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# On the Browder essential spectrum of a linear relation

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**Abstract.** For a closed linear relation on a Banach space the concepts of Browder and Browder essential spectrum are introduced and studied. If a densely defined closed linear relation $T$ has a trivial singular chain, then $T$ is Browder if and only if $T = B+K$, where $B$ is a bijective linear relation, and $K$ belongs to set $\mathcal{K}(X)$ of everywhere defined single valued compact linear operators, and left commutes with $T$. This is used to prove that the Browder essential spectrum coincides with the set $\cap \{\sigma(T+K) : K \in \mathcal{K}(X)$ and $KT \subset TK\}$.

## 1. Introduction

We will denote the set of nonnegative integers by $\mathbb{N}$. Let $X$ denote a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A multivalued linear operator on $X$ or simply a linear relation on $X$, $T : X \to X$ is a mapping from a subspace $D(T) \subset X$, called the domain of $T$, into the collection of nonempty subsets of $X$ such that $T(\alpha x_1 + \beta x_2) = \alpha T x_1 + \beta T x_2$ for all nonzero $\alpha$, $\beta$ scalars and $x_1$, $x_2 \in D(T)$. If $T$ maps the points of its domain to singletons, then $T$ is said to be a single valued linear operator or simply an operator. We denote the class of linear relations on $X$ by $LR(X)$.

A linear relation $T$ in $X$ is uniquely determined by its graph, $G(T)$ which is defined by $G(T) := \{(x, y) \in X \times X : x \in D(T),\ y \in Tx\}$. Let $T \in LR(X)$. The inverse of $T$ is the linear relation $T^{-1}$ given by $G(T^{-1}) := \{(y, x) : (x, y) \in G(T)\}$. The subspace $T^{-1}(0)$ is denoted by $N(T)$ and $T$ is called injective if

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$N(T) = \{0\}$, that is, if $T^{-1}$ is a single valued linear operator. The range of $T$ is the subspace $R(T) := T(D(T))$ and $T$ is called surjective if $R(T) = X$. When $T$ is injective and surjective we say that $T$ is bijective. We write $\alpha(T) := \dim N(T)$; $\beta(T) := \dim X/R(T)$ and the index of $T$, $k(T)$, is defined by $k(T) := \alpha(T) - \beta(T)$ provided $\alpha(T)$ and $\beta(T)$ are not both infinite. Let $M$ be a subspace of $X$ such that $M \cap D(T) \neq \emptyset$. Then the linear relations $T\mid_M$ and $T_M$ are given by $G(T\mid_M) := \{(x, y) \in G(T) : x \in M\}$ and $G(T_M) := \{(x, y) \in G(T) : x, y \in M\}$ respectively.

For $T, S \in LR(X)$ and $\lambda \in \mathbb{K}$, the linear relations $T + S$, $T \widehat{+} S$, $\lambda T$ and $TS$ are defined by $G(T+S) := \{(x, y+z) : (x, y) \in G(T),\ (x, z) \in G(S)\}$, $G(T \widehat{+} S) := \{(x+u, y+v) : (x, y) \in G(T), (u, v) \in G(S)\}$, $G(\lambda T) := \{(x, \lambda y) : (x, y) \in G(T)\}$ and $G(TS) := \{(x, y) : \exists z \in X,\ (x, z) \in G(S), (z, y) \in G(T)\}$ respectively. Since the composition of linear relations is clearly associative, for all integer $n$, $T^n$ is defined as usual with $T^0 = I$ and $T^1 = T$. The notation $T \subset S$ means that $G(T) \subset G(S)$. We say that $T$ has a trivial singular chain manifold if $R_c(T) = \{0\}$ where $R_c(T) := (\cup_{n=1}^{\infty} N(T^n)) \cap (\cup_{n=1}^{\infty} T^n(0))$.

