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On surjective ring homomorphisms between semi-simple commutative Banach algebras

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Abstract. Let A and B be semi-simple commutative Banach algebras. We give a representation of surjective ring homomorphisms from A onto B in terms of complex ring homomorphisms and injective, continuous and closed mapping between the maximal ideal spaces. As a corollary, we prove that neither the disc algebra $A(\bar{\mathbb{D}})$ nor the commutative Banach algebra of all bounded holomorphic functions $H^\infty(\mathbb{D})$ are ring homomorphic image of any semi-simple commutative regular Banach algebras. Under additional assumptions on the maximal ideal spaces, we also prove automatic linearity of ring homomorphisms.

1. Introduction and results

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over the complex number field $\mathbb{C}$. We say that a mapping $\rho : \mathcal{A} \to \mathcal{B}$ is a ring homomorphism provided that

$$\rho(f+g) = \rho(f) + \rho(g)$$
$$\rho(fg) = \rho(f)\rho(g)$$

for every $f, g \in \mathcal{A}$. By definition, ring homomorphisms need not be linear nor continuous. If, in addition, ρ is homogeneous, that is, $\rho(\lambda f) = \lambda\rho(f)$ for every $\lambda \in \mathbb{C}$ and $f \in \mathcal{A}$, then ρ is a usual homomorphism.

One might expect that ring homomorphisms are quite similar to homomorphisms. In fact, under some additional assumptions, it is known to be true. For

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example, Arnold [1] proved that a ring isomorphism between the Banach algebras of all bounded operators from an infinite dimensional Banach space to another is automatically linear, or conjugate-linear. Unfortunately, ring homomorphisms need not be linear nor conjugate-linear in general. For example, let us consider a ring homomorphism τ from $\mathbb{C}$ to $\mathbb{C}$. For simplicity, we shall call τ a ring homomorphism on $\mathbb{C}$. It is obvious that the zero mapping $\tau(z) = 0$ $(z \in \mathbb{C})$, the identity $\tau(z) = z$ $(z \in \mathbb{C})$ and the complex conjugate $\tau(z) = \overline{z}$ $(z \in \mathbb{C})$ are ring homomorphisms on $\mathbb{C}$. We call them trivial ring homomorphisms on $\mathbb{C}$. In fact, Kestelman [5] proved that there exists a non-trivial ring homomorphism on $\mathbb{C}$. It follows from a result of Charnow [2] that the cardinal number of the set of all non-trivial ring automorphisms on $\mathbb{C}$ is $2^{\mathfrak{c}}$, where $\mathfrak{c}$ denotes the cardinality of continuum. Ring homomorphisms have more surprising feature. Let $\Omega \subset \mathbb{C}$ be a region and let $H(\Omega)$ be the algebra of all holomorphic functions on Ω. In [8] it is proven that there exists an *injective* ring homomorphism from $H(\Omega)$ to $\mathbb{C}$. Thus we may regard $H(\Omega)$ as a subring of $\mathbb{C}$. Thus the study of ring homomorphic image is complicated and interesting.

Let $\bar{\mathbb{D}}$ be the closure of the open unit disc $\mathbb{D}$, and let $\mathbb{T} = \bar{\mathbb{D}} \setminus \mathbb{D}$. Molnár [9] considered ring homomorphic image of commutative C^*-algebras. More explicitly, he proved that the group algebras $L^1(\mathbb{R})$, $L^1(\mathbb{T})$ and the disc algebra $A(\bar{\mathbb{D}})$ are not ring homomorphic images of any commutative C^*-algebras. Let $1 \leq p < \infty$, n a positive integer and let G be a compact abelian group. Takahasi and Hatori [10] proved that $L^1(\mathbb{R}^n)$, $A(\bar{\mathbb{D}})$ and $C^n([a, b])$, the commutative Banach algebra of all n-times continuously differentiable functions on $[a, b]$, are not ring homomorphic image of the L^p-space $L^p(G)$.

The purpose of this paper is to generalize and unify the above results concerning ring homomorphic images. To do this, we will study surjective ring homomorphisms between semi-simple commutative Banach algebras. Kaplansky [4] studied ring isomorphisms between semi-simple Banach algebras. Although a part of Theorem 1.1 below can be deduced from [6, Corollary 2.8], just for the sake of completeness we give a direct proof. In fact, we shall prove that surjective ring homomorphisms are represented by continuous, *injective and closed* mapping between the maximal ideal spaces.

Theorem 1.1. *Let A and B be semi-simple commutative Banach algebras with maximal ideal spaces M_A and M_B, respectively. If $\rho : A \to B$ is a surjective ring homomorphism, then there exist a mapping $\Phi : M_B \to M_A$ and a partitioning $\{M_{-1}, M_1, M_d\}$ of M_B satisfying the following conditions:*

(a) *Φ is an injective, continuous and closed mapping,*

(b) *both M_{-1} and M_1 are clopen, and M_d is at most finite, and*

(c) *for each $\varphi \in M_d$, there exists a non-trivial ring automorphism τ_φ on $\mathbb{C}$ such that*

$$\widehat{\rho(f)}(\varphi) = \begin{cases} \overline{\hat{f}(\Phi(\varphi))} & \varphi \in M_{-1} \\ \hat{f}(\Phi(\varphi)) & \varphi \in M_1 \\ \tau_\varphi(\hat{f}(\Phi(\varphi))) & \varphi \in M_d \end{cases} \tag{1.1}$$

for every $f \in A$, where $\hat{\cdot}$ denotes the Gelfand transform.

