt{x}(1-x)$ and

$$\Delta_{h\varphi^\lambda}^2 f(x)=\begin{cases}f(x+h\varphi^\lambda(x))-2f(x)+f(x-h\varphi^\lambda(x)), & \text{if } x\pm h\varphi^\lambda(x)\in[0,1];\\ 0, & \text{otherwise},\end{cases}$$

$$K_{\varphi^\lambda}^2(f,t^2)=\inf_{g\in D}\{\|f-g\|\ +t^2\|\varphi^{2\lambda}g''\|\},$$

where

$$D=\{g\mid g'\in A.C._{loc},\ \|\varphi^{2\lambda}g''\|<\infty\}.$$

It is well known that $\omega_{\varphi^\lambda}^2(f,t)\sim K_{\varphi^\lambda}^2(f,t^2)$ (cf. [6]), i.e. there exists a positive constant C such that

$$C^{-1}K_{\varphi^\lambda}^2(f,t^2)\le\omega_{\varphi^\lambda}^2(f,t)\le CK_{\varphi^\lambda}^2(f,t^2).$$

Noting that it is absolutely superfluous to talk about $f(1)$, we can assume that the function space is $C_B[0,1)$ (the set of bounded and continuous functions on $[0,1)$). Now we state the direct and inverse theorems.

Theorem 1.1. *For $f\in C_B[0,1), 0\le\lambda\le1$, there holds*

$$|\widetilde{M}_n(f,x)-f(x)|\le C\omega_{\varphi^\lambda}^2(f,n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)).$$

Theorem 1.2. *For* $f \in C_B[0,1)$, $0<\alpha<2$, $0\le\lambda\le 1$, $n\ge 2$, *if*

$$|\widetilde{M}_n(f,x)-f(x)| = O(n^{-\frac{\alpha}{2}}\varphi^{\alpha(1-\lambda)}(x)),$$

then $\omega^2_{\varphi^\lambda}(f,t)=O(t^\alpha)$.

Remark 1. In this paper the notation $f(x)=O(g(x))$ means there exists a positive constant C such that $f(x)\le Cg(x)$.

Remark 2. Throughout this paper, C denotes a positive constant independent of n and x and not necessarily the same at each occurrence.

From Theorem 1.1, 1.2, we obtain the equivalence theorem.

Theorem 1.3. *For* $f \in C_B[0,1)$, $0<\alpha<2$, $0\le\lambda\le 1$, $n\ge 2$, *one has*

$$|\widetilde{M}_n(f,x)-f(x)| = O(n^{-\frac{\alpha}{2}}\varphi^{\alpha(1-\lambda)}(x)) \Leftrightarrow \omega^2_{\varphi^\lambda}(f,t)=O(t^\alpha).$$

2. The direct theorem

In order to prove Theorem 1.1, we shall need the following lemmas.

Lemma 2.1. *For* $x\in[0,1)$ *and sufficiently large* n, *there holds*

$$\widetilde{M}_n(t-x,x)=0, \tag{2.1}$$

$$\widetilde{M}_n((t-x)^2,x)=O\left(\frac{\varphi^2(x)}{n}\right), \tag{2.2}$$

$$\widetilde{M}_n((t-x)^4,x)=O\left(\frac{\varphi^4(x)}{n^2}\right). \tag{2.3}$$

Proof. By an easy calculations, we can obtain (2.1) and (2.2). Next we estimate (2.3). Writing

$$\begin{aligned}\widetilde{M}_n(t^4,x) &= \sum_{k=1}^{\infty}\frac{k(k+1)(k+2)(k+3)}{(n+k)(n+k+1)(n+k+2)(n+k+3)}m_{n,k}(x)\\ &= \varphi^4(x)\sum_{k=0}^{\infty}\frac{(n+k-3)(n+k-2)(n+k-1)(n+k)(n+k+1)}{(k+1)(n-3)(n-2)(n-1)n}\\ &\quad\cdot\frac{(k+3)(k+4)(k+5)}{(n+k+3)(n+k+4)(n+k+5)}m_{n-4,k}(x),\end{aligned}$$

$$x\widetilde{M}_n(t^3,x) = x\sum_{k=1}^{\infty}\frac{k(k+1)(k+2)}{(n+k)(n+k+1)(n+k+2)}m_{n,k}(x)$$
$$= \varphi^4(x)\sum_{k=0}^{\infty}\frac{(n+k-3)(n+k-2)(n+k-1)(n+k)(k+2)(k+3)}{(n-3)(n-2)(n-1)n(n+k+3)(n+k+2)}m_{n-4,k}(x),$$

$$x^2\widetilde{M}_n(t^2,x) = \varphi^4(x)\sum_{k=1}^{\infty}\frac{(n+k-3)(n+k-2)(n+k-1)k(k+1)}{(n-3)(n-2)(n-1)n(n+k+1)}m_{n-4,k}(x),$$

$$x^4 = \varphi^4(x)\sum_{k=0}^{\infty}\frac{(n+k-3)(n+k-2)(k-1)k}{(n-3)(n-2)(n-1)n}m_{n-4,k}(x),$$