Suppose that $X$ is a normed space and $T \in LR(X)$. Let $Q_T$ denote the quotient map from $X$ onto $X/\overline{T(0)}$. Clearly $Q_T T$ is an operator. We say that $T$ is closed if its graph is a closed subspace of $X \times X$, continuous if $\|T\| := \|Q_T T\| < \infty$, open if its inverse is continuous equivalently if its minimum modulus $\gamma(T)$ is a positive number, where $\gamma(T) := \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\|, x \in D(T)\}$, Fredholm, denoted $T \in \phi(T)$, if it is closed with $\dim N(T) < \infty$ and $R(T)$ is a closed finite codimensional subspace of $X$.

If $X$ is a complex normed space, then the resolvent set of $T$ is the set $\rho(T) := \{\lambda \in \mathbb{C} : \lambda - T$ is injective, open and has dense range $\}$. It is clear from the Closed Graph Theorem for linear relations [5, III.5.3] that if $T$ is closed and $X$ is complete then $\rho(T) = \{\lambda \in \mathbb{C} : \lambda - T$ is bijective $\}$. The spectrum of $T$ is the set $\sigma(T) := \mathbb{C} \backslash \rho(T)$.

An underlying motivation for the introduction of multivalued linear operators into Functional Analysis by J. von Neumann [12] was to aid the investigation of differential equations governed by non densely defined operators. The conjugate of such operators are linear relations. Linear relations are more convenient because one can always define the inverse, the closure and the completion of a linear relation. Interesting works on multivalued linear operators include the treatise on partial differential relations by Gromov [9], the application of multivalued methods to solution of differential equations by Favini and Yagi [6], the development of fixed point theory for linear relations to the existence of mild solutions of quasi-linear differential inclusions of evolution and also to many problems of fuzzy theory (see, for example [1], [8], [11] and [14]) and several papers on semi-Fredholm linear relations and other classes related to them (see, for example [2], [3] and [4]).

This paper deals with the Browder linear relations in a Banach space. The results obtained in this paper are mainly based on recent developments of SANDOVICI, DE SNOO and WINKLER [15]. We start the Section 2 recalling a lemma due to CROSS [5] related to linear combinations and compositions of linear relations and we present some definitions and propositions concerning the ascent and descent of a linear relation in a vector space that we shall need to obtain the main theorems. These definitions and propositions, together with their proofs, can be found in [15].

Section 3 contains the main results. We recall that if $T$ is a bounded operator in a Banach space $X$ it follows from [17, 3.1 and 3.3] that $T$ is Browder if and only if $T$ can be written in the form $T = A + K$, where $A$ is an isomorphism and $K$ belongs to the set $\mathcal{K}(X)$ of everywhere defined single valued compact linear operators, and commutes with $T$. Also, it is well known (see, for example [13]) that if $X$ is a complex Banach space then the Browder essential spectrum of $T$, $\sigma_{\mathcal{B}}(T)$, is a closed subset of $\mathbb{C}$ and $\sigma_{\mathcal{B}}(T) = \cap\{\sigma(T + K) : KT = TK$ and $K \in \mathcal{K}(X)\}$. In the next Section 3 we will show that results of the type mentioned above can be extended to closed multivalued linear operators under certain conditions.

## 2. Algebraic properties of a linear relation in a vector space

Throughout this section $T$ will be denote a linear relation in a vector space $X$.

We will make use of the following lemma concerning the laws governing the operations of addition and scalar multiplication in $LR(X)$ combined with the operations of composition and inversion, in particular the right and left distributive laws.

**Lemma 1** ([5, I.4.2]). *Let $R, S, T \in LR(X)$. Then*

(i) $\lambda(ST) = (\lambda S)T = S(\lambda T)$, $\lambda \in \mathbb{K}\backslash\{0\}$.

(ii) *If $S \subset T$ then $SR \subset TR$.*

(iii) *$(R + S)T \subset RT + ST$ with equality if $T$ is single valued.*

(iv) *$TR + TS \subset T(R + S)$ with equality if $D(T)$ is the whole space.*

Two examples which show that equality may not hold in Lemma 1 (iii) and (iv) have been constructed by Cross [5, I.4.3].

