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Finsleroid–Finsler space of involutive case and A-special relation

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Abstract. The involutive case means the framework in which the characteristic scalar $g(x)$ may vary in the direction assigned by the input 1-form b, such that $dg = \mu b$ with a scalar $\mu(x)$. Required calculation shows that in the Finsleroid–Finsler space the involutive case realizes through the A-special relation the picture that instead of the Landsberg condition $\dot{A}_{ijk} = 0$ we have the vanishing $\dot{\alpha}_{ijk} = 0$ with the normalized tensor $\alpha_{ijk} = A_{ijk}/\|A\|$. Success is predetermined by a reached possibility to write down the associated spray coefficients in the transparent form that accounts for the dependence $g = g(x)$. Interesting particular properties of the associated hv-curvature tensor come to play.

1. Introduction and motivation

Among various possible methods to specify the Finsler space, raising forth the Landsberg condition $\dot{A}_{ijk} = 0$ occupies an important geometrical role (see [1]–[3]). In the Finsleroid–Finsler space, the condition can be realized in a simple and attractive way [4], [5]. At the same time, the condition requires the Finsleroid charge g to be a constant. How should we overcome the restriction?

At the first sight, in the Finsler geometry the weak Landsberg condition $\dot{A}_i = 0$ is to be considered as being a next-step extension of the proper Landsberg condition $\dot{A}_{ijk} = 0$. However, in the Finsleroid–Finsler space both the conditions are tantamount (because of the particular representation (1.27)).

A scrupulous analysis performed has revealed a remarkable observation that an attractive method to permit $g \neq$ const is to use the nullification condition

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$\dot{\alpha}_{ijk} = 0$ with the normalized Cartan tensor α_{ijk}. Clearly, the condition is attained when the A-special relation (2.5) holds. Remarkably, the relation occurs being reachable upon assuming that the scalar $g(x)$ reveals the involutive behaviour: $dg = \mu(x)b$ (see (3.1)). The last equality can locally be written as $dg = \mu b_i(x)dx^i$, which means geometrically that the scalar $g(x)$ varies just in the direction assigned by the vector $b_i(x)$.

The Finsleroid–Finsler space can be constructed as follows. Let M be an N-dimensional C^∞ differentiable manifold, T_xM denote the tangent space to M at a point $x \in M$, and $y \in T_xM\backslash 0$ mean tangent vectors. Suppose we are given on M a positive-definite Riemannian metric $S = S(x,y)$. Denote by $R_N = (M,S)$ the obtained N-dimensional Riemannian space. Let us also assume that the manifold M admits a non-vanishing 1-form $b = b(x,y)$ which is unit: $\|b\| = \|b\|_{\text{Riemannian}} = 1$. It is convenient to use the variable

$$q = \sqrt{S^2 - b^2}. \tag{1.1}$$

With respect to natural local coordinates in the space R_N we have the local representations $\|b\| = \sqrt{a^{ij}b_ib_j}$ and $b = b_i(x)y^i$, together with $S = \sqrt{a_{ij}(x)y^iy^j}$. The covariant index of the vector b_i will be raised by means of the Riemannian rule $b^i = a^{ij}b_j$, which inverse reads $b_i = a_{ij}b^j$. The reciprocity $a^{in}a_{nj} = \delta^i{}_j$ is assumed, where $\delta^i{}_j$ stands for the Kronecker symbol. It is convenient to use the tensor $r_{ij}(x) := a_{ij}(x) - b_i(x)b_j(x)$ to have the representation $q = \sqrt{r_{ij}(x)y^iy^j}$ of the scalar (1.1). The vanishing $r_{ij}b^j = 0$ (coming from $\|b\| = 1$) reduces many expressions arisen in processes of various calculations.

