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Geometric properties of generalized Bessel functions

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Abstract. In this paper our aim is to establish some geometric properties (like univalence, starlikeness, convexity and close-to-convexity) for the generalized Bessel functions of the first kind. In order to prove our main results, we use the technique of differential subordinations developed by MILLER and MOCANU, and some classical results of OZAKI and FEJÉR.

1. Introduction and preliminary results

In 1960 KREYSZIG and TODD [14] has determined the radius of univalence of the function $z \mapsto z^{1-p}J_p(z)$, where J_p is the Bessel function of the first kind, defined by (2.2). At the same time BROWN [9] has discussed the radius of univalence of the functions $z \mapsto J_p(z)$ and $z \mapsto [J_p(z)]^{1/p}$ for certain complex values of p using the methods of NEHARI and ROBERTSON. SELINGER [22] in 1995 used differential subordinations to investigate certain geometric properties of the function $z \mapsto z^{-p/2}J_p(z^{1/2})$. Motivated by the results of SELINGER, recently the author [2] extended the results from [22] to generalized Bessel functions. However, there is a little mistake which appears in the papers [2] and [22], namely in the proofs of the main results of the above mentioned papers the investigated functions in the question should be multiplied with a factor. In this paper our

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aim is to clarify this miss-understanding (see Section 2) and motivated by the above results to generalize further the results of Selinger using the technique of differential subordinations (see Section 3). Moreover, we determine conditions of close-to-convexity and univalence of these functions using sufficient conditions of univalence due to Fejér [11] and Ozaki [18] (see Section 4). To achieve our goal in this section we recall some basic facts and preliminary results.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. An analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be convex if it is univalent and if it maps $\mathbb{D}$ conformally onto a convex domain, i.e. $f(\mathbb{D})$ is a convex domain. It is known (see [10]), that f is convex if and only if $f'(0) \neq 0$ and

$$\operatorname{Re}\,[1 + zf''(z)/f'(z)] > 0 \quad \text{for all } z \in \mathbb{D}.$$

If, in addition,

$$\operatorname{Re}\,[1 + zf''(z)/f'(z)] > \alpha \quad \text{for all } z \in \mathbb{D},$$

where $0 \leq \alpha < 1$, then f is called convex of order α.

An analytic function $g : \mathbb{D} \to \mathbb{C}$ with $g(0) = 0$, is said to be starlike if it is univalent and $g(\mathbb{D})$ is starlike with respect to the origin. The function g with $g(0) = 0$ and $g'(0) \neq 0$ is starlike (see [10]) if and only if

$$\operatorname{Re}\,[zg'(z)/g(z)] > 0 \quad \text{for all } z \in \mathbb{D}.$$

If, in addition,

$$\operatorname{Re}\,[zg'(z)/g(z)] > \alpha \quad \text{for all } z \in \mathbb{D},$$

where $0 \leq \alpha < 1$, then g is called starlike of order α (see [12]). We remark that, according to the Alexander duality theorem [1], the function $f : \mathbb{D} \to \mathbb{C}$ is convex of order α, where $0 \leq \alpha < 1$ if and only if $z \mapsto zf'(z)$ is starlike of order α.

The analytic function $f : \mathbb{D} \to \mathbb{C}$ is said to be close-to-convex (or is said to be close-to-convex with respect to the function φ), if there exists a convex function $\varphi : \mathbb{D} \to \mathbb{C}$ such that

$$\operatorname{Re}[f'(z)/\varphi'(z)] > 0 \quad \text{for all } z \in \mathbb{D}.$$

We note that, analogously an analytic function $f : \mathbb{D} \to \mathbb{C}$ is called close-to-convex of order α, where $0 \leq \alpha < 1$, if there exists a convex function $\varphi : \mathbb{D} \to \mathbb{C}$ such that

$$\operatorname{Re}[f'(z)/\varphi'(z)] > \alpha \quad \text{for all } z \in \mathbb{D}.$$

We note that every starlike (and hence convex) function of the form $z + a_2z^2 + \ldots + a_nz^n + \ldots$ is in fact close-to-convex, and every close-to-convex function is univalent. However, if a function is starlike then it is not necessary that it will be close-to-convex with respect to a particular convex function. For more details we refer the interested reader to the papers [10], [13], [18] and to the references therein.

The next lemmas will be used to prove several theorems.

Lemma 1.1 ([15]). *Let $\mathbb{E}$ be a set in the complex plane $\mathbb{C}$ and $\psi : \mathbb{C}^3 \times \mathbb{D} \mapsto \mathbb{C}$ a function, that satisfies the admissibility condition $\psi(\rho \mathrm{i}, \sigma, \mu + \nu \mathrm{i}; z) \notin \mathbb{E}$, where $z \in \mathbb{D}$, $\rho, \sigma, \mu, \nu \in \mathbb{R}$ with $\mu+\sigma \leq 0$ and $\sigma \leq -(1+\rho^2)/2$. If $h : \mathbb{D} \to \mathbb{C}$, which satisfies $h(0) = 1$, is analytic and for all $z \in \mathbb{D}$ we have $\psi\left(h(z), zh'(z), z^2h''(z); z\right) \in \mathbb{E}$, then $\operatorname{Re} h(z) > 0$ for all $z \in \mathbb{D}$. In particular, if we only have $\psi : \mathbb{C}^2 \times \mathbb{D} \mapsto \mathbb{C}$, the admissibility condition reduces to $\psi(\rho \mathrm{i}, \sigma; z) \notin \mathbb{E}$ for all $z \in \mathbb{D}$ and $\rho, \sigma \in \mathbb{R}$ with $\sigma \leq -(1 + \rho^2)/2$.*

Lemma 1.2 ([18]). *If the function $f(z) = z + a_2z^2 + \ldots + a_nz^n + \ldots$ is analytic in $\mathbb{D}$ and in addition $1 \geq 2a_2 \geq \ldots \geq na_n \geq \ldots \geq 0$ or $1 \leq 2a_2 \leq \ldots \leq na_n \leq \ldots \leq 2$, then f is close-to-convex with respect to the convex function $z \mapsto -\log(1-z)$. Moreover, if the odd function $g(z) = z+b_3z^3+\ldots+b_{2n-1}z^{2n-1}+\ldots$ is analytic in $\mathbb{D}$ and if $1 \geq 3b_3 \geq \ldots \geq (2n+1)b_{2n+1} \geq \ldots \geq 0$ or $1 \leq 3b_3 \leq \ldots \leq (2n+1)b_{2n+1} \leq \ldots \leq 2$, then g is univalent in $\mathbb{D}$.*

We note that, as Ponnusamy and Vuorinen [20] pointed out, proceeding exactly as in the proof of Lemma 1.2 one can verify directly that if the odd function g satisfies the hypothesis of Lemma 1.2, then g is close-to-convex with respect to the convex function

$$z \mapsto \frac{1}{2} \log \frac{1+z}{1-z}.$$

We end this section with the next classical and interesting results of Fejér.

Lemma 1.3 ([11]). *If the function $f(z) = a_1z+a_2z^2+\ldots+a_nz^n+\ldots$, where $a_1 = 1$ and $a_n \geq 0$ for all $n \geq 2$, is analytic in $\mathbb{D}$ and if the sequences $\{na_n\}_{n\geq1}$, $\{na_n - (n + 1)a_{n+1}\}_{n\geq1}$ both are decreasing, then f is starlike in $\mathbb{D}$. Moreover, if for the analytic function $g(z) = b_1 + b_2z + \ldots + b_{n+1}z^n + \ldots$, where $b_1 = 1$ and $b_n \geq 0$ for all $n \geq 2$, we have that $\{b_n\}_{n\geq1}$ is a convex decreasing sequence, i.e., $b_n - 2b_{n+1} + b_{n+2} \geq 0$ and $b_n - b_{n+1} \geq 0$ for all $n \geq 1$, then $\operatorname{Re}[g(z)] > 1/2$ for all $z \in \mathbb{D}$.*

It is important to note here that Nezhmetdinov and Ponnusamy [16] using the duality technique have obtained other sufficient conditions over the Maclaurin coefficients of an analytic and normalized function f that imply its starlikeness. More precisely, Nezhmetdinov and Ponnusamy [16] in particular proved that if the function $f(z) = z + a_2z^2 + \ldots + a_nz^n + \ldots$ is analytic in $\mathbb{D}$ and in addition $2 \leq 3a_2 \leq \ldots \leq (n+1)a_n \leq \ldots$ and $na_n \leq 2$ for all $n \geq 2$, or $2/3 \geq a_2 \geq 2a_3 \geq \ldots \geq (n-1)a_n \geq \ldots \geq 0$ and $na_n \geq a_2$ for all $n \geq 3$, then f is starlike in $\mathbb{D}$.