*Definition 2.* The ascent and descent of $T$ are defined by

$$a(T) := \min\{p \in \mathbb{N} : N(T^p) = N(T^{(p+1)})\}$$
$$d(T) := \min\{q \in \mathbb{N} : R(T^q) = R(T^{(q+1)})\}$$

respectively, whenever these minima exist. If no such numbers exist the ascent and descent of $T$ are defined to be $\infty$.

In [15], the authors proved that many of the entirely algebraic results concerning the ascent, descent, nullity and defect of operators in vector spaces remain valid in the context of linear relations under the additional condition that the linear relation has a trivial singular chain manifold. Some algebraic properties described in [15] that we shall need to obtain the main theorems of Section 3 are recalled.

**Lemma 3.** (i) *Let $\eta \in \mathbb{K} \setminus \{0\}$. Then $N((\eta - T)^n) \subset R(T^m)$ for all nonnegative integers $n$, $m$.*

(ii) *Let $n \in \mathbb{N}$. If $\alpha(T) < \infty$ is $\alpha(T^n) \leq n\alpha(T)$ and if $\beta(T) < \infty$ then $\beta(T^n) \leq n\beta(T)$.*

**Proposition 4.** (i) *Let $n, m \in \mathbb{N}$ and assume that $R_c(T) = \{0\}$. Then the linear relation $\psi$ from $N(T^{(n+m)})/N(T^n)$ to $R(T^n) \cap N(T^m)$ defined by $\psi[x] := T^n x$ is bijective and single valued.*

(ii) *If $N(T) \cap R(T^p) = \{0\}$ for some $p \in \mathbb{N}$, then $a(T) \leq p$ and $R_c(T) = \{0\}$. Conversely if $R_c(T) = \{0\}$ and $a(T) \leq p$ for some nonnegative integer $p$, then $N(T^m) \cap R(T^p) = \{0\}$ for all positive integer $m$.*

(iii) *If $R_c(T) = \{0\}$, $\alpha(T) = \beta(T) < \infty$ and $p = a(T) < \infty$, then $a(T) = d(T)$ and $X = R(T^p) \oplus N(T^p)$.*

*Definition 5.* Assume that $M$ and $N$ are two complementary subspaces of $X$. The linear relation $T$ is said to be completely reduced by the pair $(M, N)$, denoted $T = T_M \otimes T_N$, if it can decomposed as $T = T_M \widehat{+} T_N$ and $G(T_M) \cap G(T_N) = \{(0, 0)\}$.

In the case of operators the concept of complete reducibility goes back at least to Taylor (see, [16]) whose definition is slightly different but coincides with Definition 5 if $T$ is a single valued linear operator.

If $T$ is completely reduced by the pair $(M, N)$, then it is easy to see that $D(T) = D(T_M) \oplus D(T_N)$; $N(T) = N(T_M) \oplus N(T_N)$; $R(T) = R(T_M) \oplus R(T_N)$ and $T(0) = T_M(0) \oplus T_N(0)$.

**Proposition 6.** *If $X = R(T^p) \oplus N(T^p)$ for some nonnegative integer $p$, then $T$ is completely reduced by the pair $(R(T^p), N(T^p))$.*

**Proposition 7.** *Assume that $M$ and $N$ are two complementary subspaces of $X$ such that $T$ is completely reduced by the pair $(M, N)$. Furthermore, assume that $N \subset D(T)$, $\dim N < \infty$, $T_N$ is single valued and $T_M$ is a bijective linear relation. Let $P$ and $Q$ be the projections of $X$ onto $M$ along $N$ and onto $N$ along $M$, respectively and define the linear relations $A$ and $B$ by $G(A) := \{(x, Py - Qx) : (x, y) \in G(T)\}$ and $B := (T_N + I)Q$. Then $B$ is an everywhere defined single valued linear operator in $X$ with $\dim R(B) < \infty$, $A$ is bijective, $BA \subset AB$ and $T = A + B$.*

## 3. Browder essential spectrum of a closed linear relation

It is well known that if $n \in \mathbb{N}$ and $T$ is a bounded Fredholm operator from a Banach space $X$ into $X$ then $T^n$ is a Fredholm operator with $k(T^n) = nk(T)$. The following result gives conditions under which this property remains valid for closed linear relations and it will be used to obtain a characterisation of Browder linear relations (Theorem 10 below).