Let $g = g(x)$ be a scalar specified as follows:

$$-2 < g(x) < 2. \tag{1.2}$$

We shall apply the convenient notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \qquad G = \frac{g}{h}. \tag{1.3}$$

The *Finsleroid-characteristic quadratic form*

$$B(x,y) := b^2 + gbq + q^2 \equiv S^2 + gb\sqrt{S^2 - b^2} \tag{1.4}$$

is of the negative discriminant $D_{\{B\}} = -4h^2 < 0$ and, therefore, is positively definite. We use the function

$$\tau = 1 + gw + w^2 \equiv \frac{B}{b^2}, \tag{1.5}$$

where $w = q/b$, to produce the function

$$K = b \exp \int \frac{w dw}{\tau}. \tag{1.6}$$

Since the function (1.5) is representable in the form

$$\tau = h^2 + \left(w + \frac{g}{2}\right)^2, \tag{1.7}$$

the integration process in (1.6) is simple, namely, the result is given by the following definition.

Key Definition. The scalar function $K(x, y)$ given by the formulas

$$K(x, y) = \sqrt{B(x, y)}\, J(x, y), \quad J(x, y) = \mathrm{e}^{-\frac{1}{2} G(x) f(x,y)}, \tag{1.8}$$

where

$$f = -\arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if} \quad b \geq 0, \tag{1.9}$$

and

$$f = \pi - \arctan \frac{G}{2} + \arctan \frac{L}{hb}, \quad \text{if} \quad b \leq 0, \tag{1.10}$$

with

$$L = q + \frac{g}{2} b, \tag{1.11}$$

is called the *Finsleroid–Finsler metric function.*

The function K has been normalized such that $0 \leq f \leq \pi$ and the Finsleroid length $K(x, b^i(x))$ of the vector b^i is equal to the Riemannian length $\|b\| = 1$, such that

$$K\left(x, b^i(x)\right) = 1. \tag{1.12}$$

The zero-vector $y = 0$ is excluded from consideration. The positive (not absolute) homogeneity holds:

$$K(x, \lambda y) = \lambda K(x, y), \quad \lambda > 0, \ \forall x, \ \forall y.$$

Entailed Definitions. The arisen space $FF^{PD}_g := \{R_N;\, b_i(x);\, g(x);\, K(x, y)\}$ is called the *Finsleroid–Finsler space.* The space R_N is called the *associated Riemannian space.* Within any tangent space T_xM, the Finslerian metric function $K(x, y)$ given by the formulas (1.8)–(1.11) produces the *Finsleroid*

$$F^{PD}_{g\,\{x\}} := \big\{y \in F^{PD}_{g\,\{x\}} : y \in T_xM, \ K(x, y) \leq 1\big\}. \tag{1.13}$$

The *Finsleroid Indicatrix* $I^{PD}_{g\,\{x\}} \in T_xM$ is the boundary of the Finsleroid:

$$I^{PD}_{g\,\{x\}} := \big\{y \in I^{PD}_{g\,\{x\}} : y \in T_xM, \ K(x, y) = 1\big\}. \tag{1.14}$$

Since at $g = 0$ the FF^{PD}_g-space is Riemannian, then the body $F^{PD}_{g=0\,\{x\}}$ is a unit ball supported by the point x, and $I^{PD}_{g=0\,\{x\}}$ is a unit sphere.

The scalar $g(x)$ is called the *Finsleroid charge.* The 1-form $b = b_i(x)y^i$ is called the *Finsleroid-axis* 1*-form.*

We can evaluate straightforwardly the Finsleroid metric tensor components $g_{ij} = (1/2)\partial^2 K^2/\partial y^i \partial y^j$, together with their reciprocals g^{ij} (so that $g_{in}g^{ij} = \delta^j{}_n$, where $\delta^j{}_n$ is the Kronecker symbol). The determinant of the tensor is found to read merely

$$\det(g_{ij}) = J^{2N} \det(a_{ij}). \tag{1.15}$$

The right-hand part of (1.15) is everywhere positive.

The FF^{PD}_g-space is smooth of the class C^2, and not of the class C^3, on all of the *slit tangent bundle* $TM \setminus 0$. The FF^{PD}_g-space is smooth of the class C^∞ on all of the *b-slit tangent bundle*

$$T_b M \ := TM \setminus 0 \setminus b \setminus -b \tag{1.16}$$

(obtained by deleting out in $TM\setminus 0$ all the directions which point along, or oppose, the directions given rise to by the 1-form b).