recalling that

$$\widetilde{M}_n((t-x)^4,x) = \widetilde{M}_n(t^4,x) - 4x\widetilde{M}_n(t^3,x) + 6x^2\widetilde{M}_n(t^2,x) - 3x^4,$$

we can get the result. □

Lemma 2.2. *For $k \geq 1$ and sufficiently large n, one has*

$$C_{n-2,k-1}^{-1}\int_0^1 (1-t)^{-2}m_{n-2,k-1}(t)dt \leq C\left(\frac{n}{n+k}\right)^{-2}, \tag{2.4}$$

$$C_{n-2,k-1}^{-1}\int_0^1 (1-t)^{4}m_{n-2,k-1}(t)dt \leq C\left(\frac{n}{n+k}\right)^{4}, \tag{2.5}$$

$$C_{n-2,k-1}^{-1}\int_0^1 t^{2}m_{n-2,k-1}(t)dt \leq C\left(\frac{k}{n+k}\right)^{2}. \tag{2.6}$$

Proof. We only need to prove (2.4). The proof of (2.5)(2.6) is similar. For $k \geq 1$, there holds

$$\int_0^1 t^{k-1}(1-t)^{n-3} = \frac{(n-3)!(k-1)!}{(n+k-3)!}.$$

By the observation

$$\frac{(n+k-2)(n+k-1)}{n-1}\frac{(n+k-3)!}{(n-2)!(k-1)!}\frac{(n-3)!(k-1!)}{(n+k-3)!}\left(\frac{n}{n+k}\right)^2 \leq 4,$$

the proof of (2.4) is completed. □

Lemma 2.3 (cf. [11]). *For $l \in N_0, m \in Z$, there holds*

$$\sum_{k=1}^{\infty} m_{n,k}(x) \left(\frac{k}{n+k}\right)^{-l} \le Mx^{-l}, \sum_{k=1}^{\infty} m_{n,k}(x) \left(\frac{n}{n+k}\right)^{-m} \le M(1-x)^{-m}.$$

Remark 3. For $x \in E_n = [\frac{1}{n}, 1)$, one has

$$\sum_{k=1}^{\infty} m_{n,k}(x) \left(\frac{k}{n+k}\right)^{2} \le Mx^2.$$

Lemma 2.4. *For $f \in C[0,1]$, one has $|\widetilde{M}_n(f,x)| \le \|f\|$.*

Proof. We can get the result by direct computations. □

Lemma 2.5. *For $f \in D, 0 \le \lambda \le 1$, there holds $|\widetilde{M}_n(f,x) - f(x)| \le Cn^{-1}\varphi^{2(1-\lambda)}(x)\|\varphi^{2\lambda} f''\|$.*

Proof. It is known from the Taylor's expansion, that

$$f(t) = f(x) + f'(x)(t-x) + \int_x^t f''(v)(t-v)dv,$$

one can write

$$\widetilde{M}_n(f,x) - f(x) = f'(x)\widetilde{M}_n(t-x,x) + \widetilde{M}_n(R_2(f,t,x),x),$$

where $R_2(f,t,x) = \int_x^t f''(v)(t-v)dv$.

Since $\widetilde{M}_n(t-x,x) = 0$, we have

$$\begin{aligned}|\widetilde{M}_n(f,x) - f(x)| &= \widetilde{M}_n(R_2(f,t,x),x)\\ &= \|f''\varphi^{2\lambda}\| \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1} \int_0^1 \left|\int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)} dv\right| m_{n-2,k-1}(t)\, dt m_{n,k}(x).\end{aligned}$$

It is sufficient to show that

$$\widetilde{M}_n\left(\left|\int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)} dv\right|, x\right) = O\left(\frac{\varphi^{2-2\lambda}(x)}{n}\right). \tag{2.7}$$

For v between t and x, there holds [4]

$$\left|\int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)} dv\right| \le g_x(t) = \begin{cases} \dfrac{(t-x)^2}{\varphi^{2\lambda}(x)}, & \text{for } t \le x;\\[2ex] \dfrac{(t-x)^2}{(x(1-x)(1-t))^{\lambda}}, & \text{for } t > x.\end{cases}$$

Therefore,

$$\begin{aligned}
&\widetilde{M}_n\left(\left|\int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)}dv\right|, x\right)\\
&= \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_0^1 \left|\int_x^t \frac{|t-v|}{\varphi^{2\lambda}(v)}dv\right| m_{n-2,k-1}(t)dt m_{n,k}(x)\\
&= \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_0^x \frac{(t-x)^2}{\varphi^{2\lambda}(x)} m_{n-2,k-1}(t)dt m_{n,k}(x)\\
&\quad + \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_x^1 \frac{(t-x)^2}{(x(1-x)(1-t))^{\lambda}} m_{n-2,k-1}(t)dt m_{n,k}(x) =: \quad E_1+E_2.
\end{aligned}$$