2. Univalence, convexity and starlikeness of generalized Bessel functions

Let us consider the second-order differential equation [24, p. 38]

$$z^2w''(z) + zw'(z) + (z^2 - p^2)w(z) = 0, \tag{2.1}$$

which is called Bessel's equation, where p is an unrestricted real (or complex) number. The function J_p, which is called the Bessel function of the first kind of order p, is defined as a particular solution of (2.1). This function has the form [24, p. 40]

$$J_p(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(p+n+1)}\left(\frac{z}{2}\right)^{2n+p} \quad \text{for all } z \in \mathbb{C}. \tag{2.2}$$

The differential equation [24, p. 77]

$$z^2w''(z) + zw'(z) - (z^2 + p^2)w(z) = 0, \tag{2.3}$$

which differs from Bessel's equation only in the coefficient of w, is of frequent occurrence in problems of mathematical physics. The particular solution of (2.3) is called the modified Bessel function of the first kind of order p, and is defined by the formula [24, p. 77]

$$I_p(z) = \sum_{n\geq 0} \frac{1}{n!\Gamma(p+n+1)}\left(\frac{z}{2}\right)^{2n+p} \quad \text{for all } z \in \mathbb{C}. \tag{2.4}$$

The differential equation

$$z^2w''(z) + 2zw'(z) + [z^2 - p(p+1)]w(z) = 0, \tag{2.5}$$

which differs from equations (2.1) and (2.3) in the coefficient of $zw'(z)$ and $w(z)$, is called the spherical Bessel equation. Its particular solution is called the spherical Bessel function of the first kind of order p, and is defined by the formula

$$S_p(z) = \sum_{n\geq 0} \frac{(-1)^n}{n!\ \Gamma\left(p+n+\frac{3}{2}\right)}\left(\frac{z}{2}\right)^{2n+p} \quad \text{for all } z \in \mathbb{C}. \tag{2.6}$$

Now, let us consider the linear differential equation

$$z^2w''(z) + bzw'(z) + \left(cz^2 + d\right) w(z) = 0, \tag{2.7}$$

where $b, c, d, p \in \mathbb{C}$. If, in particular, we choose $d = d_1p^2 + d_2p + d_3$, where $d_1, d_2, d_3 \in \mathbb{C}$, then this generalizes the equations (2.1), (2.3) and (2.5). Moreover, this permits to study the Bessel, modified Bessel and spherical Bessel functions together. Due to our notations using the Frobenius method we can seek the solution of equation (2.7) in the following form:

$$w(z) = z^p \sum_{n\geq 0} a_n z^n.$$

It is easy to show that we have the following recursion between the coefficients a_n and a_{n-2} of the above infinite power series for all $n \geq 2$:

$$a_n \cdot n\,(n + 2p + b - 1) + a_n \left[(d_1 + 1)p^2 + (b + d_2 - 1)p + d_3\right] = -c \cdot a_{n-2}.$$

Letting $d_1 = -1$, $d_2 = 1 - b$ and $d_3 = 0$, the recursion becomes

$$a_n \cdot n[n + 2p + b - 1] = -c \cdot a_{n-2} \tag{2.8}$$

and the differential equation (2.7) will be the following

$$z^2w''(z) + bzw'(z) + \left[cz^2 - p^2 + (1 - b)p\right] w(z) = 0. \tag{2.9}$$

Using the recursive relation (2.8) we get that

$$\begin{aligned} w(z) &= a_0(p) \sum_{n\geq 0} \frac{(-1)^n c^n}{n! 4^n \prod_{m=1}^{n} \left(m + p + \frac{b-1}{2}\right)} z^{2n+p} \\ &= a_0(p) \sum_{n\geq 0} \frac{(-1)^n c^n \Gamma\left(p + \frac{b+1}{2}\right)}{n! 4^n \Gamma\left(n + p + \frac{b+1}{2}\right)} z^{2n+p} \end{aligned}$$

is a particular solution of equation (2.9), where $a_0 = a_0(p) \neq 0$. For convenience we denote the above particular solution with $w_p(z) = w(z)$ and then we choose

$$a_0(p) = \left[2^p \Gamma\left(p + \frac{b+1}{2}\right)\right]^{-1}$$

to obtain

$$w_p(z) = \sum_{n\geq 0} \frac{(-1)^n c^n}{n! \Gamma\left(p + n + \frac{b+1}{2}\right)} \cdot \left(\frac{z}{2}\right)^{2n+p} \quad \text{for all } z \in \mathbb{C}. \tag{2.10}$$

In what follows, by definition, any solution of the linear differential equation (2.9) will be called the generalized Bessel function of order p, while the particular solution w_p defined by (2.10) will be called the generalized Bessel function of the first kind of order p. In the study of geometric properties of these generalized Bessel functions an interesting method is the technique of differential subordinations, i.e. the application of Lemma 1.1. Thus, we would like to apply Lemma 1.1 for the analytic function $h : \mathbb{D} \to \mathbb{C}$, defined by $h(z) = w_p(z)$ and for the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$, defined by

$$\psi\left(h(z), zh'(z), z^2h''(z); z\right) = z^2h''(z) + bzh'(z) + \left[cz^2 - p^2 + (1-b)p\right] h(z),$$

with $\mathbb{E} = \{0\}$. But we have that $w_p(0) = 0$, and therefore we consider the transformation

$$u_p(z) = [a_0(p)]^{-1} z^{-p/2} w_p(z^{1/2})$$

to obtain $u_p(z) = b_0 + b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots$, where for all $n \geq 0$

$$b_n = \frac{(-1)^n c^n \Gamma\left(p + \frac{b+1}{2}\right)}{n! 4^n \Gamma\left(n + p + \frac{b+1}{2}\right)}. \tag{2.11}$$

Using the Pochhammer (or Appell) symbol, defined in terms of Euler's gamma functions, by $(\lambda)_n = \Gamma(\lambda + n)/\Gamma(\lambda) = \lambda(\lambda+1)\ldots(\lambda+n-1)$, we obtain for the function u_p the following form

$$u_p(z) = \sum_{n\geq 0} \frac{(-1)^n c^n}{4^n (\kappa)_n} \frac{z^n}{n!} = \sum_{n\geq 0} \frac{1}{(\kappa)_n \cdot n!} \left(-\frac{cz}{4}\right)^n, \tag{2.12}$$

where $\kappa = p + (b+1)/2 \neq 0, -1, -2, \ldots$. This function is analytic in $\mathbb{C}$, satisfies the condition $u_p(0) = 1$ and satisfies also the differential equation

$$4z^2 u''(z) + 2(2p + b + 1) z u'(z) + czu(z) = 0. \tag{2.13}$$

For further result on this transformation of the generalized Bessel function, which is called sometimes as the generalized and normalized Bessel function of the first kind, we refer to the recent papers [5], [6], [7, 8], where among other things some interesting functional inequalities, integral representations and extensions of some known trigonometric inequalities were established. Now, in view of the above preliminaries the function u_p and its transformations $z \mapsto zu_p(z)$ and $z \mapsto zu_p(z^2)$ can be studied with the the aid of lemmas 1.1, 1.2 and 1.3. The mistake in the papers [2] and [22] is that the Lemma 1.1 is applied to the functions

$$z \mapsto z^{-1/2} w_p(z^{1/2}), \quad z \mapsto z^{-1/2} J_p(z^{1/2}),$$

respectively, and not to the functions

$$z \mapsto [a_0(p)]^{-1} z^{-1/2} w_p(z^{1/2}), \quad z \mapsto 2^p \Gamma(p+1) z^{-1/2} J_p(z^{1/2}),$$

respectively. With other words in the proofs of the main results of [2] and [22] the factors $[a_0(p)]^{-1}$, and $2^p\Gamma(p+1)$ respectively, are missing. In this section we would like to present the correct version of the main results of the papers [2] and [22]. The reason in introducing the above generalized Bessel functions is that the Bessel and modified Bessel functions has similar properties, and because of this we would like to have an unified exposition of the results on different types of Bessel functions. More precisely, the modified Bessel function is in fact just the Bessel function with imaginary argument, and consequently it maps the unit disk into the same domain as the Bessel function. In what follows for convenience we deduce some basic results on generalized Bessel functions, which will be used in the sequel. As we will see, most of the properties of generalized Bessel functions can be deduced analogously as for the Bessel and modified Bessel functions.

The next proposition will be applied for the study of the univalence of the function u_p.

Proposition 2.14. *If $b, p, c \in \mathbb{C}$ such that $2p + b + 1 \neq 0, -2, -4, \ldots,$ and $z \in \mathbb{C}$, then for the generalized Bessel function of the first kind of order p the following recursive relations hold:*

(i) $zw_{p-1}(z) + czw_{p+1}(z) = (2p + b - 1)w_p(z)$;

(ii) $zw_p'(z) + (p + b - 1)w_p(z) = zw_{p-1}(z)$;

(iii) $zw_p'(z) + czw_{p+1}(z) = pw_p(z)$;

(iv) $[z^{-p}w_p(z)]' = -cz^{-p}w_{p+1}(z)$;

(v) $2(2p + b + 1)u_p'(z) = -cu_{p+1}(z)$.