**Proposition 8.** *Let $n \in \mathbb{N}$ and let $T \in \phi(X)$ where $X$ is a Banach space. We have:*

(i) *$T^n \in \phi(X)$.*

(ii) *If $T$ is densely defined and $R_c(T) = \{0\}$, then $k(T^n) = nk(T)$.*

Proof. (i) We first note the following elementary property

$(*)$ If $M$ and $N$ are subspaces of $X$ such that $M$ is closed and $M \subset N$, then $N$ is closed in $X$ if and only if $N/M$ is closed in $X/M$.

Since $Q_T T$ is a closed single valued with $T(0)$ closed [5, II.5.3] we have from $(*)$ that $R(Q_T T)$ is closed and since $N(Q_T T) = N(T)$ ([5, II.3.4]) is finite dimensional it follows from [7, IV.2.9] that $Q_T T R(T)$ is closed and thus applying again $(*)$ we obtain that $R(T^2)$ is closed. Also $T^2$ is a closed linear relation by virtue of [5, II.5.1 and III.5.3] and since $\dim N(T^2)$ and $\dim X/R(T^2)$ are both finite (Lemma 3 (ii)), is $T^2 \in \phi(X)$. Now continuing in this way we deduce that $T^n \in \phi(X)$.

(ii) By Proposition 4 (i), the linear relation $\psi$ from $N(T^2)/N(T)$ to $R(T) \cap N(T)$ defined by $\psi[x] := Tx$ is a bijective single valued linear operator. Hence, put $N_1 := R(T) \cap N(T)$,

$$\alpha(T^2) = \alpha(T) + \dim N_1. \tag{1}$$

Let $N_2$ be a subspace of $N(T)$ such that $N(T) = N_1 \oplus N_2$. Then

$$\alpha(T) = \dim N_1 + \dim N_2. \tag{2}$$

Now for $y \in Tx \cap N_2$, we have $y \in N_1 \cap N_2 = \{0\}$. Hence $R(T) \cap N_2 = \{0\}$.

Since $\beta(T)$ and $\dim N_2$ are both finite, $R(T)$ and $R(T) \oplus N_2$ are closed and as by hypothesis $D(T)$ is dense in $X$ it follows from [7, IV. 2.8] that

$$X = R(T) \oplus N_2 \oplus N_3 \tag{3}$$

where $N_3$ is some finite dimensional subspace of $D(T)$ so that

$$D(T) = (R(T) \cap D(T)) \oplus N_2 \oplus N_3; \quad \beta(T) = \dim N_2 + \dim N_3 \tag{4}$$

and $T$ is injective on $N_3$.

Let $y \in R(T)$. Then by (4) there exist $x_1 \in R(T) \cap D(T)$, $x_2 \in N_2$, $x_3 \in N_3$ for which $y \in Tx_1 + Tx_2 + Tx_3 \subset R(T^2) + TT^{-1}(0) + TN_3 = R(T^2) + T(0) + TN_3 \subset R(T^2) + TN_3$. Now, let $z \in R(T^2) \cap TN_3$. We have $z \in Tm$ and $z \in Tn$ where $m \in R(T)$ and $n \in N_3$. Then $0 \in T(m - n)$, so that $m - n \in N(T) \subset R(T) \oplus N_2$. Therefore $n \in (R(T) \oplus N_2) \cap N_3 = \{0\}$ (by (4)). In consequence

$$R(T) = R(T^2) + TN_3, \quad T(0) = R(T^2) \cap N_3. \tag{5}$$

Also, $\dim N_3 = \dim TN_3$ by the injectivity of $T$ on $N_3$ and [5, I.6.4]. This fact combined with (5) yield

$$\beta(T) + \dim N_3 = \beta(T^2). \tag{6}$$

Now, adding (1), (2), (4) and (6) gives $k(T^2) = 2k(T)$ and proceeding in this way we obtain the desired assertion (ii). □

In the sequel $X$ will denote a Banach space and $T$ will always denote an element of $CR(X)$, the set of all closed linear relations on $X$.