We will estimate E_1, E_2 separately. Using (2.2), we can deduce

$$\begin{aligned}
E_1 &\le \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_0^1 \frac{(t-x)^2}{\varphi^{2\lambda}(x)} m_{n-2,k-1}(t)dt m_{n,k}(x)\\
&\le \varphi^{-2\lambda}\widetilde{M}_n((t-x)^2, x) \le Cn^{-1}\varphi^{2-2\lambda}(x).
\end{aligned}$$

By Lemma 2.2 and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
E_2 &\le \sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_0^1 \frac{(t-x)^2}{(x(1-x)(1-t))^{\lambda}} m_{n-2,k-1}(t)dt m_{n,k}(x)\\
&\le \left(\sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_0^1 (t-x)^4 m_{n-2,k-1}(t)dt m_{n,k}(x)\right)^{\frac{1}{2}}\\
&\quad \cdot (x(1-x))^{-\lambda}\left(\sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_0^1 (1-t)^{-2\lambda} m_{n-2,k-1}(t)dt m_{n,k}(x)\right)^{\frac{1}{2}}\\
&\le \left(\sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_0^1 (t-x)^4 m_{n-2,k-1}(t)dt m_{n,k}(x)\right)^{\frac{1}{2}}\\
&\quad \cdot (x(1-x))^{-\lambda}\left(\sum_{k=0}^{\infty} C_{n-2,k-1}^{-1}\int_0^1 (1-t)^{-2} m_{n-2,k-1}(t)dt m_{n,k}(x)\right)^{\frac{\lambda}{2}}\\
&\le Cn^{-1}\varphi^{2-2\lambda}(x).
\end{aligned}$$

From the above two inequalities, we can get (2.7) which completed Lemma 2.5. □

PROOF OF THEOREM 1.1. From the definition of $K^2_{\varphi^\lambda}(f,t^2)$, we can choose $g \in D$ satisfying

$$\|f-g\| + \ t^2\|\varphi^{2\lambda}g''\| \le 2K^2_{\varphi^\lambda}(f,t^2). \tag{2.8}$$

By Lemma 2.4 and Lemma 2.5, there holds

$$\begin{aligned}\left|\widetilde{M}_n(f,x)-f(x)\right| &\le \left|\widetilde{M}_n(f-g,x)\right| + |f(x)-g(x)| + \left|\widetilde{M}_n(g,x)-g(x)\right| \\ &\le C_1\left[\|f-g\| + n^{-1}\varphi^{2(1-\lambda)}(x)\|\varphi^{2\lambda}g''\|\right].\end{aligned}$$

Hence by (2.8) and set $t = n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)$, one has

$$\left|\widetilde{M}_n(f,x)-f(x)\right| \le CK^2_{\varphi^\lambda}\left(f, n^{-1}\varphi^{2(1-\lambda)}(x)\right).$$

Using the relation $K^2_{\varphi^\lambda}(f,t^2) \sim \omega^2_{\varphi^\lambda}(f,t)$, we get

$$\left|\widetilde{M}_n(f,x)-f(x)\right| \le C\omega^2_{\varphi^\lambda}(f, n^{-\frac{1}{2}}\varphi^{1-\lambda}(x)). \qquad \square$$

3. The inverse theorem

For the convenience of the proof of Theorem 1.2, we need some new notations. Set

$$\begin{aligned}&\|f\|_0 = \sup_{x\in(0,1)}\left|\varphi^{\alpha(\lambda-1)}(x)f(x)\right|,\\ &C_0 = \{f \in C_B[0,1], f(0)=f(1)=0\},\\ &C^0_{\alpha,\lambda} = \{f\in C_0, \|f\|_0<\infty\},\\ &\|f\|_2 = \sup_{x\in(0,1)}\left|\varphi^{2+\alpha(\lambda-1)}(x)f''(x)\right|,\\ &C^2_{\alpha,\lambda} = \{f\in C_0, \|f\|_2<\infty, f'\in A.C._{loc}\}.\end{aligned}$$

Let us introduce a new K-functional, for $f \in C_0$

$$K^\alpha_\lambda(f,t^2) = \inf_{g\in C^2_{\alpha,\lambda}}\{\|f-g\|_0 + t^2\|g\|_2\}.$$

For the proof of the inverse theorem, we also need the following lemmas, which will be proved in the next section.

Lemma 3.1 (cf. [10]). *Let* $\varphi(x)=\sqrt{x}(1-x)$, $A_{n,2p}(x)=M_n((t-x)^{2p},x)$, $p\in N$, *we have the estimates, for* $n>2p$,

$$A_{n,2p}(x)\le C\begin{cases}\dfrac{\varphi^{2p}(x)}{n^p}, & \text{for } x\ge\dfrac{1}{n};\\ \dfrac{\varphi^2(x)(1-x)^{2p-2}}{n^{2p-1}}, & \text{for } x<\dfrac{1}{n}.\end{cases}$$

Lemma 3.2. *Let* $n\in N$, $n\ge 2$, $f\in C^0_{\alpha,\lambda}$, *then* $\|\widetilde{M}_n f\|_2\le C_1 n\|f\|_0$.

Lemma 3.3. *Let* $n\in N$, $n\ge 2$, $f\in C^2_{\alpha,\lambda}$, *then* $\|\widetilde{M}_n f\|_2\le C_2\|f\|_2$.