As a corollary from Theorem 1.1, we can prove the following two results, which generalize some results in [9, Corollary] and [10, Corollary 4]. Corollary 1.3 (b) is also a generalization of [3, Corollary 3.1]. In fact, Hatori, Ishii, the first and second authors of this paper considered the case where A and B have units.

Corollary 1.2. *Let A be a semi-simple regular commutative Banach algebra and let B be a semi-simple commutative Banach algebra. If there exists a surjective ring homomorphism $\rho : A \to B$, then B is regular.*

Corollary 1.3. *Let A and B be semi-simple commutative Banach algebras. Suppose that the maximal ideal space M_B of B is infinite and connected.*

(a) *If the maximal ideal space M_A of A is discrete, then there is no surjective ring homomorphism from A onto B.*

(b) *If there exists a surjective ring homomorphism $\rho : A \to B$, then ρ is linear or conjugate-linear.*

2. Construction of the mapping Φ

Before proving lemmas, we need a characterization of trivial ring homomorphisms on $\mathbb{C}$. The following result is well-known, so we omit a proof (For a proof, see, for example, [7, Proposition 2.1]).

Proposition 2.1. *Let τ be a ring homomorphism on $\mathbb{C}$. Then each of the following three conditions implies the other two.*

(a) *τ is trivial.*

(b) *There exist $\alpha_0, \beta_0 > 0$ such that $|z| < \alpha_0$ implies $|\tau(z)| \leq \beta_0$.*

(c) *τ is continuous at 0.*

Remark 2.1. By Proposition 2.1, we see that a ring homomorphism τ on $\mathbb{C}$ is non-trivial if and only if the following conditions are satisfied:

for each $\alpha, \beta > 0$, there exists $z \in \mathbb{C}$ with $|z| < \alpha$ but $|\tau(z)| > \beta$.

We shall use this fact several times.

Until the end of this section, A and B denote semi-simple commutative Banach algebras with maximal ideal spaces M_A and M_B, respectively. We also denote by ρ a surjective ring homomorphism from A onto B.

Definition 1. For each φ of M_B, we define the induced mapping ρ_φ from A into $\mathbb{C}$ by

$$\rho_\varphi(f) = \widehat{\rho(f)}(\varphi) \quad (f \in A),$$

where $\hat{\ }$ is the Gelfand transform. Since ρ is surjective, ρ_φ is a surjective ring homomorphism for every $\varphi \in M_B$.

Notation. Let A_e be the commutative Banach algebra obtained by adjunction of a unit element e to A. Here we notice that A_e is well-defined even for unital A. The maximal ideal space M_{A_e} of A_e is the one-point compactification $M_A \cup \{x_\infty\}$ of M_A.

Lemma 2.2. *For each $\varphi \in M_B$, there exists a unique ring homomorphism $\widetilde{\rho_\varphi}$ from A_e onto $\mathbb{C}$ with $\widetilde{\rho_\varphi}|_A = \rho_\varphi$.*

Proof. Take $\varphi \in M_B$. Since ρ_φ is surjective, there exists $a \in A$ with $\rho_\varphi(a) = 1$. Define the mapping $\widetilde{\rho_\varphi}$ from A_e to $\mathbb{C}$ by

$$\widetilde{\rho_\varphi}(f + \lambda e) = \rho_\varphi(f) + \rho_\varphi(\lambda a) \quad (f + \lambda e \in A_e).$$

By definition, $\widetilde{\rho_\varphi}|_A = \rho_\varphi$, and so $\widetilde{\rho_\varphi}$ is surjective since so is ρ_φ. By the definition of $\widetilde{\rho_\varphi}$, it is obvious that $\widetilde{\rho_\varphi}$ is additive. We shall prove that $\widetilde{\rho_\varphi}$ is multiplicative. Take $f + \lambda e, g + \mu e \in A_e$. Since $\rho_\varphi(a) = 1$, we have

$$\rho_\varphi(\lambda\mu a) = \rho_\varphi(\lambda\mu a)\rho_\varphi(a) = \rho_\varphi(\lambda a)\rho_\varphi(\mu a). \tag{2.1}$$