If $T$ is a single valued Browder operator on a Banach space (that is, $T$ is Fredholm of finite ascent and descent) it follows from a classical result of [10, 38.5] that $T$ has index zero. However, this property is not valid in the context of linear relations (see, [15]). These observations suggest the following notion.

*Definition 9.* We say that a linear relation $T$ on $X$ is Browder, denoted $T \in \mathcal{B}CR(X)$, if $T$ is Fredholm of index zero and has finite ascent and descent.

**Theorem 10.** *Let $T \in CR(X)$ have a trivial singular chain manifold. Consider the following properties:*

(i) $T \in \mathcal{B}CR(X)$.

(ii) *There exist linear relations $A$, $B$ such that $T = A + B$ with $A$ a bijection, $B \in \mathcal{K}(X)$ and $BT \subset TB$.*

*Then* (i) *implies* (ii). *If $T$ is densely defined, then* (i) *and* (ii) *are equivalent.*

Proof. (i) $\Rightarrow$ (ii) Assume that $T$ is Browder. According Proposition 4 (iii), there is a nonnegative integer $p$ such that $p := a(T) = d(T)$ and $X = R(T^p) \oplus N(T^p)$. Then by Proposition 6 $T$ is completely reduced by the pair $(R(T^p), N(T^p))$, that is, $T = T_{R(T^p)} \oplus T_{N(T^p)}$. Also $R(T^p)$ and $N(T^p)$ are both closed subspaces with $\dim N(T^p) < \infty$ (Proposition 8 (i)), $T_{R(T^p)}$ is bijective since $p = d(T)$ implies that $R(T^p) = R(T_{R(T^p)})$ and $N(T_{R(T^p)}) = N(T) \cap R(T^p) = \{0\}$ (Proposition 4 (ii)) and $T_{N(T^p)}$ is single valued since if $y \in N(T_{N(T^p)}(0)$ then clearly $y \in T_{R(T^p)}(0) \oplus T_{N(T^p)}(0) = T(0) \subset R(T^p)$. Hence $y \in N(T^p) \cap R(T^p) = \{0\}$ (Proposition 4 (ii)).

In this situation, we can apply Proposition 7 to obtain two linear relations $A$ and $B$, with $A$ bijective, $B$ bounded finite rank operator such that $BA \subset AB$ and $T = A + B$. Furthermore, $BT = B(A + B) = BA + BB$ (Lemma 1 (iv)) $\subset AB + BA = (A + B)B$ (Lemma 1 (iii)) $= TB$. Therefore (ii) holds, as desired.

(ii)$\Rightarrow$ (i) Suppose $T = A + B$ with $T$, $A$ and $B$ as in the hypothesis of (ii). Then

$$T \text{ is a Fredholm linear relation of index zero.} \tag{7}$$

Follows immediately from [4], [9].

$$BA \subset AB;\ TA \subset AT \text{ and } BT^n \subset T^nB \quad \text{for all positive integer } n. \tag{8}$$

Indeed, $BA = B(T - B) = BT - BB$ (Lemma 1 (i) and (iv)) $\subset TB - BB = (T - B)B$ (Lemma 1 (i) and (iii)) $= AB$. This fact implies that $TA \subset AT$ since $TA = (A + B)A \subset AA + BA$ (Lemma 1 (iii)) $\subset AA + AB \subset A(A + B)$ (Lemma 1 (iv)) $= AT$. We prove that $BT^n \subset T^nB$ by induction. For $n = 1$ is trivial. Assume the property to be valid for $n$. Then $BT^{(n+1)} = BT^nT \subset T^nBT$ (by the induction hypothesis and Lemma 1 (ii)) $\subset T^{(n+1)}B$. Hence (8) holds.