Proof. (i) If we compute the expression $w_{p-1}(z) + w_{p+1}(z)$, then we have that

$$w_{p-1}(z) + w_{p+1}(z) = \sum_{n\geq 0} \frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b-1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p-1}$$

$$+ \sum_{n\geq 0} \frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b+3}{2}\right)} \left(\frac{z}{2}\right)^{2n+p+1} = \frac{1}{\Gamma\left(p+\frac{b-1}{2}\right)} \left(\frac{z}{2}\right)^{p-1}$$

$$+ \sum_{n\geq 1} \left[\frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b-1}{2}\right)} + \frac{(-1)^{n-1} c^{n-1}}{(n-1)!\Gamma\left(p+n+\frac{b+1}{2}\right)}\right] \left(\frac{z}{2}\right)^{2n+p-1}$$

$$= \frac{2p+b-1}{z} \left[\frac{1}{\Gamma\left(p+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^p + \sum_{n\geq 1} \frac{(-1)^n c^n}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2n+p}\right]$$

$$+\frac{2p+b-1}{z}\left[\frac{2}{2p+b-1}\sum_{n\geq 1}\frac{(-1)^n c^{n-1}(c-1)}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p}\right].$$

Consequently, we obtain that

$$\begin{aligned}
&w_{p-1}(z)+w_{p+1}(z)\\
&\quad=\frac{2p+b-1}{z}\left[w_p(z)+\frac{2}{2p+b-1}\sum_{m\geq 0}\frac{(-1)^{m+1}c^m(c-1)}{m!\Gamma\left(p+m+1+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2m+p+2}\right]\\
&\quad=\frac{2p+b-1}{z}\left[w_p(z)+\frac{z}{2p+b-1}(1-c)w_{p+1}(z)\right],
\end{aligned}$$

which implies that $zw_{p-1}(z)+czw_{p+1}(z)=(2p+b-1)w_p(z)$ holds, as we required.

(ii) Analogously, if we compute the expression $w_{p-1}(z)-w_{p+1}(z)$, then we have

$$\begin{aligned}
w_{p-1}(z)-w_{p+1}(z)&=\frac{2p+b-1}{2\Gamma\left(p+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{p-1}\\
&+\sum_{n\geq 1}\frac{(-1)^n c^n\left(p+n+\frac{b-1}{2}+\frac{n}{c}\right)}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p-1}=\frac{p}{2\Gamma\left(p+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{p-1}\\
&+\frac{p+b-1}{2\Gamma\left(p+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{p-1}+\sum_{n\geq 1}\frac{(-1)^n c^n\left(n+\frac{p}{2}\right)}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p-1}\\
&+\sum_{n\geq 1}\frac{(-1)^n c^n(p+b-1)}{2\cdot n!\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p-1}+\sum_{n\geq 1}\frac{(-1)^n c^{n-1}}{(n-1)!\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p-1}\\
&=w_p'(z)+\frac{p+b-1}{z}w_p(z)-w_{p+1}(z),
\end{aligned}$$

and thus we obtain the second recursive relation.

(iii) Combining the recursive relations (i) and (ii), we get that

$$zw_p'(z)+(p+b-1)w_p(z)+zw_{p-1}(z)+czw_{p+1}(z)=(2p+b-1)w_p(z),$$

which implies that $zw_p'(z)+czw_{p+1}(z)=pw_p(z)$.

(iv) Using the third recursive relation we obtain

$$\begin{aligned}
[z^{-p}w_p(z)]'&=z^{-2p}[w_p'(z)z^p-pz^{p-1}w_p(z)]\\
&=z^{-p-1}[zw_p'(z)-w_p(z)]=-cz^{-p}w_{p+1}(z).
\end{aligned}$$

(v) For convenience, we use part (iv). Since from definition and from part (iv) we have $w_p(z) = [a_0(p)]z^p u_p(z^2)$ and $[z^{-p}w_p(z)]' = -cz^{-p}w_{p+1}(z)$, we get

$$2zu_p'(z^2) = [u_p(z^2)]' = [a_0(p)]^{-1}[z^{-p}w_p(z)]' = -[a_0(p)]^{-1}cz^{-p}w_{p+1}(z).$$

But $w_{p+1}(z) = [a_0(p+1)]z^{p+1}u_{p+1}(z^2)$, therefore

$$2zu_p'(z^2) = -cz[a_0(p)]^{-1}[a_0(p+1)]u_{p+1}(z^2). \tag{2.15}$$

Now, if we compute the expression $[a_0(p+1)]/[a_0(p)]$, using the relation $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$, then we obtain $(2p+b+1)[a_0(p+1)] = [a_0(p)]$, thus by relation (2.15) the proof is complete. □

We note that part (v) of Proposition 2.14 in fact can de deduced directly from the definition of the function u_p. We have included in this paper the other parts of Proposition 2.14 just because the results stated above were used among other things in the forthcoming papers [4], [7] of the author.

The next result contains conditions for the function u_p to be univalent, convex, starlike in the unit disk, and provides the correction of the main results from [22] and [2].

Theorem 2.16. *If $b, c, p \in \mathbb{R}$ and $\kappa = p + (b+1)/2$, then the functions w_p and u_p satisfy the following properties:*

(i) *If $\kappa \geq |c|/4 + 1$, then $\operatorname{Re} u_p(z) > 0$ for all $z \in \mathbb{D}$;*

(ii) *If $\kappa \geq |c|/4$ and $c \neq 0$, then u_p is univalent in $\mathbb{D}$;*

(iii) *If $\kappa \geq |c|/4 + 1/2$ and $c \neq 0$, then u_p is convex in $\mathbb{D}$;*

(iv) *If $\kappa \geq |c|/4 + 3/2$ and $c \neq 0$, then $z \mapsto zu_p(z)$ is starlike in $\mathbb{D}$;*

(v) *If $\kappa \geq |c|/2 + 1$ and $c \neq 0$, then $z \mapsto zu_p(z)$ is starlike of order $1/2$ in $\mathbb{D}$;*

(vi) *If $\kappa \geq |c|/2 + 1$ and $c \neq 0$, then $z \mapsto z^{1-p}w_p(z)$ is starlike in $\mathbb{D}$.*

Proof. (i) Clearly when $c = 0$ we have $u_p(z) \equiv 1$, thus $\operatorname{Re} u_p(z) > 0$ for all $z \in \mathbb{D}$. Now suppose that $\kappa \geq |c|/4 + 1$ and $c \neq 0$. Put $h = u_p$. Since h satisfies (2.13), we have

$$4z^2h''(z) + 4\kappa zh'(z) + czh(z) = 0. \tag{2.17}$$

If we consider $\psi(r, s, t; z) = 4t + 4\kappa s + czr$ and $\mathbb{E} = \{0\}$, then from the differential equation (2.17) we have $\psi\left(h(z), zh'(z), z^2h''(z); z\right) \in \mathbb{E}$ for all $z \in \mathbb{D}$. Next we will use Lemma 1.1 to prove that $\operatorname{Re} h(z) > 0$ for all $z \in \mathbb{D}$. If we put $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, then

$$\operatorname{Re}\psi(\rho\mathrm{i}, \sigma, \mu + \nu\mathrm{i}; x + \mathrm{i}y) = 4(\mu + \sigma) + 4(\kappa - 1)\sigma - c\rho y \quad \text{for all } \rho, \sigma, \mu, \nu \in \mathbb{R}.$$

Let $\rho, \sigma, \mu, \nu \in \mathbb{R}$ satisfy $\mu + \sigma \le 0$ and $\sigma \le -(1+\rho^2)/2$. Since $\kappa > 1$, we have

$$\operatorname{Re} \psi\left(\rho \mathrm{i}, \sigma, \mu + \nu \mathrm{i}; x + \mathrm{i}y\right) \le -2(\kappa - 1)\rho^2 - cy\rho - 2(\kappa - 1).$$

Set $Q_1(\rho) = -2(\kappa - 1)\rho^2 - cy\rho - 2(\kappa - 1)$. This value will be strictly negative for all real ρ, because the discriminant Δ of $Q_1(\rho)$ satisfies

$$\Delta := c^2 y^2 - 16(\kappa - 1)^2 < c^2 - 16(\kappa - 1)^2 \le 0$$

whenever $y \in (-1, 1)$. Consequently ψ satisfies the admissibility condition of Lemma 1.1. Hence by Lemma 1.1 we conclude $\operatorname{Re} h(z) = \operatorname{Re} u_p(z) > 0$ for all $z \in \mathbb{D}$.