Note also that

$$\rho_\varphi(\mu f) = \rho_\varphi(\mu f)\rho_\varphi(a) = \rho_\varphi(f)\rho_\varphi(\mu a) \tag{2.2}$$

since ρ_φ is multiplicative. By the same reasoning, we have $\rho_\varphi(\lambda g) = \rho_\varphi(g)\rho_\varphi(\lambda a)$. It follows that

$$\begin{aligned}
\widetilde{\rho_\varphi}((f + \lambda e)(g + \mu e)) &= \widetilde{\rho_\varphi}(fg + \mu f + \lambda g + \lambda\mu e) \\
&= \rho_\varphi(fg + \mu f + \lambda g) + \rho_\varphi(\lambda\mu a) \\
&= \rho_\varphi(f)\rho_\varphi(g) + \rho_\varphi(f)\rho_\varphi(\mu a) + \rho_\varphi(g)\rho_\varphi(\lambda a) \\
&\quad + \rho_\varphi(\lambda a)\rho_\varphi(\mu a) \qquad \text{(by (2.1) and (2.2))} \\
&= \{\rho_\varphi(f) + \rho_\varphi(\lambda a)\}\{\rho_\varphi(g) + \rho_\varphi(\mu a)\} \\
&= \widetilde{\rho_\varphi}(f + \lambda e)\,\widetilde{\rho_\varphi}(g + \mu e).
\end{aligned}$$

This proves that ρ_φ is multiplicative. We thus conclude that $\widetilde{\rho_\varphi}$ is a surjective ring homomorphism from A_e onto $\mathbb{C}$ with $\widetilde{\rho_\varphi}|_A = \rho_\varphi$.

Finally, we prove the uniqueness of $\widetilde{\rho_\varphi}$. Let ${\rho_\varphi}^* : A_e \to \mathbb{C}$ be another ring homomorphism with ${\rho_\varphi}^*|_A = \rho_\varphi$. Note, for each $\lambda \in \mathbb{C}$, that

$$ {\rho_\varphi}^*(\lambda e) = {\rho_\varphi}^*(\lambda e)\,\rho_\varphi(a) = {\rho_\varphi}^*(\lambda a) = \rho_\varphi(\lambda a) $$

since $\rho_\varphi(a) = 1$. For each $f + \lambda e \in A_e$, we have

$$ {\rho_\varphi}^*(f + \lambda e) = {\rho_\varphi}^*(f) + {\rho_\varphi}^*(\lambda e) = \rho_\varphi(f) + \rho_\varphi(\lambda a) = \widetilde{\rho_\varphi}(f + \lambda e), $$

which proves the uniqueness. This completes the proof. □

Lemma 2.3. *Let $\widetilde{\rho_\varphi}$ be from Lemma 2.2 for each $\varphi \in M_B$. There exists unique $\psi \in M_{A_e} \setminus \{x_\infty\}$ with $\ker \widetilde{\rho_\varphi} = \ker \psi$. For such ψ, we have $\ker \rho_\varphi = \ker(\psi|_A)$.*

Proof. Take $\varphi \in M_B$. By Lemma 2.2, there is a unique ring homomorphism $\widetilde{\rho_\varphi}$ from A_e onto $\mathbb{C}$ with $\widetilde{\rho_\varphi}|_A = \rho_\varphi$. We show that the kernel $\ker \widetilde{\rho_\varphi}$ is an algebra ideal. Since ρ_φ preserve both additions and multiplications, it is enough to show that $\lambda f \in \ker \widetilde{\rho_\varphi}$ whenever $\lambda \in \mathbb{C}$ and $f \in \ker \widetilde{\rho_\varphi}$. Take $\lambda \in \mathbb{C}$ and $f \in \ker \widetilde{\rho_\varphi}$. Since $\widetilde{\rho_\varphi}(f) = 0$, for $a \in A$ with $\widetilde{\rho_\varphi}(a) \neq 0$, we have

$$ \widetilde{\rho_\varphi}(\lambda f)\widetilde{\rho_\varphi}(a) = \widetilde{\rho_\varphi}(f)\widetilde{\rho_\varphi}(\lambda a) = 0. $$

It follows that $\widetilde{\rho_\varphi}(\lambda f) = 0$ since $\widetilde{\rho_\varphi}(a) \neq 0$. Thus $\lambda f \in \ker \widetilde{\rho_\varphi}$, and so $\ker \widetilde{\rho_\varphi}$ is an algebra ideal of A.

Note that $\ker \widetilde{\rho_\varphi}$ is a proper algebra ideal since $\widetilde{\rho_\varphi}|_A = \rho_\varphi$ is non-zero. There exists $\psi \in M_{A_e}$ with $\ker \widetilde{\rho_\varphi} \subset \ker \psi$. We shall prove that $\ker \widetilde{\rho_\varphi} = \ker \psi$. Take $u_0 \in A_e$ with $u_0 \notin \ker \widetilde{\rho_\varphi}$. Since $\widetilde{\rho_\varphi}$ is surjective, there is $v_0 \in A_e$ such that $\widetilde{\rho_\varphi}(v_0) = 1/\,\widetilde{\rho_\varphi}(u_0)$. Then

$$ \widetilde{\rho_\varphi}(u_0 v_0 - e) = \widetilde{\rho_\varphi}(u_0)\widetilde{\rho_\varphi}(v_0) - \widetilde{\rho_\varphi}(e) = 0, $$

and so $u_0 v_0 - e \in \ker \widetilde{\rho_\varphi} \subset \ker \psi$. Thus we have $\psi(u_0)\psi(v_0) = 1$, which implies $u_0 \notin \ker \psi$. This proves $\ker \psi \subset \ker \widetilde{\rho_\varphi}$, and so $\ker \widetilde{\rho_\varphi} = \ker \psi$.