Lemma 3.4 (cf. [4]). *For* $0\le\beta<1, 0<h\le\frac{1}{8}$, *one has*

$$\iint_{-\frac{h}{2}}^{\frac{h}{2}}\frac{dsdt}{\varphi^{2\beta}(x+s+t)}\le\frac{Mh^2}{\max\{\varphi(x\pm h),\varphi(x)\}^{2\beta}},\ (x\in[h,1-h]).$$

Proof of Theorem 1.2. From the definition of $K^{\alpha}_{\lambda}(f,t^2)$, for $\widetilde{M}_n(f,x)\in C^2_{\alpha,\lambda}$, we have

$$K^{\alpha}_{\lambda}(f,t^2)\le\|\widetilde{M}_n(f,x)-f(x)\|_0+t^2\|\widetilde{M}_n(f,x)\|_2, \tag{3.1}$$

and setting $t=n^{-\frac{1}{2}}$, we can choose $g\in C^2_{\alpha,\lambda}$ such that

$$\|f-g\|_0+\ n^{-1}\|g\|_2\le 2K^{\alpha}_{\lambda}(f,n^{-1}). \tag{3.2}$$

Noting the condition

$$\left|\widetilde{M}_n f-f\right|=O(n^{-\frac{\alpha}{2}}\varphi^{\alpha(1-\lambda)}(x)),$$

which implies that

$$\|\widetilde{M}_n f-f\|_0\le Cn^{-\frac{\alpha}{2}}. \tag{3.3}$$

By Lemma 3.2, Lemma 3.3 and (3.2), one has

$$\begin{aligned}\|\widetilde{M}_n f\|_2&\le\|\widetilde{M}_n(f-g)\|_2+\|\widetilde{M}_n g\|_2\le C(n\|f-g\|_0+\|g\|_2)\\&\le Cn(\|f-g\|_0+n^{-1}\|g\|_2)\le 2CnK^{\alpha}_{\lambda}(f,n^{-1}).\end{aligned} \tag{3.4}$$

We have by (3.1), (3.3), (3.4)

$$K^{\alpha}_{\lambda}(f,t^2)\le C\left(n^{-\frac{\alpha}{2}}+t^2nK^{\alpha}_{\lambda}(f,n^{-1})\right).$$

Applying the Berens–Lorentz Lemma [1, $P122$], we can get $K^{\alpha}_{\lambda}(f,t^2) \le Ct^{\alpha}$.
For $\lambda = 1$, $K^{\alpha}_{\lambda}(f,t^2) = K^2_{\varphi^\lambda}(f,t^2)$, then $\omega^2_{\varphi^\lambda}(f,t) = O(t^\alpha)$.
For $0 \le \lambda < 1$, now we estimate $|\Delta^2_{h\varphi^\lambda} f(x)|$.
(i) Fixed $h \in (0, \frac{1}{8}), x \in [h, 1-h]$ and for any $f \in C^0_{\alpha,\lambda}$, one has

$$\begin{aligned}\left|\Delta^2_{h\varphi^\lambda} f(x)\right| &\le \left|f(x+h\varphi^\lambda(x))\right| + 2\left|f(x)\right| + \left|f(x-h\varphi^\lambda(x))\right| \\ &\le 4\|f\|_0 M(x)^{\alpha(1-\lambda)},\end{aligned}$$

where $M(x) := \max\{|\varphi(x+h\varphi^\lambda(x))|,\ |\varphi(x)|,\ |\varphi(x-h\varphi^\lambda(x))|\}$.
(ii) Using Lemma 3.4, for any $g \in C^2_{\alpha,\lambda}$, one has

$$\begin{aligned}\left|\Delta^2_{h\varphi^\lambda} g(x)\right| &= \left|\iint_{-\frac{h\varphi^\lambda}{2}}^{\frac{h\varphi^\lambda}{2}} g''(x+s+t)dsdt\right| \\ &\le \|g\|_2 \iint_{-\frac{h\varphi^\lambda}{2}}^{\frac{h\varphi^\lambda}{2}} \left|\varphi^{-2+\alpha(1-\lambda)}(x+s+t)\right| dsdt \\ &\le C\|g\|_2 h^2 M(x)^{(\alpha-2)(1-\lambda)}.\end{aligned}$$

Using (i), (ii) and (3.2), there holds

$$\begin{aligned}|\Delta^2_{h\varphi^\lambda} f(x)| &\le |\Delta^2_{h\varphi^\lambda}(f(x)-g(x))| + |\Delta^2_{h\varphi^\lambda} g(x)| \\ &\le C_1\left[\|f-g\|_0 M(x)^{\alpha(1-\lambda)} + \|g\|_2 h^2 M(x)^{(\alpha-2)(1-\lambda)}\right] \\ &\le C_2 M(x)^{\alpha(1-\lambda)} K^\alpha_\lambda\left(f, \frac{h^2}{M(x)^{2(1-\lambda)}}\right) \le Ch^\alpha.\end{aligned}$$