(ii) When $\kappa \ge |c|/4$ and $c \ne 0$, then the above result implies $\operatorname{Re} u_{p+1}(z) > 0$ for all $z \in \mathbb{D}$. Using part (v) of Proposition 2.14 we conclude that

$$\operatorname{Re}\left[-\frac{4\kappa}{c} u_p'(z)\right] = \operatorname{Re} u_{p+1}(z) > 0 \quad \text{for all } z \in \mathbb{D}.$$

This in turn implies that u_p is close-to-convex with respect to the function $\varphi(z) = -(cz)/(4\kappa)$. Now, since every close to convex function is univalent, it follows that u_p is univalent in $\mathbb{D}$. We note that the univalence of the function u_p follows also from the classical Noshiro–Warschawski theorem [17], [23] (see also [10]), which states that if $f : \mathbb{E} \subseteq \mathbb{C} \to \mathbb{C}$ is analytic, $\mathbb{E}$ is a convex domain and $\operatorname{Re}[f'(z)] > 0$ there, then f is univalent in $\mathbb{E}$. Namely, from the Noshiro–Warschawski theorem the function $z \mapsto -4\kappa u_p(z)/c$ is univalent in $\mathbb{D}$, and consequently the function $z \mapsto u_p(z)$ is univalent too in $\mathbb{D}$, since the multiplication of a non-zero constant do not disturb the univalence.

(iii) Since $\kappa > |c|/4$ and $c \ne 0$, part (i) implies that $\operatorname{Re} u_{p+1}(z) > 0$ for all $z \in \mathbb{D}$. According to part (v) of Proposition 2.14 it follows that $u_p'(z) \ne 0$ for all $z \in \mathbb{D}$. Define $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = 1 + \frac{z u_p''(z)}{u_p'(z)}.$$

The function q is analytic in $\mathbb{D}$ and $q(0) = 1$. Since u_p satisfies the differential equation (2.13), we have $4z u_p''(z) + 4\kappa u_p'(z) + c u_p(z) = 0$. If we differentiate both sides of this equation, we obtain

$$4z u_p'''(z) + 4(\kappa + 1) u_p''(z) + c u_p'(z) = 0.$$

Suppose that $z \neq 0$. We know that $u_p'(z) \neq 0$, therefore if we divide both sides of this equation with $u_p'(z)$, and multiply with z, we obtain

$$4\left[\frac{zu_p'''(z)}{u_p''(z)}\right]\left[\frac{zu_p''(z)}{u_p'(z)}\right] + 4(\kappa+1)\left[\frac{zu_p''(z)}{u_p'(z)}\right] + cz = 0. \tag{2.18}$$

Now we differentiate logarithmically and multiply with z on both sides of the equation $q(z) - 1 = [zu_p''(z)]/u_p'(z)$. Thus we obtain

$$\frac{zq'(z)}{q(z)-1} = 1 + \frac{zu_p'''(z)}{u_p''(z)} - [q(z)-1],$$

and therefore

$$\frac{zu_p'''(z)}{u_p''(z)} = \frac{zq'(z) + q^2(z) - 3q(z) + 2}{q(z)-1}.$$

In view of (2.18) this result reveals that q satisfies the following differential equation:

$$4zq'(z) + 4q^2(z) + 4(\kappa-2)q(z) + cz - 4(\kappa-1) = 0. \tag{2.19}$$

Obviously, this equation is also valid when $z = 0$.

If we use $\psi(r,s;z) = 4s + 4r^2 + 4(\kappa-2)r + cz - 4(\kappa-1)$ and $\mathbb{E} = \{0\}$, then (2.19) implies $\psi\left(q(z), zq'(z); z\right) \in \mathbb{E}$ for all $z \in \mathbb{D}$. Now we use Lemma 1.1 to prove that $\operatorname{Re} q(z) > 0$ for all $z \in \mathbb{D}$. For $z = x + \mathrm{i}y \in \mathbb{D}$ (with $x, y \in \mathbb{R}$) and $\rho, \sigma \in \mathbb{R}$ satisfying $\sigma \leq -(1+\rho^2)/2$, we obtain

$$\begin{aligned}\operatorname{Re}\psi(\rho\mathrm{i}, \sigma; x+\mathrm{i}y) &= 4\sigma - 4\rho^2 + cx - 4(\kappa-1)\\ &\leq -6\rho^2 + cx - 2(2\kappa-1) < |c| - 2(2\kappa-1) \leq 0.\end{aligned}$$

By Lemma 1.1 we conclude that $\operatorname{Re} q(z) > 0$ for all $z \in \mathbb{D}$, which shows that u_p is convex in $\mathbb{D}$.

(iv) According to part (iii) of this theorem the function u_{p-1} is convex. By Alexander's duality theorem [1] it follows that $z \mapsto zu_{p-1}'(z)$ is starlike in $\mathbb{D}$. But, on the other hand, part (v) of Proposition 2.14 yields

$$czu_p(z) = -4(\kappa-1)zu_{p-1}'(z).$$

Consequently, it results that $z \mapsto zu_p(z)$ is starlike too in $\mathbb{D}$.

(v) According to part (i) of this theorem we have $\operatorname{Re} u_p(z) > 0$ for all $z \in \mathbb{D}$, hence $u_p \neq 0$ for all $z \in \mathbb{D}$. Define $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = 1 + 2\frac{zu_p'(z)}{u_p(z)}.$$

The function q is analytic in $\mathbb{D}$ and $q(0) = 1$. Assume that $z \neq 0$. Because u_p satisfies the equation (2.13), it satisfies the following equation too:

$$4\left[\frac{zu_p''(z)}{u_p'(z)}\right]\left[\frac{zu_p'(z)}{u_p(z)}\right] + 4\kappa\left[\frac{zu_p'(z)}{u_p(z)}\right] + cz = 0. \tag{2.20}$$

In what follows we proceed as in part (iii), we differentiate logarithmically and multiply with z the expression $[q(z) - 1]/2 = [zu_p'(z)]/u_p(z)$, and thus we obtain

$$\frac{zu_p''(z)}{u_p'(z)} = \frac{2zq'(z) + q^2(z) - 4q(z) + 3}{2(q(z) - 1)}.$$

In view of (2.20) this result reveals that q satisfies the following differential equation:

$$2zq'(z) + q^2(z) + 2(\kappa - 2)q(z) + cz - 2(\kappa - 3/2) = 0, \tag{2.21}$$

which is also valid when $z = 0$.

If $\psi(r, s; z) = 2s + r^2 + 2(\kappa - 2)r + cz - 2(\kappa - 3/2)$ and $\mathbb{E} = \{0\}$, then (2.21) implies $\psi\left(q(z), zq'(z); z\right) \in \mathbb{E}$ for all $z \in \mathbb{D}$. We use Lemma 1.1 to prove that $\operatorname{Re} q(z) > 0$ for all $z \in \mathbb{D}$. For $z = x + \mathrm{i}y \in \mathbb{D}$ with $x, y \in \mathbb{R}$, and $\rho, \sigma \in \mathbb{R}$ satisfying $\sigma \leq -(1 + \rho^2)/2$, we obtain

$$\begin{aligned}\operatorname{Re}\psi(\rho\mathrm{i}, \sigma; x + \mathrm{i}y) &= 2\sigma - \rho^2 + cx - 2(\kappa - 3/2)\\ &\leq -2\rho^2 + cx - 2(\kappa - 1) < |c| - 2(\kappa - 1) \leq 0.\end{aligned}$$

By Lemma 1.1 we conclude that $\operatorname{Re} q(z) > 0$ for all $z \in \mathbb{D}$. Now consider the function $g_p : \mathbb{D} \to \mathbb{C}$, defined by $g_p(z) = zu_p(z)$. Since

$$\frac{zg_p'(z)}{g_p(z)} = \frac{1}{2} + \frac{1}{2}q(z),$$

it follows that

$$\operatorname{Re}\left[\frac{zg_p'(z)}{g_p(z)}\right] > \frac{1}{2} \quad \text{for all } z \in \mathbb{D},$$

which shows that g_p is starlike of order $1/2$.