The associated Riemannian space provides us with the Riemannian covariant derivative $\nabla_i b_j \ := \partial_i b_j - b_k a^k{}_{ij}$, where $a^k{}_{ij} \ := (1/2)a^{kn}(\partial_j a_{ni} + \partial_i a_{nj} - \partial_n a_{ji})$ are the respective Christoffel symbols.

By means of attentive (lengthy) evaluations we can find the induced (geodesic) *spray coefficients*

$$G^k = \gamma^k{}_{ij} y^i y^j, \tag{1.17}$$

where $\gamma^k{}_{ij}$ stand for the Finslerian Christoffel symbols constructed from the metric tensor g_{ij}, trace each possible cancellation or reduction, and eventually arrive at the following assertion.

The explicit form of the spray coefficients of the Finsleroid–Finsler space reads

$$G^k = g(qa^{kj} - gv^k b^j)y^h(\nabla_h b_j - \nabla_j b_h) + \frac{g}{q} v^k y^h y^m \nabla_h b_m + a^k{}_{mn} y^m y^n + E^k, \tag{1.18}$$

where $v^k = y^k - bb^k$ and

$$\begin{aligned} E^k &= \bar{M}(yg)y^k + \frac{1}{2}K^2(yg)\frac{\partial \bar{M}}{\partial y^h} g^{kh} - \frac{1}{2}\bar{M}K^2 g_h g^{kh} \\ &= \bar{M}(yg)y^k + \frac{2Kq^2}{gNB}(yg)A^k - \frac{1}{2}\bar{M}K^2 g_h g^{kh} \end{aligned} \tag{1.19}$$

with $(yg) = y^h g_h$ and $g_h = \partial g/\partial x^h$. The scalar $\bar{M}$ is defined by the equality $\partial K^2/\partial g = \bar{M}K^2$.

In obtaining the coefficients (1.19) we have used the derivatives

$$\frac{\partial h}{\partial g} = -\frac{1}{4}G, \quad \frac{\partial G}{\partial g} = \frac{1}{h^3}, \quad \frac{\partial\left(\frac{G}{h}\right)}{\partial g} = \frac{1}{h^4}\left(1 + \frac{g^2}{4}\right), \tag{1.20}$$

$$\frac{\partial f}{\partial g} = -\frac{1}{2h} + \frac{b}{B}\left(\frac{1}{4}Gq + \frac{1}{2h}b\right), \tag{1.21}$$

and

$$\bar{M} = -\frac{1}{h^3}f + \frac{1}{2}\frac{G}{hB}q^2 + \frac{1}{h^2B}bq, \quad \frac{\partial \bar{M}}{\partial y^h} = \frac{4q^2}{gNBK}A_h. \tag{1.22}$$

Elucidating the structure of the associated Cartan tensor

$$A_{ijk} := \frac{K}{2}\frac{\partial g_{ij}}{\partial y^k} \tag{1.23}$$

leads to numerous simple representations (written explicitly in [6]) involving the vector $A_k = g^{ij}A_{ijk}$.

Having been evaluated from the Finsleroid–Finsler metric function K, the norm $\|A\| = \sqrt{A^k A_k}$ proves to be independent of vectors y, namely we obtain

$$\|A\| = \frac{N}{2}|g(x)|. \tag{1.24}$$

It is convenient to construct the *normalized Cartan tensor*

$$\alpha_{ijk} := \frac{1}{\|A\|}A_{ijk} \tag{1.25}$$

and the vector

$$\alpha_k := \frac{1}{\|A\|}A_k \tag{1.26}$$

which is of the unit length: $\alpha_h\alpha^h = 1$. With using the angular Finslerian metric tensor $h_{ij} = g_{ij} - (1/K^2)y_iy_j$, where $y_i = K\partial K/\partial y^i$, we have

$$\alpha_{ijk} = \frac{1}{N}\left(h_{ij}\alpha_k + h_{ik}\alpha_j + h_{jk}\alpha_i - \alpha_i\alpha_j\alpha_k\right) \tag{1.27}$$

everywhere in the Finsleroid–Finsler space.