$$R(T^n) \subset AR(T^n) \quad \text{for all positive integer } n. \tag{9}$$

The proof will be given by induction. First consider the case $n = 1$. Since $I_X \subset AA^{-1}$ (as $A$ is surjective) is $T = TI_X \subset TAA^{-1}$ and thus $R(T) \subset$

$R(TAA^{-1}) = TAR(A^{-1}) = TAD(A) = TAD(T) \subset ATD(T)$ (by (8)) $= AR(T)$. Assume that the property is satisfied for $n$. Then $R(T^{(n+1)}) = TR(T^n) \subset TAR(T^n) \subset ATR(T^n)$ (by (8)) $= AR(T^{(n+1)})$. Hence (9) holds.

We observe that since $A$ is closed by virtue of [5, II.5.16] and it is bijective by hypothesis, the Closed Graph Theorem for linear relations [5, III.5.3] assures that $A$ is open, so that, there is a positive number $\gamma$ for which $\gamma\|x\| \leq \|Ax\|, x \in D(A) = D(T)$. Assume that $x \in D(A)$ and $z \in R(T^n)$. Then $z \in AR(T^n)$ (by (9)) and hence there exists $y \in R(T^n)$ with $z \in Ay$. Thus we have that $\gamma d(x, R(T^n)) \leq \gamma\|x - y\| \leq \|A(x - y)\| := \|Q_A A(x - y)\| = \|Q_T Ax - Q_T z\|$ (the last equality is obtained upon noting that $z \in Ay \Leftrightarrow Ay = z + A(0)$ ([5, I.2.8]) and $T(0) = A(0)$ ) and since this holds for all $z \in R(T^n)$, we obtain that

$$\begin{gathered}\text{There is } \gamma > 0 \text{ such that } \gamma d(x, R(T^n)) \leq d(Q_T Ax, Q_T R(T^n)) \\ \text{for all } x \in D(T).\end{gathered} \tag{10}$$

Assume that $T$ had infinite descent. Then there would be a bounded sequence $(x_n)$ in $R(T^n)$ for which $1 \leq d(x_n, R(T^{(n+1)}))$. Assume $m > n > 0$. Then $Tx_n \subset TR(T^n) = R(T^{(n+1)})$; $Bx_m \in BR(T^m) = R(BT^m) \subset R(T^m B)$ (by (8)) $\subset R(T^m) \subset R(T^{(n+1)})$ and $Bx_n - Bx_m + A(0) = Bx_n - Bx_m + Ax_n - Ax_n$, so that $Q_T Bx_n - Q_T Bx_m = -Q_T Ax_n + Q_T Tx_n - Q_T Bx_m$. These properties combined with (10) yield

$$\begin{aligned}\gamma &\leq \gamma d(x_n, R(T^{(n+1)})) \\ &\leq d(Q_T Ax_n, Q_T R(T^{(n+1)})) \leq \|Q_T Bx_n - Q_T Bx_m\|\end{aligned} \tag{11}$$

which contradicts the compactness of $Q_T B$, so $T$ must have finite descent.

Let $q := d(T) < \infty$ and suppose that $T$ is densely defined. Then as $T$ is a Fredholm linear relation with index zero (by (7)) it follows from Proposition 8 (ii) that $k(T^q) = 0$ and consequently $q = a(T)$. Therefore the result is proved. □

In the rest of this section we assume that $X$ is a complex Banach space.

The class $\mathcal{B}CR(X)$ motivates the corresponding Browder essential spectrum defined as follows:

*Definition 11.* The Browder essential spectrum of $T \in CR(X)$ is the set

$$\sigma_{\mathcal{B}}(T) := \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{B}CR(X)\}$$

(As it is usual, we write $\lambda - T := \lambda I_X - T$, $\lambda \in \mathbb{C}$).

It is very well known that the Browder essential spectrum of a bounded operator is a closed subset of the complex plane. The following result indicates that again the additional condition "$T$ has a trivial singular chain manifold" permits to prove the validity of the above property in our general situation.