Since $\widetilde{\rho_\varphi}|_A = \rho_\varphi$, we have

$$ \ker \rho_\varphi = \ker(\widetilde{\rho_\varphi}|_A) = (\ker \widetilde{\rho_\varphi}) \cap A = (\ker \psi) \cap A = \ker(\psi|_A). $$

In particular, $\psi|_A$ is non-zero. Thus $\psi \in M_{A_e} \setminus \{x_\infty\}$. □

Definition 2. By Lemma 2.3, for each $\varphi \in M_B$, there exists a unique element $\Phi(\varphi) \in M_{A_e} \setminus \{x_\infty\}$ with $\ker \widetilde{\rho_\varphi} = \ker \Phi(\varphi)$. We may regard Φ as a mapping from M_B to $M_{A_e} \setminus \{x_\infty\}$.

Definition 3. For each $\varphi \in M_B$, we consider the mapping $\tau_\varphi : \mathbb{C} \to \mathbb{C}$ defined by

$$\tau_\varphi(\lambda) = \widetilde{\rho_\varphi}(\lambda e) \quad (\lambda \in \mathbb{C}),$$

where $\widetilde{\rho_\varphi}$ is from Lemma 2.2.

Lemma 2.4. *For each $\varphi \in M_B$, let τ_φ be from Definition 3. Then τ_φ is a ring automorphism on $\mathbb{C}$ with*

$$\rho_\varphi(f) = \tau_\varphi(\hat{f}(\Phi(\varphi))) \tag{2.3}$$

for every $f \in A$. If, in addition, $\rho_\varphi(f) \neq 0$, then

$$\tau_\varphi(\lambda) = \frac{\rho_\varphi(\lambda f)}{\rho_\varphi(f)} \tag{2.4}$$

for every $\lambda \in \mathbb{C}$.

Proof. Take $\varphi \in M_B$. By the definition of Φ, we have, for each $f \in A$,

$$f - \hat{f}(\Phi(\varphi))e \in \ker \Phi(\varphi) = \ker \widetilde{\rho_\varphi},$$

and so

$$0 = \widetilde{\rho_\varphi}(f) - \widetilde{\rho_\varphi}(\hat{f}(\Phi(\varphi))e) = \rho_\varphi(f) - \tau_\varphi(\hat{f}(\Phi(\varphi))).$$

This proves $\rho_\varphi(f) = \tau_\varphi(\hat{f}(\Phi(\varphi)))$ for every $f \in A$.

Next, we show that τ_φ is a ring automorphism. By the definition of τ_φ, it is obvious that τ_φ is a non-zero ring homomorphism. We see that τ_φ is injective: for if there were $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq \lambda_2$ and $\tau_\varphi(\lambda_1) = \tau_\varphi(\lambda_2)$, then we would have

$$\tau_\varphi(\lambda) = \tau_\varphi(\lambda_1 - \lambda_2)\tau_\varphi\left(\frac{\lambda}{\lambda_1 - \lambda_2}\right) = 0$$

for all $\lambda \in \mathbb{C}$, since $\tau_\varphi(\lambda_1 - \lambda_2) = \tau_\varphi(\lambda_1) - \tau_\varphi(\lambda_2) = 0$. This is a contradiction since τ_φ is non-zero. We need to prove the surjectivity of τ_φ. Since ρ_φ is surjective, for each $\lambda \in \mathbb{C}$, there is $a \in A$ with $\rho_\varphi(a) = \lambda$. By (2.3), we have $\tau_\varphi(\hat{a}(\Phi(\varphi))) = \lambda$, and so τ_φ is surjective. Thus τ_φ is a ring automorphism.

Finally, for each $\lambda \in \mathbb{C}$ and $f \in A$ with $\rho_\varphi(f) \neq 0$, we have

$$\rho_\varphi(\lambda f) = \widetilde{\rho_\varphi}(\lambda f) = \widetilde{\rho_\varphi}(\lambda e)\,\widetilde{\rho_\varphi}(f) = \tau_\varphi(\lambda)\rho_\varphi(f).$$

This proves (2.4), and so the proof is complete. □

Lemma 2.5. *Let Φ be the mapping from Definition 2. Then Φ is injective.*

Proof. Take $\varphi_0, \varphi_1 \in M_B$ with $\varphi_0 \neq \varphi_1$. There is $b \in B$ with $\hat{b}(\varphi_0) = 0$ and $\hat{b}(\varphi_1) = 1$ since B is semi-simple. Choose $a \in A$ so that $\rho(a) = b$; this is possible since ρ is surjective. Then $\rho_{\varphi_0}(a) = \hat{b}(\varphi_0) = 0$ and $\rho_{\varphi_1}(a) = \hat{b}(\varphi_1) = 1$. By Lemma 2.4, we have

$$\tau_{\varphi_0}(\hat{a}(\Phi(\varphi_0)) = 0 \quad \text{and} \quad \tau_{\varphi_1}(\hat{a}(\Phi(\varphi_1)) = 1,$$

where τ_φ ($\varphi \in M_B$) is the mapping from Definition 3. Note that $\tau_\varphi(0) = 0$ and $\tau_\varphi(1) = 1$ for every non-trivial ring homomorphism. Since τ_φ is injective by Lemma 2.4, we have $\hat{a_0}(\Phi(\varphi_0)) = 0$ and $\hat{a_0}(\Phi(\varphi_1)) = 1$. We thus conclude $\Phi(\varphi_0) \neq \Phi(\varphi_1)$, and so Φ is injective. □