In our analysis, an important role is played by the tensor

$$H_{ij} = h_{ij} - \alpha_i\alpha_j, \tag{1.28}$$

which obviously possesses the nullification properties

$$H_{ij}y^j = 0, \qquad H_{ij}A^j = 0. \tag{1.29}$$

The respective h-curvature tensor $R^i{}_k$ can be constructed from the above spray coefficients according to the general rule, that is,

$$K^2R^i{}_k = 2\frac{\partial \bar{G}^i}{\partial x^k} - y^j\frac{\partial \bar{G}^i{}_k}{\partial x^j} - \bar{G}^i{}_n\bar{G}^n{}_k + 2\bar{G}^n\bar{G}^i{}_{nk} \tag{1.30}$$

(see [2]), where

$$\bar{G}^i = \frac{1}{2}G^i, \quad \bar{G}^i{}_k = \frac{1}{2}\frac{\partial G^i}{\partial y^k}, \quad \bar{G}^i{}_{nk} = \frac{1}{2}\frac{\partial G^i{}_k}{\partial y^n}, \tag{1.31}$$

and $a_n{}^i{}_{km}$ stands for the Riemannian curvature tensor of the associated Riemannian space.

In Section 2 we indicate the interesting implications of the A-special relation, including the simplification of the skew-part of the hv-curvature tensor in which the indicatrix curvature tensor comes to play. It is the part that enters the current which is the right-hand side of the covariant conservation law (2.22).

In Section 3 the involutive case is formulated. The case entails the A-special relation under the b-parallel condition.

In Conclusions several important ideas motivated our approach are emphasized.

2. *A*-special relation

By means of the over-dot we conveniently denote the action of the operator $_{|m}l^m$, such that

$$\dot{A}_i = A_{i|m}l^m, \quad \dot{A}_{ijk} = A_{ijk|m}l^m, \qquad \dot{\alpha}_i = \alpha_{i|m}l^m, \quad \dot{\alpha}_{ijk} = \alpha_{ijk|m}l^m, \tag{2.1}$$

with $_{|m}$ meaning the horizontal covariant derivative; $l^i = y^i/K$. The dotted tensor $\dot{A}_{ijk}$ is identical to that used in [2], that is, $\dot{A}_{ijk}$ is the horizontal covariant derivative of the Cartan tensor A_{ijk} along the distinguished (horizontal) direction $l^i\frac{\partial}{\partial x^i}$. Let us set forth the nullification

$$\dot{\alpha}_{ijk} = 0. \tag{2.2}$$

Whenever the representation (1.27) is valid, the condition (2.2) is equivalent to the vanishing

$$\dot{\alpha}_i = 0 \tag{2.3}$$

of the normalized vector (1.26).

Denoting

$$\gamma_k = \frac{1}{2A^h A_h}\,(A^m A_m)_{|k}, \quad \gamma = \frac{1}{2A^h A_h}\,(A^m A_m)_{|k} l^k, \tag{2.4}$$

and assuming that the A-*special relation*

$$A_{i|k} = \gamma_k A_i + \eta H_{ik} \tag{2.5}$$

holds, where η is a scalar, we are obviously entitled to write the equality

$$\alpha_{i|k} = \frac{1}{\|A\|}\eta H_{ik}. \tag{2.6}$$

Since $H_{ik}y^k = 0$ (see (1.29)), from (2.5) we directly conclude that

$$\dot{A}_i = \gamma A_i, \tag{2.7}$$

which is obviously *tantamount to* (2.3). From the representation (1.27) of the tensor α_{ijk} we obtain

$$A_{ijk|l} = \gamma_l A_{ijk} + \eta \frac{1}{N}\big(H_{ij}H_{kl} + H_{ik}H_{jl} + H_{jk}H_{il}\big), \tag{2.8}$$

which entails

$$\dot{A}_{ijk} = \gamma A_{ijk}. \tag{2.9}$$

Also, (2.3) entails the nullification

$$\dot{H}_{jk} = 0 \tag{2.10}$$

(consider the definition (1.28) of the tensor H_{jk} and take into account that $h_{jk|l}$=0 in any Finsler space), where $\dot{H}_{jk} = H_{jk|m}l^m$.