If we take supremum over $h : 0 < h \le t$, we have

$$\omega^2_{\varphi^\lambda}(f,t) = O(t^\alpha).$$

Noting that $\widetilde{M}_n(f,x) - f(x) = \widetilde{M}_n(f_0,x) - f_0(x)$, where $f_0(x) = f(x) - f(0)$, the proof of Theorem 1.2 is completed. □

4. Proofs of Lemma 3.2 and Lemma 3.3

We will need the following two representations of derivations of $\widetilde{M}_n(f,x)$ which are given by [4] or simple computations.

$$\left(\widetilde{M}_n f\right)''(x) = \sum_{k=0}^{\infty} \Phi_{n,k}(f) r_{n,k} m_{n,k}(x), \quad x \in (0,1), \tag{4.1}$$

where

$$r_{n,k} := \frac{1}{x^2}\left[\left(k-\frac{(n+1)x}{1-x}\right)^2-\left(k-\frac{(n+1)x}{1-x}\right)\right]-\frac{n+1}{x(1-x)^2},$$

$$\left(\widetilde{M}_n f\right)''(x) = (1-x)^{-2}\sum_{k=0}^{\infty}\Delta_{n,k}(f)m_{n,k}(x),\ x\in(0,1), \tag{4.2}$$

where

$$\Delta_{n,k}(f) = (n+k+1)[(n+k+2)\Phi_{n,k+2}(f)-2(n+k+1)\Phi_{n,k+1}(f)+(n+k)\Phi_{n,k}(f)].$$

Proof of Lemma 3.2. Supposing that $E_n = \left[\frac{1}{n},1\right)$, we will estimate E_n and E_n^c separately.

Case I. For $x \in E_n$, in view of (4.1), by the Hölder inequality,

$$\begin{aligned}
&\left|\varphi^{2+\alpha(\lambda-1)}(x)\big(\widetilde{M}_n f\big)''(x)\right| \\
&\quad\le \left|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)^2\Phi_{n,k}(f)m_{n,k}(x)\right| \\
&\quad\quad+\left|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)\Phi_{n,k}(f)m_{n,k}(x)\right| \\
&\quad\quad+\left|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}(n+1)\Phi_{n,k}(f)m_{n,k}(x)\right| := T_1+T_2+T_3.
\end{aligned}$$

First using the Cauchy–Schwarz inequality, the Hölder inequality and the relation (2.6), we have

$$\begin{aligned}
&C_{n-2,k-1}^{-1}\int_0^1\varphi^{\alpha(1-\lambda)}(t)m_{n-2,k-1}(t)dt \le \left(C_{n-2,k-1}^{-1}\int_0^1\varphi^2(t)m_{n-2,k-1}(t)dt\right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\le\left(C_{n-2,k-1}^{-1}\int_0^1 t^2 m_{n-2,k-1}dt\right)^{\frac{\alpha(1-\lambda)}{4}}\cdot\left(C_{n-2,k-1}^{-1}\int_0^1(1-t)^4 m_{n-2,k-1}(t)dt\right)^{\frac{\alpha(1-\lambda)}{4}} \\
&\le C\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}.
\end{aligned}$$

Next we estimate T_3, using the Hölder inequality twice and Lemma 2.3, we obtain

$$T_3 = \left|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}(n+1)\Phi_{n,k}(f)m_{n,k}(x)\right|$$

$$\le Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}C_{n-2,k-1}^{-1}\int_0^1\varphi^{\alpha(1-\lambda)}(t)m_{n-2,k-1}(t)dt m_{n,k}(x)$$

$$\le C_1n\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n,k}(x)$$

$$\le C_1n\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^2m_{n,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}}$$

$$\cdot\left(\sum_{k=1}^{\infty}\left(\frac{n}{n+k}\right)^4m_{n,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}}\le C_1n\|f\|_0.$$

For T_2, using the Hölder inequality and Lemma 2.2, Lemma 2.3, there holds

$$T_2=\left|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)\Phi_{n,k}(f)m_{n,k}(x)\right|$$

$$\le C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\left|\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)\right|$$

$$\cdot C_{n-2,k-1}^{-1}\int_0^1\varphi^{\alpha(1-\lambda)}(t)m_{n-2,k-1}(t)dt\cdot m_{n,k}(x)$$

$$\le\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\left|\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)\right|\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}$$

$$\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n,k}(x).$$

Noticing that

$$\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)m_{n,k}(x)\le(n+2)(m_{n+2,k-1}(x)+m_{n,k}(x)),$$

we have

$$T_2\le\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}(n+2)(m_{n+2,k-1}(x)+m_{n,k}(x))\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}$$

$$\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}:=Q_1+Q_2.$$

The estimate of Q_2 is similar to T_3, so we only need to estimate Q_1.