Definition 4. We define the subsets M_{-1}, M_1 and M_d of M_B by

$$\begin{aligned} M_{-1} &= \{\varphi \in M_B : \tau_\varphi(\lambda) = \bar{\lambda}\ (\lambda \in \mathbb{C})\}, \\ M_1 &= \{\varphi \in M_B : \tau_\varphi(\lambda) = \lambda\ (\lambda \in \mathbb{C})\} \quad \text{and} \\ M_d &= \{\varphi \in M_B : \tau_\varphi \text{ is non-trivial}\}. \end{aligned}$$

By definition, $\{M_{-1}, M_1, M_d\}$ is a partitioning of M_B, that is, M_{-1}, M_1 and M_d are mutually disjoint subsets of M_B with $M_{-1} \cup M_1 \cup M_d = M_B$.

From Lemma 2.6 to 2.8, $\{M_{-1}, M_1, M_d\}$ will denote the partitioning of M_B from Definition 4.

Lemma 2.6. *Both M_{-1} and M_1 are closed subsets of M_B.*

Proof. We show that $\mathrm{cl}(M_k) \subset M_k$ for $k = \pm 1$, where $\mathrm{cl}(M_k)$ denotes the closure of M_k in M_B. Take $\varphi \in \mathrm{cl}(M_k)$ and let $\{\varphi_\alpha\}$ be a net in M_k converging to φ. Choose $a \in A$ so that $\widehat{\rho(a)}(\varphi) = \rho_\varphi(a) \neq 0$. Since $\widehat{\rho(a)}$ is continuous on M_B, $\rho_{\varphi_\alpha}(a) = \widehat{\rho(a)}(\varphi_\alpha)$ converges to $\rho_\varphi(a) \neq 0$. So, without loss of generality we may assume $\rho_{\varphi_\alpha}(a) \neq 0$ for every α. It follows from (2.4) that

$$\tau_{\varphi_\alpha}(\lambda) = \frac{\rho_{\varphi_\alpha}(\lambda a)}{\rho_{\varphi_\alpha}(a)} \to \frac{\rho_\varphi(\lambda a)}{\rho_\varphi(a)} = \tau_\varphi(\lambda). \tag{2.5}$$

Since $\varphi_\alpha \in M_k$, (2.5) implies that $\tau_\varphi(\lambda) = \bar{\lambda}$ if $k = -1$, and $\tau_\varphi(\lambda) = \lambda$ if $k = 1$. Thus $\varphi \in M_k$ for $k = \pm 1$, and the proof is complete. □

Lemma 2.7. *M_d is an open and at most finite subset of M_B.*

Proof. By Lemma 2.6, $M_d = M_B \setminus (M_{-1} \cup M_1)$ is open. Assume to the contrary that M_d contains a countable subset $\{\varphi_n\}_{n=1}^{\infty}$ with $\varphi_i \neq \varphi_j$ $(i \neq j)$. Set, for each $n \in \mathbb{N}$, the set of all natural numbers, $\psi_n = \Phi(\varphi_n)$. By Lemma 2.5, Φ is injective, and so $\psi_i \neq \psi_j$ $(i \neq j)$. Since $\varphi_n \in M_d$, the ring homomorphism τ_{φ_n} from Definition 3 is non-trivial. For simplicity, we will write τ_n instead of τ_{φ_n}. By (2.3), we have

$$\rho_{\varphi_n}(f) = \tau_n(\hat{f}(\Phi(\varphi_n))) = \tau_n(\hat{f}(\psi_n)) \tag{2.6}$$

for each $f \in A$.

Take $a_1 \in A$ with $\hat{a_1}(\psi_1) = 1$. Since τ_1 is non-trivial, there exists $\lambda_1 \in \mathbb{C}$ with $|\lambda_1| < (2\|a_1\|)^{-1}$ and $|\tau_1(\lambda_1)| > 2$ (cf. Remark 2.1). Set $f_1 = \lambda_1 a_1 \in A$. Then

$$\|f_1\| < 2^{-1} \quad \text{and} \quad |\tau_1(\hat{f}_1(\psi_1))| > 2.$$

By induction, we shall prove that, for each $n \in \mathbb{N}$ with $n \geq 2$, there exists $f_n \in A$ such that

$$\|f_n\| < 2^{-n}, \quad |\tau_n(\hat{f}_n(\psi_n))| > 2^n + \left|\tau_n\left(\sum_{k=1}^{n-1} \hat{f}_k(\psi_n)\right)\right|$$

and that

$$\hat{f}_n(\psi_1) = \hat{f}_n(\psi_2) = \cdots = \hat{f}_n(\psi_{n-1}) = 0.$$

Take $a_2 \in A$ with $\hat{a_2}(\psi_1) = 0$ and $\hat{a_2}(\psi_2) = 1$. Since τ_2 is non-trivial, there exists $\lambda_2 \in \mathbb{C}$ such that

$$|\lambda_2| < \frac{1}{2^2\|a_2\|} \quad \text{and} \quad |\tau_2(\lambda_2)| > 2^2 + |\tau_1(\hat{f}_1(\psi_1))|.$$