The hv-curvature tensor

$$P_{jikl} := -\big(A_{ijl|k} - A_{jkl|i} + A_{kil|j}\big) + A_{ij}{}^u \dot{A}_{ukl} - A_{jk}{}^u \dot{A}_{uil} + A_{ki}{}^u \dot{A}_{ujl} \tag{2.11}$$

(this representation is tantamount to the definition (3.4.11) on p. 56 of the book [2]) gets essentially reduced upon plugging (2.8) and (2.9), namely it reads

$$P_{jikl} = -\big(A_{ijl}\gamma_k - A_{jkl}\gamma_i + A_{kil}\gamma_j\big) - \eta\frac{1}{N}\big(H_{ij}H_{kl} + H_{ik}H_{jl} + H_{jk}H_{il}\big) \\ + \gamma(A_{ij}{}^u A_{ukl} - A_{jk}{}^u A_{uil} + A_{ki}{}^u A_{ujl}). \tag{2.12}$$

Let us consider the skew-part

$$P^{[ji]}{}_{kl} \ := \frac{1}{2}(P^{ji}{}_{kl} - P^{ij}{}_{kl}). \tag{2.13}$$

From (2.12) it follows that

$$P_{[ji]kl} = A_{jkl}\gamma_i - A_{kil}\gamma_j + \gamma(A_{ki}{}^u A_{ujl} - A_{jk}{}^u A_{uil}). \tag{2.14}$$

The curvature of indicatrix is well-known to be described by the tensor

$$\hat{R}_i{}^j{}_{mn} := \frac{1}{K^2}(A_h{}^j{}_m A_i{}^h{}_n - A_h{}^j{}_n A_i{}^h{}_m). \tag{2.15}$$

By comparing (2.15) with (2.14) we may write

$$P_{[ji]kl} = A_{jkl}\gamma_i - A_{kil}\gamma_j + \gamma K^2 \hat{R}_{jikl}. \tag{2.16}$$

In the Finsleroid–Finsler space, the tensor (2.15) possesses the representation

$$K^2 \hat{R}_{ijmn} = \frac{1}{N^2}(A^k A_k)\Big(h_{in}h_{mj} - h_{im}h_{nj}\Big). \tag{2.17}$$

Thus the following assertion is valid.

Theorem 2.1. *If the A-special relation* (2.5) *holds together with the representation* (1.27) *of the normalized Cartan tensor, then the skew-part of the hv-curvature tensor is constructed according to* (2.16).

In any Finsler space, the identity

$$g^{jl}\Big(R_j{}^i{}_{il|t} + R_j{}^i{}_{lt|i} + R_j{}^i{}_{ti|l}\Big) = P^{li}{}_{iu}R^u{}_{lt} + P^{li}{}_{lu}R^u{}_{ti} + P^{li}{}_{tu}R^u{}_{il} \tag{2.18}$$

(see the formula (3.5.3) on p. 58 of the book [2]) holds and the tensor $R^u{}_{il}$ is skew-symmetric with respect to the subscripts. We can write the identity as

$$g^{jl}\Big(R_j{}^i{}_{il|t} + R_j{}^i{}_{lt|i} + R_j{}^i{}_{ti|l}\Big) = 2P^{[li]}{}_{iu}R^u{}_{lt} - P^{[li]}{}_{tu}R^u{}_{li}, \tag{2.19}$$

so that the covariant divergence of the tensor

$$\rho_{ij} \ := \frac{1}{2}(R_i{}^m{}_{mj} + R^m{}_{ijm}) - \frac{1}{2}g_{ij}R^{mn}{}_{nm} \tag{2.20}$$

is given by

$$\rho^i{}_{j|i} = -P^{[lm]}{}_{mu}R^u{}_{lj} + \frac{1}{2}P^{[lm]}{}_{ju}R^u{}_{lm} \tag{2.21}$$

which can be written as

$$\rho^i{}_{j|i} = J_j \tag{2.22}$$

with

$$J_j = P^{[lm]}{}_{ku}\left(-R^u{}_{lj}\delta^k{}_m + \frac{1}{2}R^u{}_{lm}\delta^k{}_j\right). \tag{2.23}$$