$$Q_1=\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}(n+2)m_{n+2,k-1}(x)\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}$$

$$\le Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n+2,k-1}(x)$$

$$\le Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\alpha(1-\lambda)}m_{n+2,k-1}(x)\right)^{\frac{1}{2}}$$

$$\cdot\left(\sum_{k=1}^{\infty}\left(\frac{n}{n+k}\right)^{2\alpha(1-\lambda)}m_{n+2,k-1}(x)\right)^{\frac{1}{2}}$$

$$\le Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{2}m_{n+2,k-1}(x)\right)^{\frac{\alpha(1-\lambda)}{4}}$$

$$\cdot\left(\sum_{k=1}^{\infty}\left(\frac{n}{n+k}\right)^{4}m_{n+2,k-1}(x)\right)^{\frac{\alpha(1-\lambda)}{4}}$$

$$= Cn\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left(\sum_{k=0}^{\infty}\left(\frac{k+1}{n+k+1}\right)^{2}m_{n+2,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}}$$

$$\cdot\left(\sum_{k=0}^{\infty}\left(\frac{n}{n+k+1}\right)^{4}m_{n+2,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}}.$$

By Lemma 2.3 and Remark 3, we get

$$\sum_{k=0}^{\infty}\left(\frac{k+1}{n+k+1}\right)^{2}m_{n+2,k}(x)$$

$$= 2\sum_{k=1}^{\infty}\left(\frac{k}{n+k+2}\right)^{2}m_{n+2,k}(x)+\frac{1}{(n+1)^2}(1-x)^{n+3}\le Cx^2 \quad (\text{for } x\in E_n),$$

and

$$\sum_{k=0}^{\infty}\left(\frac{n}{n+k+1}\right)^{4}m_{n+2,k}(x)$$

$$= \sum_{k=1}^{\infty}\left(\frac{n}{n+k+1}\right)^{4}m_{n+2,k}(x)+\left(\frac{n}{n+1}\right)^{4}(1-x)^{n+3}$$

$$\le 2\sum_{k=1}^{\infty}\left(\frac{n+2}{n+k+2}\right)^{4}m_{n+2,k}(x)+(1-x)^4\le C(1-x)^4,$$

from which $Q_1\le Cn\|f\|_0$. Therefore, $T_2\le Cn\|f\|_0$.

Finally we estimate T_1.

$$\begin{aligned}
T_1 &= \left|\varphi^{\alpha(\lambda-1)}(x)\sum_{k=0}^{\infty}\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)^2\Phi_{n,k}(f)m_{n,k}(x)\right| \\
&\le \|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)^2 \\
&\quad\cdot\frac{(n+k-2)(n+k-1)}{n-1}\int_0^1\varphi^{\alpha(1-\lambda)}(t)m_{n-2,k-1}(t)dtm_{n,k}(x) \\
&\le C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\sum_{k=1}^{\infty}\frac{(1-x)^2}{x}\left(k-\frac{(n+1)x}{1-x}\right)^2\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n,k}(x) \\
&\le C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\left\{\sum_{k=1}^{\infty}\frac{1}{x}[k-(n+k)x]^2\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\right. \\
&\quad\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n,k}(x)+x\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad\left.\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n,k}(x)\right\}:=P_1+P_2.
\end{aligned}$$

Similarly we estimate P_1, P_2 respectively.

$$\begin{aligned}
P_1 &= C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)n^2\sum_{k=1}^{\infty}\frac{1}{x}\left(\frac{k}{n+k}-x\right)^2\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}} \\
&\quad\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)-2}m_{n,k}(x) \\
&\le C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)\frac{n^2}{x}\left(\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^2m_{n,k}(x)\right)^{\frac{\alpha(1-\lambda)}{4}} \\
&\quad\cdot\left(\sum_{k=1}^{\infty}\left(\frac{k}{n+k}-x\right)^8m_{n,k}(x)\right)^{\frac{1}{4}} \\
&\quad\cdot\left(\sum_{k=1}^{\infty}\left(\frac{n}{n+k}\right)^{-8}m_{n,k}(x)\right)^{\frac{2-\alpha(1-\lambda)}{8}}\le Cn\|f\|_0,
\end{aligned}$$

and

$$P_2 = C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)x\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)}m_{n,k}(x)\le Cn\|f\|_0.$$

From the estimate of T_1, T_2 and T_3, we obtain the required result.

Case II. For $x\in E_n^c=(0,\frac{1}{n})$.

For $f\in C^0_{\alpha,\lambda}$, the representation (4.2) shows that

$$\begin{aligned}
&\left|\varphi^{2+\alpha(\lambda-1)}(x)\big(\widetilde{M}_nf\big)''(x)\right|\\
&\quad=\Big|\varphi^{\alpha(\lambda-1)}(x)x\sum_{k=1}^{\infty}(n+k+1)\big[(n+k+2)\Phi_{n,k+2}(f)-2(n+k+1)\Phi_{n,k+1}(f)\\
&\qquad+(n+k)\Phi_{n,k}(f)\big]m_{n,k}(x)\Big|\\
&\quad\le\varphi^{\alpha(\lambda-1)}(x)x\Bigg[\left|\sum_{k=1}^{\infty}(n+k+1)(n+k+2)\Phi_{n,k+2}(f)m_{n,k}(x)\right|\\
&\qquad+\left|\sum_{k=1}^{\infty}2(n+k+1)^2\Phi_{n,k+1}(f)m_{n,k}(x)\right|\\
&\qquad+\left|\sum_{k=1}^{\infty}(n+k+1)(n+k)\Phi_{n,k}(f)]m_{n,k}(x)\right|\Bigg]:=I_1+I_2+I_3.
\end{aligned}$$