(vi) Define the function $h_p : \mathbb{D} \to \mathbb{C}$ by $h_p(z) = z^{1-p}w_p(z)$. Since $h_p(z) = a_0(p)zu_p(z^2)$, where $a_0(p) = [2^p\Gamma(\kappa)]^{-1}$, it follows that

$$\frac{zh_p'(z)}{h_p(z)} = 2\left[\frac{z^2g_p'(z^2)}{g_p(z^2)} - \frac{1}{2}\right].$$

But from part (v) of this theorem we know that g_p is starlike of order 1/2. Thus we conclude that

$$\operatorname{Re}\left[\frac{zh_p'(z)}{h_p(z)}\right] > 0 \quad \text{for all } z \in \mathbb{D},$$

and hence h_p is starlike in $\mathbb{D}$. □

We note that clearly the results of Theorem 2.16 are still valid if we replace the function $z \mapsto u_p(z) = [a_0(p)]^{-1}z^{-1/2}w_p(z^{1/2})$ with the function $z \mapsto z^{-1/2}w_p(z^{1/2})$. For the function $z \mapsto z^{-1/2}w_p(z^{1/2})$ the above geometric properties were established in [2], but in the proofs there was overlooked the condition $h(0) = 1$ of Lemma 1.1. In a forthcoming paper [3], using the Cauchy–Buniakowski–Schwarz inequality, the author has extended the results of Theorem 2.16 to the case when the parameters b, p, c are complex. We would like take the opportunity to note that in [3, Theorem 2.1] the expression $2\operatorname{Im}\kappa - 1$ should be replaced with $2\operatorname{Im}\kappa$. Namely, the corrected version of the main results from [3] is that for $b, p, c \in \mathbb{C}$ the followings are true:

(i) If $\operatorname{Re}\kappa \geq |c|/4 + 1$, then $\operatorname{Re} u_p(z) > 0$ for all $z \in \mathbb{D}$;

(ii) If $\operatorname{Re}\kappa \geq |c|/4$ and $c \neq 0$, then u_p is univalent in $\mathbb{D}$;

(iii) If $\operatorname{Re}\kappa \geq |c|/4 + (\operatorname{Im}\kappa)^2/6 + 1/2$ and $c \neq 0$, then u_p is convex in $\mathbb{D}$;

(iv) If $\operatorname{Re}\kappa \geq |c|/4 + (\operatorname{Im}\kappa)^2/6 + 3/2$ and $c \neq 0$, then $z \mapsto zu_p(z)$ is starlike in $\mathbb{D}$;

(v) If $\operatorname{Re}\kappa \geq |c|/2 + (\operatorname{Im}\kappa)^2/4 + 1$ and $c \neq 0$, then $z \mapsto zu_p(z)$ is starlike of order 1/2 in $\mathbb{D}$;

(vi) If $\operatorname{Re}\kappa \geq |c|/2 + (\operatorname{Im}\kappa)^2/4 + 1$ and $c \neq 0$, then $z \mapsto z^{1-p}w_p(z)$ is starlike in $\mathbb{D}$.

Taking into account the above results we have the following particular cases:

2.1. Bessel functions. Choosing $b = c = 1$, $d_1 = -1$ and $d_2 = d_3 = 0$ in (2.7), we obtain the Bessel differential equation (2.1) and the Bessel function of the first kind of order p, defined by (2.2) satisfies this equation. In particular, the results of [3, Theorem 2.1] becomes:

Corollary 2.22. *Let $\mathcal{J}_p : \mathbb{D} \to \mathbb{C}$ be defined by $\mathcal{J}_p(z) = 2^p\Gamma(p+1)z^{-p}J_p(z)$. Then the following assertions are true:*

(i) *If $\operatorname{Re} p \geq 1/4$, then $\operatorname{Re}\mathcal{J}_p(z^{1/2}) > 0$ for all $z \in \mathbb{D}$;*

(ii) *If $\operatorname{Re} p \geq -3/4$, then $z \mapsto \mathcal{J}_p(z^{1/2})$ is univalent in $\mathbb{D}$;*

(iii) *If $\operatorname{Re} p \geq -1/4 + (\operatorname{Im} p)^2/6$, then $z \mapsto \mathcal{J}_p(z^{1/2})$ is convex in $\mathbb{D}$;*

(iv) *If $\operatorname{Re} p \geq 3/4 + (\operatorname{Im} p)^2/6$, then $z \mapsto z\mathcal{J}_p(z^{1/2})$ is starlike in $\mathbb{D}$;*

(v) *If $\operatorname{Re} p \geq 1/2 + (\operatorname{Im} p)^2/4$, then $z \mapsto z\mathcal{J}_p(z^{1/2})$ is starlike of order 1/2 in $\mathbb{D}$;*

(vi) *If* $\operatorname{Re} p \geq 1/2 + (\operatorname{Im} p)^2/4$, *then* $z \mapsto z^{1-p} J_p(z)$ *is starlike in* $\mathbb{D}$.

Clearly, as in the general case, the results of Corollary 2.22 are still valid if we use the function $z \mapsto z^{-p/2} J_p(z^{1/2})$ instead of $z \mapsto \mathcal{J}_p(z^{1/2})$. For p real, Selinger in [22] found the above geometric properties of the function $z \mapsto z^{-p/2} J_p(z^{1/2})$, and analogously in the proofs of the main results of [22] there is overlooked the fact that the function $z \mapsto z^{-p/2} J_p(z^{1/2})$ does not maps 0 into 1. We note that Brown [9, Theorem 3] has proved that if the complex number $p = \operatorname{Re} p + \mathrm{i} \operatorname{Im} p$ satisfy one of the following conditions

$$\operatorname{Im} p \leq \operatorname{Re} p \in [0,1) \quad \text{or} \quad \operatorname{Re} p \geq 1,\ 2\operatorname{Re} p - 1 > (\operatorname{Im} p)^2,$$

then the normalized Bessel function $z \mapsto z^{1-p} J_p(z)$ is univalent in every circle $|z| < r = \rho_\mu^*$, where $\mu^2 = \operatorname{Re}[p^2]$, $\mu > 0$, and ρ_μ^* is the smallest positive zero of the function $rJ_\mu'(r) + [\operatorname{Re}(1-p)] \cdot J_\mu(r)$. In particular, when p is real the function $z \mapsto z^{1-p} J_p(z)$ is starlike in $|z| < \rho_\mu^*$, but is not univalent in any larger circle. The method used by Brown is completely different than the method of differential subordinations, and it is worth mentioning that in the case of the unit disk our result from part (vi) of Corollary 2.22 slightly improves the above result of Brown because we have that

$$\operatorname{Re} p \geq 1/2 + (\operatorname{Im} p)^2/2 \geq 1/2 + (\operatorname{Im} p)^2/4,$$

i.e. if $\operatorname{Re} p \geq 1/2 + (\operatorname{Im} p)^2/4$, then the normalized Bessel function $z \mapsto z^{1-p} J_p(z)$ is still starlike and hence univalent in $\mathbb{D}$.

2.2. Modified Bessel functions. Taking $b = 1$, $c = d_1 = -1$ and $d_2 = d_3 = 0$ in (2.7), we obtain the differential equation (2.3), which has solution the modified Bessel function of the first kind of order p, defined by (2.4). For the function $\mathcal{I}_p : \mathbb{D} \to \mathbb{C}$, defined by

$$\mathcal{I}_p(z) = 2^p \Gamma(p+1) z^{-p} I_p(z),$$

the properties are the same like for the function $\mathcal{J}_p$, because in this case we have $|c| = 1$. More precisely, we have the following results.

Corollary 2.23. *The following assertions are true:*

(i) *If* $\operatorname{Re} p \geq 1/4$, *then* $\operatorname{Re} \mathcal{I}_p(z^{1/2}) > 0$ *for all* $z \in \mathbb{D}$*;*

(ii) *If* $\operatorname{Re} p \geq -3/4$, *then* $z \mapsto \mathcal{I}_p(z^{1/2})$ *is univalent in* $\mathbb{D}$*;*

(iii) *If* $\operatorname{Re} p \geq -1/4 + (\operatorname{Im} p)^2/6$, *then* $z \mapsto \mathcal{I}_p(z^{1/2})$ *is convex in* $\mathbb{D}$*;*

(iv) *If* $\operatorname{Re} p \geq 3/4 + (\operatorname{Im} p)^2/6$, *then* $z \mapsto z\mathcal{I}_p(z^{1/2})$ *is starlike in* $\mathbb{D}$*;*

(v) *If* $\operatorname{Re} p \geq 1/2 + (\operatorname{Im} p)^2/4$, *then* $z \mapsto z\mathcal{I}_p(z^{1/2})$ *is starlike of order* $1/2$ *in* $\mathbb{D}$*;*

(vi) *If* $\operatorname{Re} p \geq 1/2 + (\operatorname{Im} p)^2/4$, *then* $z \mapsto z^{1-p} I_p(z)$ *is starlike in* $\mathbb{D}$.

2.3. Spherical Bessel functions. If we take $b = 2$, $c = 1$, $d_1 = d_2 = -1$ and $d_3 = 0$ in (2.7), we obtain the spherical Bessel differential equation (2.5). The spherical Bessel function of the first kind of order p, defined by (2.6) satisfies the previous differential equation and the next result holds.