Owing to $y^j{}_{|i} = 0$, contracting (2.23) by y^j makes us conclude from (2.22) that in terms of the vector

$$\rho^i := \rho^i{}_j y^j \tag{2.24}$$

and the scalar

$$\Upsilon := J_j y^j \tag{2.25}$$

the equality

$$\rho^i{}_{|i} = \Upsilon \tag{2.26}$$

holds. Taking into account the identities $h^m{}_k y^k = 0$, $R^u{}_{lj} l^j = R^u{}_l$, and $R^u{}_l y^l = 0$, together with $P^{[lm]}{}_{ku} y^k = 0$, from (2.23) and (2.25) we conclude that

$$\Upsilon = -P^{[lm]}{}_{mu} R^u{}_l. \tag{2.27}$$

In the Finsleroid–Finsler space under study, we should use here the above theorem, obtaining simply

$$\Upsilon = (\gamma_l A_u - \gamma_m A^m{}_{lu})R^{ul} - \frac{1}{4}(N-2)g^2\gamma K R^u{}_u. \tag{2.28}$$

3. Finsleroid–Finsler space upon involution

Let us set forth the *involution condition*

$$dg = \mu b, \qquad \mu = \mu(x) \tag{3.1}$$

(in terms of local coordinates the first equality reads $g_i = \mu b_i$ with $g_i = \partial g/\partial x^i$), and formulate the following definition.

Definition. The arisen space

$$IFF^{PD}_g := \{FF^{PD}_g \text{ with } dg = \mu(x)b\} \tag{3.2}$$

is called the *involutive Finsleroid–Finsler space*, and $\mu(x)$ is called the *involution scalar*.

In the space (3.2), the quantities defined in (2.4) become simply $\gamma_k = g_k/g$ and $\gamma = g_k l^k/g$, so that the A-special relation (2.5) takes on the form

$$A_{i|k} = \frac{1}{g} g_k A_i + \eta H_{ik}. \tag{3.3}$$

We say that the space FF_g^{PD} is *b-parallel*, if the 1-form b is parallel in the sense of the associated Riemannian space, which reads $\nabla b = 0$ (that is, $\nabla_i b_j = 0$ with respect to local coordinates).

It proves that the following theorem is valid.

Theorem 3.1. *In the b-parallel involutive space IFF_g^{PD} the A-special relation* (3.3) *holds.*

As a direct consequence of the above theorem,

$$\{\nabla b = 0 \text{ and } dg = \mu b\} \implies \dot{\alpha}_i = 0. \tag{3.4}$$

From (2.4) and (3.1) we have

$$\gamma = \frac{1}{K} \frac{\mu}{g} b. \tag{3.5}$$

With this formula, the tensor (2.16) takes on the explicit representation

$$P_{[ji]kl} = \mu \frac{1}{g} \big(A_{jkl} b_i - A_{kil} b_j + bK \hat{R}_{jikl} \big). \tag{3.6}$$

To verify the above theorem, we note that the condition (3.1) entails $b(bg) = (yg)$, $(bg) = \mu$, $(yg) = \mu b$, where $(bg) = b^i g_i$, and obtain the equality

$$K g^{kj} g_j = \frac{2bw}{Ng} (bg) A^k + b(bg) l^k, \tag{3.7}$$

so that the representation (1.19) is simplified to read

$$E^k = \frac{1}{2} \bar{M} (yg) y^k - \widehat{M} K \frac{1}{Ng} w(yg) A^k, \tag{3.8}$$

where $\widehat{M} = \bar{M} - (2/B) b^2 w$. The simple equality

$$\frac{\partial \bar{M}_i}{\partial g} = -4 \frac{bq^3}{B^2} \frac{2}{KNg} A_i \tag{3.9}$$

can be obtained from (1.20)–(1.22). The quantity η entered the right-hand part of (3.3) can be taken from the formula (C.19) of [6], namely, we have explicitly

$$\eta = \frac{1}{4K}\mu \widehat{M}\frac{Ng}{2}\frac{B}{q} + \frac{1}{4K}\mu M\frac{Ng}{2}\left(\frac{b(2b+gq)}{2q} + q + gb\right). \tag{3.10}$$

The scalar (2.28) can now be written in the form

$$\Upsilon = \mu\left[\frac{1}{g}(b_l A_n - b_m A^m{}_{ln})R^{nl} - \frac{1}{4}(N-2)gbR^n{}_n\right] \tag{3.11}$$

which is proportional to the involution scalar μ.