The methods of estimating I_1, I_2, I_3 are similar, we make an example of I_1.

$$\begin{aligned}
I_1=&\left|\varphi^{\alpha(\lambda-1)}(x)x\sum_{k=1}^{\infty}(n+k+1)(n+k+2)C^{-1}_{n-2,k+1}\int_0^1 f(t)m_{n-2,k+1}(t)dtm_{n,k}(x)\right|\\
\le&\|f\|_0\varphi^{\alpha(\lambda-1)}(x)x\sum_{k=1}^{\infty}(n+k+1)(n+k+2)C^{-1}_{n-2,k+1}\int_0^1\frac{m_{n-2,k+1}(t)dt}{\varphi^{\alpha(\lambda-1)}(t)}m_{n,k}(x)\\
\le&\|f\|_0\varphi^{\alpha(\lambda-1)}(x)xn(n+1)\sum_{k=1}^{\infty}C^{-1}_{n,k+1}\int_0^1\varphi^{\alpha(1-\lambda)}(t)(1-t)^{-2}m_{n,k+1}(t)dtm_{n,k}(x)\\
\le&C\|f\|_0\varphi^{\alpha(\lambda-1)}(x)xn(n+1)\sum_{k=1}^{\infty}\left(\frac{k}{n+k}\right)^{\frac{\alpha(1-\lambda)}{2}}\cdot\left(\frac{n}{n+k}\right)^{\alpha(1-\lambda)-2}m_{n,k}(x).
\end{aligned}$$

Using the Hölder inequality and Lemma 2.3, and noticing that $x < \frac{1}{n}$, $(1-x)^{-2} < \left(1-\frac{1}{n}\right)^{-2} \leq 4 (n \geq 2)$, there holds

$$I_1 \leq C\|f\|_0 \varphi^{\alpha(\lambda-1)}(x) x n(n+1) \left(\sum_{k=1}^{\infty} \frac{k}{n+k} m_{n,k}(x)\right)^{\alpha(1-\lambda)} \cdot \left(\sum_{k=1}^{\infty} \left(\frac{n}{n+k}\right)^{-2} m_{n,k}(x)\right)^{\frac{2-\alpha(1-\lambda)}{2}} \leq C_1 n\|f\|_0.$$

□

Proof of Lemma 3.3. In view of (4.2) and

$$m'_{n,k}(x) = \frac{n+1}{(1-x)^2}\left[m_{n+1,k-1}(x) - m_{n+1,k}(x)\right],$$

$$m''_{n,k}(x) = \frac{1}{(1-x)^2}\big[(n+k)(n+k+1) m_{n,k}(x) - 2(n+k)^2 m_{n,k-1}(x) + (n+k)(n+k-1) m_{n,k-2}(x)\big],$$

using the integration by parts, we have

$$\Delta_{n,k}(f) = \int_0^1 f(t) n m''_{n,k+1}(t) dt = n \int_0^1 f''(t) m_{n,k+1}(t) dt.$$

Furthermore, there holds

$$\big(\widetilde{M}_n f\big)''(x) = (1-x)^{-2} \sum_{k=0}^{\infty} n \int_0^1 f''(t) m_{n,k+1}(t) dt m_{n,k}(x).$$

To prove the lemma, for $f \in C^2_{\alpha,\lambda}$, we estimate

$$\left|\varphi^{2+\alpha(\lambda-1)} \big(\widetilde{M}_n f\big)''(x)\right| = \left|\varphi^{\alpha(\lambda-1)}(x) x \sum_{k=1}^{\infty} n \int_0^1 f''(t) m_{n,k+1}(t) dt m_{n,k}(x)\right|$$

$$\leq \|f\|_2 \varphi^{\alpha(\lambda-1)}(x) x \sum_{k=1}^{\infty} n \int_0^1 \varphi^{-[2+\alpha(\lambda-1)]}(t) m_{n,k+1}(t) dt m_{n,k}(x) := H.$$

Noticing that

$$x \frac{n(n+1)}{(n+k+2)(n+k+3)} m_{n,k}(x) = \frac{(n+1)(n+k)(n+k-1)}{(n+k+2)(n+k+3)(n-1)} m_{n-1,k}(x) \varphi^2(x)$$

$$\leq \frac{n+1}{n-1} m_{n-1,k}(x)\varphi^2(x) \leq 3m_{n-1,k}(x)\varphi^2(x), \quad (\ for\ \ n \geq 2),$$

we can deduce

$$H \leq 3\|f\|_2 \varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} C_{n,k+1}^{-1} \int_0^1 \varphi^{-[2+\alpha(\lambda-1)]}(t) m_{n,k+1}(t) dt m_{n-2,k}(x).$$

$$\leq 3\|f\|_2 \varphi^{\alpha(\lambda-1)}(x) \sum_{k=1}^{\infty} \left(C_{n,k+1}^{-1} \int_0^1 t^{-2} m_{n,k+1}(t) dt \right)^{\frac{2-\alpha(1-\lambda)}{4}}$$

$$\cdot \left(C_{n,k+1}^{-1} \int_0^1 (1-t)^{-4} m_{n,k+1}(t) dt \right)^{\frac{2-\alpha(1-\lambda)}{4}} m_{n-2,k}(x).$$