Corollary 2.24. *Let* $\mathcal{S}_p : \mathbb{D} \to \mathbb{C}$ *be defined by* $\mathcal{S}_p(z) = 2^p \Gamma(p + 3/2) z^{-p} S_p(z)$. *Then the following assertions are true:*

(i) *If* $\operatorname{Re} p \geq -1/4$, *then* $\operatorname{Re} \mathcal{S}_p(z^{1/2}) > 0$ *for all* $z \in \mathbb{D}$*;*

(ii) *If* $\operatorname{Re} p \geq -5/4$, *then* $z \mapsto \mathcal{S}_p(z^{1/2})$ *is univalent in* $\mathbb{D}$*;*

(iii) *If* $\operatorname{Re} p \geq -3/4 + (\operatorname{Im} p)^2/6$, *then* $z \mapsto \mathcal{S}_p(z^{1/2})$ *is convex in* $\mathbb{D}$*;*

(iv) *If* $\operatorname{Re} p \geq 1/4 + (\operatorname{Im} p)^2/6$, *then* $z \mapsto z\mathcal{S}_p(z^{1/2})$ *is starlike in* $\mathbb{D}$*;*

(v) *If* $\operatorname{Re} p \geq (\operatorname{Im} p)^2/4$, *then* $z \mapsto z\mathcal{S}_p(z^{1/2})$ *is starlike of order* $1/2$ *in* $\mathbb{D}$*;*

(vi) *If* $\operatorname{Re} p \geq (\operatorname{Im} p)^2/4$, *then* $z \mapsto z^{1-p} S_p(z)$ *is starlike in* $\mathbb{D}$.

3. Convexity and starlikeness of order α of generalized Bessel functions

The following results contains conditions for the function u_p to be close-to-convex, convex and starlike of order α in the unit disk. These are another generalizations of Theorem 2.16.

Theorem 3.1. *If* $0 \leq \alpha < 1/2$ *and* $b, p, c \in \mathbb{R}$, *then the following assertions are true:*

(i) *If* $4\kappa \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c| + 1$, *then* $\operatorname{Re} u_p(z) > \alpha$ *for all* $z \in \mathbb{D}$*;*

(ii) *If* $4\kappa \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c|$ *and* $c \neq 0$, *then* u_p *is close-to-convex of order* α *in* $\mathbb{D}$.

Proof. (i) First assume that $c = 0$. Then $u_p(z) \equiv 1$, and consequently $\operatorname{Re} u_p(z) > \alpha$ for all $z \in \mathbb{D}$. Now suppose that $\kappa \geq (1 - \alpha)(1 - 2\alpha)^{-1/2}|c|/4 + 1$ and $c \neq 0$. Define the function $h : \mathbb{D} \to \mathbb{C}$ by

$$h(z) = \frac{u_p(z) - \alpha}{1 - \alpha}.$$

Since u_p satisfies (2.13), h will satisfy the following differential equation:

$$4z^2 h''(z) + 4\kappa z h'(z) + cz \left[h(z) + \frac{\alpha}{1 - \alpha} \right] = 0. \tag{3.2}$$

Using $\psi(r,s,t;z) = 4t + 4\kappa s + cz[r + \alpha/(1-\alpha)]$ and $\mathbb{E} = \{0\}$, we see that equation (3.2) implies $\psi\left(h(z), zh'(z), z^2h''(z); z\right) \in \mathbb{E}$ for all $z \in \mathbb{D}$. Next we use Lemma 1.1 to prove that $\operatorname{Re} h(z) > 0$ for all $z \in \mathbb{D}$. For $z = x + \mathrm{i}y$, where $x, y \in \mathbb{R}$, we have

$$\operatorname{Re}\psi\left(\rho\mathrm{i}, \sigma, \mu + \nu\mathrm{i}; x + \mathrm{i}y\right) = 4(\mu + \sigma) + 4(\kappa - 1)\sigma - c\rho y + \alpha cx/(1-\alpha)$$

for all $\rho, \sigma, \mu, \nu \in \mathbb{R}$. Let $\rho, \sigma, \mu, \nu \in \mathbb{R}$ satisfy $\mu + \sigma \leq 0$ and $\sigma \leq -(1+\rho^2)/2$. Since $\kappa - 1 > 0$, we obtain

$$\operatorname{Re}\psi\left(\rho\mathrm{i}, \sigma, \mu + \nu\mathrm{i}; x + \mathrm{i}y\right) \leq -2(\kappa-1)\rho^2 - cy\rho - 2(\kappa-1) + \alpha cx/(1-\alpha).$$

Set $Q_1(\rho) = -2(\kappa-1)\rho^2 - cy\rho - 2(\kappa-1) + \alpha cx/(1-\alpha)$. This value is strictly negative for all real ρ, because the discriminant Δ_1 of $Q_1(\rho)$ satisfies

$$\begin{aligned}\Delta_1 &= c^2y^2 - 16(\kappa-1)^2 + 8\alpha cx(\kappa-1)/(1-\alpha)\\ &< c^2(1-x^2) - 16(\kappa-1)^2 + 8\alpha cx(\kappa-1)/(1-\alpha) =: Q_2(x) \leq 0,\end{aligned}$$

whenever $x^2 + y^2 < 1$ and the discriminant Δ_2 of $Q_2(x)$ is negative. Δ_2 has the form

$$\Delta_2 = 4c^2\left[-16\frac{1-2\alpha}{(1-\alpha)^2}(\kappa-1)^2 + c^2\right]$$

and this is negative if and only if we have $\kappa \geq (1-\alpha)(1-2\alpha)^{-1/2}|c|/4 + 1$. Hence by Lemma 1.1 we conclude that

$$\operatorname{Re} h(z) = \operatorname{Re}\left[\frac{1}{1-\alpha}(u_p(z) - \alpha)\right] > 0 \quad \text{for all } z \in \mathbb{D},$$

and this implies that $\operatorname{Re} u_p(z) > \alpha$ for all $z \in \mathbb{D}$, as we required.

(ii) Now, suppose that $\kappa \geq (1-\alpha)(1-2\alpha)^{-1/2}|c|/4$ and $c \neq 0$. Then the above result implies $\operatorname{Re} u_{p+1}(z) > \alpha$ for all $z \in \mathbb{D}$. Using again part (v) of Proposition 2.14 we conclude that

$$\operatorname{Re}\left[\left(-\frac{4\kappa}{c}\right)u_p'(z)\right] = \operatorname{Re} u_{p+1}(z) > \alpha \quad \text{for all } z \in \mathbb{D},$$

i.e. u_p is close-to-convex of order α in $\mathbb{D}$ with respect to the function $\varphi(z) = -(cz)/(4\kappa)$. □

Theorem 3.3. *If $0 \leq \alpha < 1$ and $b, p, c \in \mathbb{R}$ such that $c \neq 0$ and $4\alpha^2 + (|c| - 6)\alpha + 2 \geq 0$, then the functions w_p and u_p have the following properties:*

(i) *If $4(1-\alpha)\kappa \geq |c| + 2(1-\alpha)(1-2\alpha)$, then u_p is convex of order α in $\mathbb{D}$;*

(ii) *If* $4(1-\alpha)\kappa \geq |c| + 2(1-\alpha)(3-2\alpha)$, *then* $z \mapsto zu_p(z)$ *is starlike of order* α *in* $\mathbb{D}$;

(iii) *If* $4(1-\alpha)\kappa \geq |c| + 2(1-\alpha)(3-2\alpha)$ *and* $\alpha \neq 0$, *then* $z \mapsto z^{[2(1-\alpha)-p]/[2\alpha]} w_p\left(z^{1/[2\alpha]}\right)$ *is starlike in* $\mathbb{D}$.

Proof. (i) The equality

$$\frac{|c| + 2(2\alpha^2 - 3\alpha + 1)}{4(1-\alpha)} = \frac{|c|}{4} + \frac{4\alpha^2 + (|c| - 6)\alpha + 2}{4(1-\alpha)}$$

implies $\kappa \geq |c|/4$. By applying part (i) of Theorem 2.16 we conclude that $\operatorname{Re} u_{p+1}(z) > 0$ for all $z \in \mathbb{D}$. According to part (v) of Proposition 2.14 it follows that $u_p'(z) \neq 0$. Define $q : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = 1 + \frac{zu_p''(z)}{(1-\alpha)u_p'(z)}.$$

The function q is analytic in $\mathbb{D}$ and $q(0) = 1$. Since u_p satisfies the differential equation (2.13) it can be shown, as in the proof of Theorem 2.16, that q satisfies the following differential equation:

$$4(1-\alpha)zq'(z) + 4(1-\alpha)^2 q^2(z) + 2(1-\alpha)e_1 q(z) + cz - 2(1-\alpha)e_2 = 0, \quad (3.4)$$

where $e_1 = 2\kappa + 4(\alpha - 1)$ and $e_2 = 2\kappa + 2(\alpha - 1)$.