The respective *involutive curvature tensor* $\breve{R}^i{}_k$ is constructed according to

$$K^2\breve{R}^i{}_k = 2\frac{\partial \bar{E}^i}{\partial x^k} - y^j\frac{\partial \bar{E}^i{}_k}{\partial x^j} - \bar{E}^i{}_n\bar{E}^n{}_k + 2\bar{E}^n\bar{E}^i{}_{nk} + y^n a_n{}^i{}_{km}y^m, \tag{3.12}$$

where

$$\bar{E}^i = \frac{1}{2}E^i, \qquad \bar{E}^i{}_k = \frac{1}{2}\frac{\partial E^i}{\partial y^k}, \qquad \bar{E}^i{}_{nk} = \frac{1}{2}\frac{\partial E^i{}_k}{\partial y^n}, \tag{3.13}$$

and $a_n{}^i{}_{km}$ stands for the Riemannian curvature tensor of the associated Riemannian space. Methodologically, the tensor $\breve{R}^i{}_k$ is of a novel geometrical type, being created by the gradient of the Finsleroid charge and constructed from the involutive spray coefficients E^k.

4. Conclusions

The Finsleroid–Finsler space involves a characteristic scalar, $g(x)$, such that the vanishing of the scalar reduces the space to a Riemannian space. Any dimension $N \geq 2$ is admissible. The Finsleroid as being defined according to (1.13) is *rotund*, namely it is a body of revolution about the axis assigned by the input 1-form b.

Varying $g(x)$ entails varying the form as well as the curvature of the Finsleroid. The Landsberg case of the Finsleroid–Finsler space implies strictly $g = \text{const}$, as a direct consequence of the norm value (1.24). To set a liberty to the scalar $g(x)$, we must overcome the restrictive case. It proves that a fruitful idea is to substitute the condition $\dot{\alpha}_{ijk} = 0$ with the Landsberg condition $\dot{A}_{ijk} = 0$ proper. Would one assume $\|A\| = \text{const}$, one observes that $\dot{\alpha}_{ijk} = 0$ implies $\dot{A}_{ijk} = 0$. In the Finsleroid–Finsler space under study, $g \neq \text{const}$ implies $\|A\| \neq \text{const}$ (see (1.24)). The involutive approach realizes an interesting particular way of dependence of g

on x, enabling at the same time to obtain sufficiently simple representations for the spray coefficients and entailed tensors.

Examining the conservation law for the fundamental tensor ρ_{ij} results in the equalities (2.20)–(2.23). In the Landsberg case, the hv-curvature tensor P_{ijkl} is well-known to be totally symmetric in all four of its indices (see p. 60 in [2]), such that the skew-part $P^{[lm]}{}_{ku}$, and whence the current J_j arisen from the right-hand part of the conservation law (2.22), vanishes identically. Under the A-special condition, however, the tensor $P^{[lm]}{}_{ku}$ is meaningful, being expressed through the indicatrix curvature tensor in accordance with (2.16), so that the current J_j is no more the nought. In the involutive case, the scalar μ can be factored out the expression of the current J_j (see particularly (3.11)), so that we may say that *the involution creates the current in the Finsleroid–Finsler space.*

The spray coefficients (1.18) include the part E^k which involves the gradient of $g(x)$. When $g = \text{const}$, the coefficients E^k vanish identically, in which case (1.18) coincide with the spray coefficients proposed in [5]. The involutive curvature tensor (3.12)–(3.13) is meaningful even if the associated Riemannian space is flat and the 1-form b is parallel.

All the calculation details which have underlined the present paper, and also convenient explicit representations of the involved derivative tensor $A_{i|j}$ and the curvature tensor $R^i{}_k$, can be found in [6]. It would be appealing to develop in future the extensions which can go over the b-parallel case $\nabla b = 0$.

Various Finslerian ideas of applications (see [7]–[9]) can well be matched to the ($g \neq \text{const}$)-Finsleroid–Finsler space.

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