Set $f_2 = \lambda_2 a_2 \in A$. Then $\hat{f}_2(\psi_1) = 0$ and $\hat{f}_2(\psi_2) = \lambda_2$. It follows that

$$\|f_2\| < 2^{-2}, \quad \hat{f}_2(\psi_1) = 0 \quad \text{and} \quad |\tau_2(\hat{f}_2(\psi_2))| > 2^2 + |\tau_1(\hat{f}_1(\psi_1))|.$$

Suppose that there are $f_k \in A$ $(k = 2, \cdots, n-1)$ with

$$\|f_k\| < 2^{-k}, \quad \hat{f}_k(\psi_1) = \cdots = \hat{f}_k(\psi_{k-1}) = 0 \quad \text{and}$$

$$|\tau_k(\hat{f}_k(\psi_k))| > 2^k + \left|\tau_k\left(\sum_{j=1}^{k-1} \hat{f}_j(\psi_k)\right)\right|.$$

Choose $a_n \in A$ so that $\hat{a_n}(\psi_n) = 1$ and

$$\hat{a_n}(\psi_1) = \cdots = \hat{a_n}(\psi_{n-1}) = 0.$$

In fact, take $b_i \in A$, for each i $(1 \le i \le n-1)$, with $\hat{b_i}(\psi_i) = 0$ and $\hat{b_i}(\psi_n) = 1$. Then $\Pi_{i=1}^{n-1} b_i \in A$ is the desired element. Since τ_n is non-trivial, there is $\lambda_n \in \mathbb{C}$ with

$$|\lambda_n| < \frac{1}{2^n \|a_n\|} \quad \text{and} \quad |\tau_n(\lambda_n)| > 2^n + \left| \tau_n \left(\sum_{j=1}^{n-1} \hat{f}_j(\psi_n) \right) \right|.$$

Set $f_n = \lambda_n a_n \in A$. Then

$$\|f_n\| < 2^{-n}, \quad \hat{f}_n(\psi_1) = \cdots = \hat{f}_n(\psi_{n-1}) = 0.$$

Since $\hat{a_n}(\psi_n) = 1$, we have $\hat{f}_n(\psi_n) = \lambda_n$, and so

$$|\tau_n(\hat{f}_n(\psi_n))| > 2^n + \left| \tau_n \left(\sum_{j=1}^{n-1} \hat{f}_j(\psi_n) \right) \right| \tag{2.7}$$

as desired.

Since $\|f_n\| < 2^{-n}$, the series $\sum_{n=1}^{\infty} f_n$ converges to an element, say $f_0 \in A$. We have, for each $n \in \mathbb{N}$, $\hat{f}_0(\psi_n) = \sum_{k=1}^{n} \hat{f}_k(\psi_n)$ since $\hat{f}_k(\psi_n) = 0$ for each $k = n+1, n+2, \cdots$. By (2.6), we have, for each $n \in \mathbb{N}$,

$$\begin{aligned} |\rho_{\varphi_n}(f_0)| = |\tau_n(\hat{f}_0(\psi_n))| &= \left| \tau_n \left(\sum_{k=1}^{n} \hat{f}_k(\psi_n) \right) \right| \\ &= \left| \sum_{k=1}^{n} \tau_n \left(\hat{f}_k(\psi_n) \right) \right| \ge |\tau_n(\hat{f}_n(\psi_n))| - \left| \tau_n \left(\sum_{k=1}^{n-1} \hat{f}_k(\psi_n) \right) \right|, \end{aligned}$$

and so, by (2.7),

$$|\widehat{\rho(f_0)}(\varphi_n)| = |\rho_{\varphi_n}(f_0)| > 2^n.$$

Since $\widehat{\rho(f_0)}$ is bounded on M_B, we now reach a contradiction. We thus proved that M_d is at most finite subset of M_B. □

Lemma 2.8. *The mapping $\Phi : M_B \to M_{A_e} \setminus \{x_\infty\}$ is continuous.*

Proof. Let $\varphi_0 \in M_B$ and let $\{\varphi_\alpha\} \subset M_B$ be a net converging to φ_0. We prove that $\Phi(\varphi_\alpha)$ converges to $\Phi(\varphi_0)$. If $\varphi_0 \in M_d$, then $\{\varphi_0\}$ is open by Lemma 2.6 and 2.7. So, we may assume that $\varphi_\alpha = \varphi_0$ for each α. Thus $\Phi(\varphi_\alpha) = \Phi(\varphi_0)$, and so $\Phi(\varphi_\alpha)$ converges to $\Phi(\varphi_0)$.

Next, we consider the case where $\varphi_0 \in M_k$ for $k = \pm 1$. By Lemma 2.6 and 2.7, M_k is clopen of M_B. Thus we may assume $\{\varphi_\alpha\} \subset M_k$ for each α.