**Theorem 12.** *Let $T \in CR(X)$ such that $R_c(T) = \{0\}$. Then the Browder essential spectrum of $T$ is a closed subset of $\mathbb{C}$.*

Proof. Let $\lambda \in \mathbb{C}\backslash\sigma_{\mathcal{B}}(T)$. We shall assume without loss of generality that $\lambda = 0$. Thus $T$ is Fredholm of index zero and there is $p \in \mathbb{N}$ such that $p = a(T) = d(T)$ (Proposition 4 (iii)).

We first prove that

$$X = R(\eta - T) + R(T^p), \quad \eta \in \mathbb{C}\backslash\{0\} \tag{$\odot$}$$

This assertion is obtained upon noting that $X = R(T^p) \oplus N(T^p)$ (Proposition 4 (iii)) and $N(T^p) = N((\eta - (\eta - T))^p) \subset R(\eta - T)$ (Lemma 3 (i)).

Now, for $\eta \neq 0$ we define $T_o := T \mid_{R(T^p)}$ and $\eta_o := \eta I \mid_{R(T^p)}$. Then since $T$ is closed and $R(T^p)$ is closed (Proposition 8 (i)), $T_o$ is a closed linear relation. Moreover, $T_o$ is injective (as $N(T_o) = N(T) \cap R(T^p) = \{0\}$ by Proposition 4 (ii)), $R(T_o) = R(T^p)$ (as $d(T) = p < \infty$) and thus by the Closed Graph Theorem for linear relations [5, III.5.4], $T_o$ is open, that is, $\gamma(T_o) > 0$.

Let $0 <\mid \eta \mid< \gamma(T_o)$. We have that $\alpha(\eta_o - T_o) \leq \alpha(T_o)$ ([5, III.7.4]) and since $N(\eta_o - T_o) = N(\eta - T) \cap R(T^p)$ with $N(\eta - T) \subset R(T^p)$ (Lemma 3 (i)), it follows that $a(\eta - T) = \alpha(\eta - T) = 0$. Also, as $R(T^p) = (\eta - T)R(T^p) \subset R(\eta - T)$, the property ($\odot$) yields immediately to $X = R(\eta - T)$, that is, $d(\eta - T) = \beta(\eta - T) = 0$. Consequently, $\sigma_{\mathcal{B}}(T)$ is closed, as desired. □

*Remark 13.* We note that there exists a closed densely defined operator $T$ (so that $R_c(T) = \{0\}$) such that $\sigma(T) = \emptyset$ (see, [5, VI.2.7]) and hence such that $\sigma_{\mathcal{B}}(T) = \emptyset$.

**Theorem 14.** *Let $T \in CR(X)$ be densely defined such that $R_c(T) = \{0\}$. Then*

$$\sigma_{\mathcal{B}}(T) = \cap\{\sigma(T + K) : K \in \mathcal{K}(X) \text{ and } KT \subset TK\}.$$

Proof. We observe that by [15, 7.1]

$$\text{For all scalar } \eta,\ R_c(\eta - T) = \{0\} \text{ if and only if } R_c(T) = \{0\}. \tag{$\dagger$}$$

Let $\lambda \notin \cap\{\sigma(T + K) : K \in \mathcal{K}(X)$ and $KT \subset TK\}$. Then there exists $K \in \mathcal{K}(X)$ and $KT \subset TK$ for which $\lambda \in \rho(T + K)$. Thus $\lambda - (T + K)$ is bijective, $\lambda - T = \lambda - (T + K) + K$ is clearly a closed densely defined linear relation such that $K(\lambda - T) \subset (\lambda - T)K$ and $R_c(\lambda - T) = \{0\}$ and thus by Theorem 10 (ii) $\Rightarrow$ (i) we deduce that $\lambda - T$ is a Browder linear relation.

The other inclusion follows immediately from ($\dagger$) together Theorem 10 (i) $\Rightarrow$ (ii). □

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