Using the method in the proof of Lemma 2.2, we also get

$$C_{n,k+1}^{-1} \int_0^1 t^{-2} m_{n,k+1}(t) dt \leq C \left(\frac{k}{n+k-2} \right)^{-2},$$

$$C_{n,k+1}^{-1} \int_0^1 (1-t)^{-4} m_{n,k+1}(t) dt \leq C \left(\frac{n-2}{n+k-2} \right)^{-4}.$$

Hence using the Hölder inequality and Lemma 2.3, one has

$$H \leq C\|f\|_2 \varphi^{\alpha(\lambda-1)+2}(x) \sum_{k=1}^{\infty} \left(\frac{k}{n+k-2} \right)^{\frac{\alpha(1-\lambda)-2}{2}}$$

$$\cdot \left(\frac{n-2}{n+k-2} \right)^{\alpha(1-\lambda)-2} m_{n-2,k}(x)$$

$$\leq C\|f\|_2 \varphi^{\alpha(\lambda-1)+2}(x) \left(\sum_{k=1}^{\infty} \left(\frac{k}{n+k-2} \right)^{\alpha(1-\lambda)-2} m_{n-2,k}(x) \right)^{\frac{1}{2}}$$

$$\cdot \left(\sum_{k=1}^{\infty} \left(\frac{n-2}{n+k-2} \right)^{2(\alpha(1-\lambda)-2)} m_{n-2,k}(x) \right)^{\frac{1}{2}}$$

$$\leq C\|f\|_2 \varphi^{\alpha(\lambda-1)+2}(x) \left(\sum_{k=1}^{\infty} \left(\frac{k}{n+k-2} \right)^{-2} m_{n-2,k}(x) \right)^{\frac{2-\alpha(1-\lambda)}{4}}$$

$$\cdot \left(\sum_{k=1}^{\infty} \left(\frac{n-2}{n+k-2} \right)^{-4} m_{n-2,k}(x) \right)^{\frac{2-\alpha(1-\lambda)}{4}}$$

$$\leq C\|f\|_2 \varphi^{\alpha(\lambda-1)+2}(x) x^{-\frac{2-\alpha(1-\lambda)}{2}} (1-x)^{-(2-\alpha(1-\lambda))} \leq C\|f\|_2. \qquad \square$$

Acknowledgment. The authors are extremely thankful to the referees for their useful recommendations and valuable remarks.

References

[1] U. Abel, The complete asymptotic expansion for the Meyer-König and Zeller operators, *J. Math. Anal. Appl.* **208** (1997), 109–119.

[2] U. Abel, The moments for Meyer-König and Zeller operators, *J. Approx. Theory* **82** (1995), 352–361.

[3] A. H. Alkemade, The second moment for the Meyer-König and Zeller operators, *J. Approx. Theory* **40** (1984), 261–273.

[4] M. Becker and R. J. Nessel, A global approximation theorem for Meyer-König and Zeller operators, *Math. Z.* **160** (1978), 195–206.

[5] Z. Ditzian, Direct estimate for Berstein polynomials, *J. Approx. Theory* **79** (1994), 165–166.

[6] Z. Ditzian and V. Totik, Moduli of Smoothness, *Springer-Verlag*, *New York*, 1987.

[7] M. Heilmann, Eigenfunctions of Durrmeyer-type modifications of Meyer-König and Zeller operators, *J. Approx. Theory* **125** (2003), 63–73.

[8] W. Meyer-König and K. Zeller, Bernsteinsche Potenzreihen, *Studia Math.* **19** (1960), 89–84.

[9] F. Taşdelen and A. Erençin, The generalization of bivariate Meyer-König and Zeller operators, *J. Math. Anal. Appl.* **331** (2007), 727–735.

[10] S. Guo and Q. Qi, The moments for Meyer-König and Zeller operators, *Appl. Math. Lett.* **20** (2007), 719–722.

[11] V. Totik, Approximation by Meyer-König and Zeller operators, *Math. Z.* **182** (1983), 425–446.

[12] V. Totik, Uniform approximation by Baskakov and Meyer-König and Zeller operators, *Period. Math. Hungar.* **14** (1983), 209–228.

[13] X. Zeng, Bounds for Bernstein basis functions and Meyer-König and Zeller basis functions, *J. Math. Anal. Appl.* **219** (1998), 364–376.

QIULAN QI
COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE
HEBEI NORMAL UNIVERSITY
SHIJIAZHUANG 050016
PEOPLE'S REPUBLIC OF CHINA

E-mail: qiqiulan@hebtu.edu.cn

JUAN LIU
COLLEGE OF MATHEMATICS AND INFORMATION SCIENCE
HEBEI NORMAL UNIVERSITY
SHIJIAZHUANG 050016
PEOPLE'S REPUBLIC OF CHINA

(Received June 7, 2007; revised September 27, 2007)