If

$$\psi(r, s; z) = 4(1-\alpha)s + 4(1-\alpha)^2 r^2 + 2(1-\alpha)e_1 r + cz - 2(1-\alpha)e_2$$

and $\mathbb{E} = \{0\}$, then (3.4) implies $\psi\left(q(z), zq'(z); z\right) \in \mathbb{E}$ for all $z \in \mathbb{D}$. We use Lemma 1.1 to prove that $\operatorname{Re} q(z) > 0$ for all $z \in \mathbb{D}$. For $z = x + \mathrm{i}y \in \mathbb{D}$ (with $x, y \in \mathbb{R}$) and $\rho, \sigma \in \mathbb{R}$ satisfying $\sigma \leq -(1+\rho^2)/2$, we obtain

$$\begin{aligned}
\operatorname{Re} \psi(\rho\mathrm{i}, \sigma; x + \mathrm{i}y) &= 4(1-\alpha)\sigma - 4(1-\alpha)^2\rho^2 + cx - 2(1-\alpha)e_2 \\
&\leq -2(1-\alpha)(3-2\alpha)\rho^2 + cx - 2(1-\alpha)(1+e_2) \\
&< |c| + 2(1-\alpha)(1-2\alpha) - 4(1-\alpha)\kappa \leq 0.
\end{aligned}$$

By Lemma 1.1 we conclude that $\operatorname{Re} q(z) > 0$ for all $z \in \mathbb{D}$. This result implies

$$\operatorname{Re}\left[1 + \frac{zu_p''(z)}{u_p'(z)}\right] = (1-\alpha)\operatorname{Re} q(z) + \alpha > \alpha \qquad \text{for all } z \in \mathbb{D},$$

which shows that u_p is convex of order α in $\mathbb{D}$.

(ii) Since

$$\kappa - 1 \geq \frac{|c| + 2(1-\alpha)(3-2\alpha)}{4(1-\alpha)} - 1 = \frac{|c| + 2(1-\alpha)(1-2\alpha)}{4(1-\alpha)},$$

it follows from part (i) of this theorem that u_{p-1} is convex of order α. By applying the general version of ALEXANDER's duality theorem [1] we conclude that $z \mapsto zu'_{p-1}(z)$ is starlike of order α in $\mathbb{D}$. According to part (v) of Proposition 2.14 this implies the function $z \mapsto zu_p(z)$ is also starlike of order α in $\mathbb{D}$, as we required.

(iii) Define the functions $g_p, h_p : \mathbb{D} \to \mathbb{C}$ by

$$g_p(z) = zu_p(z) \quad \text{and} \quad h_p(z) = z^{[2(1-\alpha)-p]/[2\alpha]} w_p\left(z^{1/[2\alpha]}\right),$$

respectively. Since $h_p(z) = a_0(p) z^{(1-\alpha)/\alpha} u_p\left(z^{1/\alpha}\right)$, where $a_0(p) = [2^p \Gamma(\kappa)]^{-1}$, it follows that

$$\frac{zh'_p(z)}{h_p(z)} = \frac{1}{\alpha}\left[\frac{z^{1/\alpha} g'_p(z^{1/\alpha})}{g_p(z^{1/\alpha})} - \alpha\right].$$

Finally, because g_p is starlike of order α, we deduce that h_p is starlike in $\mathbb{D}$. □

We note that if we choose $\alpha = 0$ in Theorem 3.1, then we reobtain parts (i) and (ii) of Theorem 2.16. Analogously, if we take $\alpha = 0$ in part (i) of Theorem 3.3, then we get part (iii) of Theorem 2.16. Parts (iv) and (v) of Theorem 2.16 are particular cases of part (ii) of Theorem 3.3, and choosing $\alpha = 1/2$ from part (iii) of Theorem 3.3 we reobtain part (vi) of Theorem 2.16. It is also worth mentioning that results similar to those given in Theorem 3.1 were obtained by PONNUSAMY and VUORINEN [20], [21] for Gaussian and confluent hypergeometric functions.

4. Close-to-convexity of the generalized Bessel functions

Motivated by the papers of PONNUSAMY and VUORINEN [20], [21], we discuss in this section a few conditions concerning the parameters of u_p, which guarantee the close-to-convexity with respect to the convex functions $f_1, f_2 : \mathbb{D} \to \mathbb{C}$, defined by

$$f_1(z) := -\log(1-z) \quad \text{and} \quad f_2(z) := \frac{1}{2}\log\frac{1+z}{1-z}.$$

Moreover, our aim is to improve some of the main results of Section 2.

Theorem 4.1. *If $c < 0$ and $b, p \in \mathbb{R}$, then u_p has the following properties:*

(i) *If* $\kappa \geq -c/2$, *then* $z \mapsto zu_p(z)$ *is close-to-convex with respect to the function* f_1.

(ii) *If* $\kappa \geq -3c/4$, *then* $z \mapsto zu_p(z^2)$ *is close-to-convex with respect to the function* f_2.

Proof. (i) Set $f(z) = zu_p(z) = z+b_1z^2+b_2z^3+\ldots+b_{n-1}z^n+\ldots$, where b_n is defined by (2.11). Clearly we have $b_{n-1} > 0$ for all $n \geq 2$ and $2b_1 = -c/(2\kappa) \leq 1$. From the definition of the ascending factorial notation we observe that (we use the formula $(\kappa)_n = (\kappa+n-1)(\kappa)_{n-1}$)

$$b_n = -\frac{c}{4n(\kappa+n-1)}b_{n-1}.$$

We use Lemma 1.2 to prove that f is close-to-convex with respect to the function $f_1(z) = -\log(1-z)$. Therefore we need to show that $\{nb_{n-1}\}_{n\geq 1}$ is a decreasing sequence. By a short computation we obtain

$$nb_{n-1} - (n+1)b_n = b_{n-1}\left[n + \frac{c(n+1)}{4n(\kappa+n-1)}\right] = \frac{b_{n-1}\cdot U_1(n)}{4n(\kappa+n-1)},$$

where $U_1(n) = 4n^3+4(\kappa-1)n^2+cn+c$. Using the inequalities $n^3 \geq 3n^2-3n+1$ and $n^2 \geq 2n-1$, we obtain

$$\begin{aligned}U_1(n) \geq 4(\kappa+2)n^2 + (c-12)n + c + 4 \geq [8(\kappa+2) + c - 12]n\\ - 4(\kappa+2) + c + 4 \geq U_1(1) = 2(2\kappa+c) \geq 0.\end{aligned}$$

because $\kappa+2>0$ and $4(2\kappa+1)+c>0$ by the assumptions. This implies that $nb_{n-1}-(n+1)b_n \geq 0$ for all $n \geq 1$, thus, $\{nb_{n-1}\}_{n\geq 1}$ is a decreasing sequence. By Lemma 1.2 it follows that f is close-to-convex with respect to the convex function $-\log(1-z)$.

(ii) Set $g(z) = zu_p(z^2) = z+b_3z^3+\ldots+b_{2n-1}z^{2n-1}+\ldots$, where b_n is defined by (2.11). Therefore we have $3b_1 = -(3c)/(4\kappa) \leq 1$ and $b_{2n-1} > 0$ for all $n \geq 2$. We want to show that $\{(2n-1)b_{2n-1}\}_{n\geq 2}$ is a decreasing sequence. Fix $n \geq 2$. Then we have

$$(2n-1)b_{2n-1} - (2n+1)b_{2n+1} = \frac{b_{2n-1}\cdot U_2(n)}{4n(\kappa+n-1)},$$

where $U_2(n) = 8n^3+8(\kappa-3/2)n^2-4(\kappa-c/2-1)n+c$. Using the inequalities $n^3 \geq 3n^2-3n+1$ and $n^2 \geq 2n-1$, we obtain

$$U_2(n) \geq 8(\kappa+3/2)n^2 - 4(\kappa-c/2+5)n + c + 8$$

$$\geq 12(\kappa + c/6 + 1/3)n - 8(\kappa + 3/2) + c + 8$$
$$\geq 4(\kappa + 3c/4) \geq 0.$$

Hence $\{(2n-1)b_{2n-1}\}_{n\geq 2}$ is a decreasing sequence. But the function f_2 is convex, so by applying Lemma 1.2 the desired conclusion follows. $\square$

Observe that choosing $c = -1$ and $b = 1$ in Theorem 4.1 we get the following sufficient condition of close-to-convexity: if $p \geq -1/2$, then $z \mapsto z\mathcal{I}_p(z^{1/2})$ is close-to-convex with respect to the function f_1, while if $p \geq -1/4$, then $\mapsto z\mathcal{I}_p(z)$ is close-to-convex with respect to the function f_2. In particular, the function $z \mapsto zJ_0(\mathrm{i}z)$ is close-to-convex with respect to the function f_2, because $z\mathcal{I}_0(z) = zI_0(z) = zJ_0(iz)$ satisfies the condition of part (ii) of Theorem 4.1. We note that this property of the Bessel function of the first kind of zero order was pointed out also by Ponnusamy and Vuorinen [20, p. 83].

It is also worth mentioning that, since every close-to-convex function is univalent, the above results of Theorem 4.1 imply that under the corresponding hypothesis the functions $z \mapsto zu_p(z)$ and $z \mapsto zu_p(z^2)$ are univalent in $\mathbb{D}$. In what follows we would like to improve the range of univalence (when the parameter c is negative) from Theorem 2.16, by using an interesting idea of Ponnusamy [19].