By the definition of M_k for $k = \pm 1$, τ_{φ_α} is the complex conjugate for each α when $k = -1$, and τ_{φ_α} is the identity for each α when $k = 1$. Note that $\rho_{\varphi_\alpha}(f)$ converges to $\rho_{\varphi_0}(f)$ for each $f \in A$ since $\widehat{\rho(f)}$ is continuous on M_B. It follows from (2.3) that$\hat{f}(\Phi(\varphi_\alpha))$ converges to $\hat{f}(\Phi(\varphi_0))$ for every $f \in A$. Thus $\hat{u}(\Phi(\varphi_\alpha)) \to \hat{u}(\Phi(\varphi_0))$ for each $u \in A_e$. By the definition of the Gelfand topology, we conclude that $\Phi(\varphi_\alpha)$ converges to $\Phi(\varphi_0)$. $\square$

3. Proofs and application

Proof of Theorem 1.1. Let Φ and $\{M_{-1}, M_1, M_d\}$ be from Definitions 2 and 4, respectively. Then Φ is an injective and continuous mapping by Lemmas 2.5 and 2.8. It follows from Lemmas 2.6 and 2.7 that M_{-1} and M_1 are clopen, and M_d is at most finite. Let τ_φ be from Definition 3 for each $\varphi \in M_d$. By (2.3) and Definition 4, ρ is of the form (1.1).

It remains to be proved that Φ is a closed mapping. We define a mapping $\tilde{\Phi} : M_{B_e} \to M_{A_e}$ by

$$\tilde{\Phi}(\varphi) = \begin{cases} \Phi(\varphi) & \varphi \in M_B \\ x_\infty & \varphi = y_\infty \end{cases}$$

where $\{x_\infty\} = M_{A_e} \setminus M_A$ and $\{y_\infty\} = M_{B_e} \setminus M_B$. Here we notice that for each $f \in A \subset A_e$, $\hat{f}$, as a function on M_{A_e}, is 0 at x_∞. The same remark holds for $b \in B \subset B_e$ and y_∞. We observe that $\tilde{\Phi}$ is continuous: by definition, it is enough to prove the continuity of $\tilde{\Phi}$ at y_∞. Let $\{\varphi_\alpha\} \subset M_{B_e}$ be a net converging to y_∞. By Lemma 2.7, $M_{B_e} \setminus M_d$ is an open neighborhood of y_∞, and so we may assume $\{\varphi_\alpha\} \subset M_{B_e} \setminus M_d$. Take $f \in A$. By the definition of $\tilde{\Phi}$, we have

$$\hat{f}(\tilde{\Phi}(\varphi_\alpha)) = \begin{cases} \hat{f}(\Phi(\varphi_\alpha)) & \varphi_\alpha \in M_B \setminus M_d \\ \hat{f}(x_\infty) = 0 & \varphi_\alpha = y_\infty. \end{cases} \tag{3.1}$$

On the other hand, since $\varphi_\alpha \notin M_d$, it follows from (2.3) that

$$|\widehat{\rho(f)}(\varphi_\alpha)| = |\rho_{\varphi_\alpha}(f)| = \begin{cases} |\hat{f}(\Phi(\varphi_\alpha))| & \varphi_\alpha \in M_B \setminus M_d \\ |\widehat{\rho(f)}(y_\infty)| = 0 & \varphi_\alpha = y_\infty. \end{cases}$$

By (3.1), we have, for each α,

$$|\hat{f}(\tilde{\Phi}(\varphi_\alpha))| = |\widehat{\rho(f)}(\varphi_\alpha)|. \tag{3.2}$$

Since $\widehat{\rho(f)}$ is continuous on M_{B_e}, $\widehat{\rho(f)}(\varphi_\alpha)$ converges to $\widehat{\rho(f)}(y_\infty) = 0$. It follows from (3.2) that $\hat{f}(\tilde{\Phi}(\varphi_\alpha))$ converges to $0 = \hat{f}(\tilde{\Phi}(y_\infty))$. Since $f \in A$ was arbitrary, we see that $\hat{u}(\tilde{\Phi}(\varphi_\alpha))$ converges to $\hat{u}(\tilde{\Phi}(y_\infty))$ for every $u \in A_e$. By the definition of the Gelfand topology, $\tilde{\Phi}(\varphi_\alpha)$ converges to $\tilde{\Phi}(y_\infty)$. We thus conclude that $\tilde{\Phi} : M_{B_e} \to M_{A_e}$ is continuous.

Take a closed subset F of M_B. Then $F \cup \{y_\infty\} \subset M_{B_e}$ is compact. Since $\tilde{\Phi}$ is continuous on M_{B_e}, $\tilde{\Phi}(F \cup \{y_\infty\}) = \Phi(F) \cup \{x_\infty\}$ is compact in M_{A_e}, and so $\Phi(F) \subset M_{A_e} \setminus \{x_\infty\}$ is closed in M_A. This proves that Φ is a closed mapping. □

Recall that a commutative Banach algebra A is *regular* if and only if for each pair F, ψ_0 of closed subset $F \subset M_A$ and $\psi_0 \in M_A \setminus F$, there exists $f \in A$ with $\hat{f}(\psi_0) = 1$ and $\hat{f}(\psi) = 0$ for every $\psi \in F$.

Proof of Corollary 1.2. Take $\varphi_0 \in M_B$ and closed $F \subset M_B$ with $\varphi_0 \notin F$. Let Φ be an injective and closed mapping from Theorem 1.1. Then $\Phi(F) \subset M_{A_e} \setminus \{x_\infty\}$ is closed with $\Phi(\varphi_0) \notin \Phi(F)$. Since A is regular, there exists $f_0 \in A$ with $\hat{f}_0(\Phi(\varphi_0)) = 1$ and $\hat{f}_0(\Phi(\varphi)) = 0$ for every $\varphi \in F$. Recall that if τ_φ is a non-trivial ring homomorphism, then $\tau_\varphi(r) = r$ for every $r \in \mathbb{Q}$ and $\varphi \in M_B$. By (2.3), we have $\widehat{\rho(f_0)}(\varphi_0) = \rho_{\varphi_0}(f_0) = 1$ and $\widehat{\rho(f_0)}(\varphi) = \rho_\varphi(f_0) = 0$ for every $\varphi \in F$, and so B is regular. □

Proof of Corollary 1.3. (a) Assume to the contrary that there is a surjective ring homomorphism $\rho : A \to B$. Let Φ be from Theorem 1.1. Then M_B is homeomorphic to $\Phi(M_B) \subset M_A$. By hypothesis, M_A is discrete, and so is M_B. Now we reach a contradiction since M_B is infinite and connected.

(b) Let $\{M_{-1}, M_1, M_d\}$ be from Theorem 1.1. Then M_{-1}, M_1 are clopen, and M_d is at most finite. Since M_B is assumed to be infinite and connected, it follows that $M_B = M_{-1}$, or $M_B = M_1$. So, by Theorem 1.1, there exists an injective, continuous and closed mapping $\Phi : M_B \to M_A$ with $\widehat{\rho(f)}(\varphi) = \overline{\hat{f}(\Phi(\varphi))}$ for every $f \in A$ and $\varphi \in M_B$, or $\widehat{\rho(f)}(\varphi) = \hat{f}(\Phi(\varphi))$ for every $f \in A$ and $\varphi \in M_B$. Since B is semi-simple, we have that ρ is conjugate-linear, or linear, respectively. □

Example 1. Let $\mathbb{D}$ and $\bar{\mathbb{D}}$ be the open unit disc and the closure of $\mathbb{D}$, respectively. Let $A(\bar{\mathbb{D}})$ be the disc algebra, that is, the uniform algebra of all complex-valued continuous functions on $\bar{\mathbb{D}}$, which are holomorphic in $\mathbb{D}$. Let $H^\infty(\mathbb{D})$ be the commutative Banach algebra of all bounded holomorphic functions on $\mathbb{D}$. Neither $A(\bar{\mathbb{D}})$ nor $H^\infty(\mathbb{D})$ are regular. By Corollary 1.2, both $A(\bar{\mathbb{D}})$ and $H^\infty(\mathbb{D})$ can not be the ring homomorphic images of any semi-simple regular commutative Banach algebra A. The case where $A = C_0(X)$ was proved by Molnár [9, Corollary].

Example 2. Let $n \in \mathbb{N}$ and let $C^n([a,b])$ be the set of all n-times continuously differentiable complex-valued functions on a closed interval $[a,b]$. Then $C^n([a,b])$ is a semi-simple commutative Banach algebra with respect to the pointwise operations and the norm $\|f\|_n = \sum_{k=0}^n \left\|f^{(k)}\right\|_\infty /k!$ for $f \in C^n([a,b])$.

If ρ is a surjective ring homomorphism from $C^n([a,b])$ onto itself, then ρ is of the form

$$\rho(f)(x) = \overline{f(\Phi(x))} \quad (f \in C^n([a,b]),\ x \in [a,b]), \tag{3.3}$$

or

$$\rho(f)(x) = f(\Phi(x)) \quad (f \in C^n([a,b]),\ x \in [a,b]). \tag{3.4}$$

Here, $\Phi \in C^n([a,b])$ is injective and closed. For if ρ is a surjective ring homomorphism from $C^n([a,b])$ onto itself, then by the Proof of Corollary 1.3 (b), there exists an injective, continuous and closed mapping Φ from $[a,b]$ into itself such that ρ is of the form (3.3), or (3.4). If we take $f = \mathrm{Id}$, the identity function, then we have $\Phi \in C^n([a,b])$.

Example 3. Let $1 \le p \le \infty$ and let G be a compact abelian group. Then the L^p-space $L^p(G)$ is a commutative Banach algebra with respect to convolution as a multiplication. The maximal ideal space of $L^p(G)$ is the dual group $\hat{G}$ of G for each $1 \le p \le \infty$. Let B be a semi-simple commutative Banach algebra with infinite and connected maximal ideal space. By Corollary 1.3 (a), B can not be the ring homomorphic image of $L^p(G)$ since $\hat{G}$ is discrete. The case where $B = L^1(\mathbb{R}^n), A(\bar{\mathbb{D}}), C^n([a,b])$ was obtained by [10, Corollary 4].

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