Let $f : \mathbb{D} \to \mathbb{C}$ be of the form $f(z) = z + a_2z^2 + \ldots + a_nz^n + \ldots$. The Alexander transform $\Lambda_f : \mathbb{D} \to \mathbb{C}$ of f is defined by

$$\Lambda_f(z) = \int_0^z \frac{f(t)}{t}\,\mathrm{dt} = z + \sum_{n\geq 2} \frac{a_n}{n} z^n.$$

The following result completes part (i) of Theorem 3.1 and contains some properties of the Alexander transform of the function $z \mapsto zu_p(z)$, which will be helpful to improve the range of univalence from Theorem 2.16.

Theorem 4.2. *Let b, p be arbitrary real numbers and let $c < 0$. If $4\kappa \geq -(c+2) + \sqrt{c^2/2 - 4c + 4}$, then the Alexander transform of the function $z \mapsto zu_p(z)$ is close-to-convex with respect to the function $-\log(1-z)$ and it is starlike in $\mathbb{D}$. Moreover, we have that $\operatorname{Re} u_p(z) > 1/2$ holds for all $z \in \mathbb{D}$.*

Proof. From (2.11) we have

$$f(z) = zu_p(z) = z + \sum_{n\geq 2} b_{n-1}z^n = \sum_{n\geq 1} \frac{(-c/4)^{n-1}z^n}{(\kappa)_{n-1}(n-1)!}.$$

So in this case the corresponding Alexander transform takes the form

$$\Lambda_f(z) = \sum_{n\geq 1} A_nz^n, \text{ where } A_n = \frac{b_{n-1}}{n} = \frac{(-c/4)^{n-1}}{(\kappa)_{n-1}n!} \quad \text{for all } n \geq 1.$$

Obviously we have $A_1 = 1$. Because $c < 0$ and $4\kappa \geq 2\lambda(c) > -c > 0$, we also have $A_n > 0$ for all $n \geq 2$, where

$$\lambda(c) := \frac{-(c+2) + \sqrt{c^2/2 - 4c + 4}}{2}.$$

Next we prove that the sequence $\{nA_n\}_{n\geq 1}$ is decreasing. Fix any $n \geq 1$. From the definition of the Pochhammer symbol it follows

$$(n+1)A_{n+1} = -\frac{c}{4(\kappa + n - 1)} \cdot A_n. \tag{4.3}$$

Using (4.3) we have

$$nA_n - (n+1)A_{n+1} = \frac{U_1(n) \cdot A_n}{4(\kappa + n - 1)}, \tag{4.4}$$

where $U_1(n) = 4n^2 + 4(\kappa - 1)n + c$. Since $n^2 \geq 2n - 1$ and $4\kappa > -c$, we have

$$U_1(n) \geq 4(\kappa + 1)n + c - 4 \geq U_1(1) = 4\kappa + c > 0.$$

Consequently, (4.4) yields $nA_n > (n+1)A_{n+1}$. This shows that the sequence $\{nA_n\}_{n\geq 1}$ is strictly decreasing.

Next, we show that the sequence $\{nA_n - (n+1)A_{n+1}\}_{n\geq 1}$, is also decreasing. For convenience we denote $B_n = nA_n - (n+1)A_{n+1}$ for each $n \geq 1$. Fix any $n \geq 1$. Using (4.4), we find that

$$B_n - B_{n+1} = \frac{U_2(n) \cdot A_n}{2(n+1)(\kappa + n)(\kappa + n - 1)},$$

where

$$U_2(n) = 2n^4 + 4\kappa n^3 + D_1 n^2 + D_2 n + D_3,$$

$$D_1 = 2\kappa^2 + 2\kappa + c - 2, \quad D_2 = 2\kappa^2 + (c-2)\kappa + c, \quad D_3 = (c + 8\kappa)c/8.$$

Our aim is to show that $U_2(n) > 0$. First we observe that the inequality $n^4 \geq 4n^3 - 6n^2 + 4n - 1$ holds. By using this inequality we obtain $U_2(n) \geq V(n)$, where

$$V(n) = 4(\kappa + 2)n^3 + (D_1 - 12)n^2 + (D_2 + 8)n + D_3 - 2.$$

Clearly, the coefficient of n^3 in the above expression is nonnegative, since $\kappa > 0$. Therefore using that $n^3 \geq 3n^2 - 3n + 1$, we obtain $V(n) \geq W(n)$, where

$$\begin{aligned} W(n) &= D_4 n^2 + D_5 n + D_6, \\ D_4 &= 2\kappa^2 + 14\kappa + c + 10, \quad D_5 = 2\kappa^2 + (c - 14)\kappa + c - 16, \\ D_6 &= c^2/8 + (c+4)\kappa + 6. \end{aligned}$$

Now, we observe that D_4 is also nonnegative, because

$$\kappa \geq [\lambda(c)]/2 > -c/4 > [-7 + \sqrt{29 - 2c}]/2,$$

where the value $[-7 + \sqrt{29 - 2c}]/2$ is the greatest root of the equation $D_4 = 0$. Similarly $n^2 \geq 2n - 1$, therefore $W(n) \geq X(n)$, where $X(n) = D_7 n + D_8$, $D_7 = 2D_4 + D_5$ and $D_8 = D_6 - D_4$. Analogously, by the hypothesis, we can deduce easily that $D_7 = 6\kappa^2 + (c + 14)\kappa + 3c + 4 > 0$. Indeed, the relation

$$\kappa \geq [\lambda(c)]/2 > -c/4 > [-(c + 14) + \sqrt{c^2 - 44c + 100}]/12 =: \kappa_c$$

(here κ_c is the greatest root of the equation $D_7 = 0$) implies that D_7 is nonnegative, and leads to $X(n) \geq X(1)$. In this case

$$X(1) = D_4 + D_5 + D_6 = 4\kappa^2 + 2(c + 2)\kappa + c^2/8 + 2c$$

is also positive, because $\kappa \geq \lambda(c)/2 > -c/4 > 0$. Thus, we have proved a chain of inequalities

$$U_2(n) \geq V(n) \geq W(n) \geq X(n) \geq X(1) > 0,$$

which implies $B_n - B_{n+1} > 0$. Thus the sequence $\{nA_n - (n + 1)A_{n+1}\}_{n\geq 1}$ is strictly decreasing. By Lemma 1.3 we deduce that Λ_f is starlike in $\mathbb{D}$.

The sequence $\{nA_n\}_{n\geq 1}$ is strictly decreasing and $2A_2 = b_1 = -c/(4\kappa) < 1$. Thus it follows by Lemma 1.2 that Λ_f is close-to-convex with respect to $-\log(1 - z)$. Now, we apply Lemma 1.3 to prove that $\operatorname{Re} u_p(z) > 1/2$ for all $z \in \mathbb{D}$. For this consider $g = u_p$. Therefore we have $C_n = b_{n-1} = nA_n$ for all $n \geq 1$ and thus the sequence $\{C_n\}_{n\geq 1}$ is strictly decreasing. In addition we have $C_n - 2C_{n+1} + C_{n+2} = B_n - B_{n+1} > 0$ for all $n \geq 1$. Thus, Lemma 1.3 yields the asserted property, which completes the proof. □

An important consequence of Theorem 4.2 is the following result:

Corollary 4.5. *If b, p are arbitrary real numbers and $c < 0$ such that $\kappa \geq -c/4 - 1$ and $\kappa \neq 0$, then the function u_p is univalent in $\mathbb{D}$.*

Proof. By the proof of Theorem 4.2 the Alexander transform

$$\int_0^z u_{p+1}(t)\,\mathrm{dt}$$

is close-to-convex with respect to the convex function $-\log(1 - z)$ if $\kappa + 1 \geq -c/4$, and therefore, in particular, it is univalent. Using the relation $4\kappa u_p'(z) = -cu_{p+1}(z)$, we have

$$\int_0^z u_{p+1}(t)\,\mathrm{dt} = -\frac{4\kappa}{c}\int_0^z u_p'(t)\,\mathrm{dt} = -\frac{4\kappa}{c}[u_p(z) - 1].$$

Consequently the function $z \mapsto -4\kappa[u_p(z) - 1]/c$ is univalent in $\mathbb{D}$. Since the addition of a constant and the multiplication by a nonzero quantity do not disturb the univalence, we immediately deduce that u_p is univalent in $\mathbb{D}$. This completes the proof. $\square$

From part (ii) of Theorem 2.16 u_p is univalent in $\mathbb{D}$ when $\kappa \geq |c|/4$ and $c \neq 0$. If we consider $c < 0$, then the above conditions become $\kappa \geq -c/4$. Since $-c/4 > -c/4 - 1$, it follows that the result of Corollary 4.5 is better than the above result. Moreover, recently it was proved in the forthcoming paper [4] that the result of Corollary 4.5 can be improved too. Namely, using directly the second part of Lemma 1.3, it can be shown that [4, Theorem 3.11] if $\kappa \geq -c/8 - 1$ for $b, p \in \mathbb{R}$, $c \in [-37, 0)$, then u_p is univalent in $\mathbb{D}$.

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