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Matrix transformations on the matrix domains of triangles in the spaces of strongly C_1-summable and bounded sequences

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Abstract. Let w_0^p, w^p and w_∞^p be the sets of sequences that are strongly summable to zero, summable and bounded of index $p \geq 1$ by the Cesàro method of order 1, which were introduced by Maddox [I. J. Maddox, On Kuttner's theorem, J. London Math. Soc. **43** (1968), 285–290]. We study the matrix domains $w_0^p(T) = (w_0^p)_T$, $w^p(T) = (w^p)_T$ and $w_\infty^p(T) = (w_\infty^p)_T$ of arbitrary triangles T in w_0^p, w^p and w_∞^p, determine their β-duals, and characterize matrix transformations on them into the spaces c_0, c and ℓ_∞.

1. Introduction and preliminary results

Let ω denote the set of all complex sequences $x = (x_k)_{k=1}^\infty$. As usual, we write ℓ_∞, c, c_0 and ϕ for the sets of all bounded, convergent, null and finite sequences, and $\ell_p = \{x \in \omega : \sum_{k=1}^\infty |x_k|^p < \infty\}$ for $1 \leq p < \infty$. Let e and $e^{(n)}$ $(n = 1, 2, \dots)$ be the sequences with $e_k = 1$ for all $k \in \mathbb{N}$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$, where $\mathbb{N}$ denotes the set of positive integers.

A subspace X of ω is said to be an *BK space* if it is a Banach space with continuous coordinates $P_n : X \to \mathbb{C}$ $(n = 1, 2, \dots)$, where $P_n(x) = x_n$ for all $x \in X$. A BK space $X \supset \phi$ is said to have AK if every sequence $x = (x_k)_{k=1}^\infty \in X$ has a unique representation $x = \sum_{k=1}^\infty x_k e^{(k)}$.

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If X is a subset of ω then $X^\beta = \{a \in \omega : \sum_{k=1}^\infty a_k x_k$ converges for all $x \in X\}$ is called the *β-dual of X*.

Let $X \neq \{\theta\}$ be a Banach space and $S_X = \{x \in X : \|x\| = 1\}$ and $B_X = \{x \in X : \|x\| < 1\}$ be the unit sphere and open unit ball in X. Then X^* denotes the Banach space of all continuous linear functionals on X with its norm given by $\|f\| = \sup_{x \in S_X} |f(x)| = \sup_{x \in B_X} |f(x)|$.

Let $A = (a_{nk})_{n,k=1}^\infty$ be an infinite matrix of complex numbers and $x = (x_k)_{k=1}^\infty \in \omega$. We write $A_n = (a_{nk})_{k=1}^\infty$ for the sequence in the n-th row of A, and $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ and $Ax = (A_n x)_{n=1}^\infty$ provided the series $\sum_{k=1}^\infty a_{nk} x_k$ converges for all $n \in \mathbb{N}$. If X is a subset of ω then $X_A = \{x \in \omega : Ax \in X\}$ is the *matrix domain of A in X*. Given subsets X and Y of ω, we write (X, Y) for the class of all matrices A with $X \subset Y_A$, that is $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

An infinite matrix $T = (t_{nk})_{n,k=1}^\infty$ is said to be a *triangle* if $t_{nn} \neq 0$ for all $n \in \mathbb{N}$ and $t_{nk} = 0$ for $k > n$. We will frequently use the following well-known result that every triangle T has a unique inverse S which also is a triangle, and $T(Sx) = (TS)(x) = x$ for all $x \in \omega$ ([29, 1.4.8, p. 9] and [8, Remark 22 (a), p. 22]). Throughout, let T denote a triangle, and S its inverse.

Maddox [17] introduced and studied the following sets of sequences that are strongly summable and bounded with index p ($1 \leq p < \infty$) by the Cesàro method of order 1

$$w_0^p = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n |x_k|^p = 0 \right\}, \qquad w_\infty^p = \left\{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^n |x_k|^p < \infty \right\}$$

and

$$w^p = \left\{ x \in \omega : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n |x_k - \xi|^p = 0 \text{ for some } \xi \in \mathbb{C} \right\}.$$

Throughout we use the convention that every term with a subscript less than one is equal to zero.

We write $\sum_\nu = \sum_{k=2^\nu}^{2^{\nu+1}-1}$ and $\max_\nu = \max_{2^\nu \leq k \leq 2^{\nu+1}-1}$ for $\nu = 0, 1, \ldots$. The following result is known.

Proposition 1.1. ([20, Proposition 3.44, p. 207]) *Let $1 \leq p < \infty$. Then the sets w_0^p, w^p and w_∞^p are BK spaces with the (equivalent) norms*

$$\|x\|_{w_\infty^p} = \sup_{\nu \in \mathbb{N}} \left(\frac{1}{2^\nu} \sum\nolimits_\nu |x_k|^p \right)^{1/p} \quad \text{and} \quad \|x\|_{w_\infty^p}^\dagger = \sup_{n \in \mathbb{N}} \left(\frac{1}{n} \sum_{k=1}^n |x_k|^p \right)^{1/p};$$

w_0^p is a closed subspace of w^p and w^p is a closed subspace of w_∞^p; w_0^p has AK; every sequence $x = (x_k)_{k=1}^\infty \in w^p$ has a unique representation $x = \xi e + \sum_{k=1}^\infty (x_k - \xi) e^{(k)}$, where $\xi \in \mathbb{C}$ is the strong C_1-limit *of the sequence x, that is*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - \xi|^p = 0. \tag{1.1}$$

In the past, several authors studied matrix transformations on sequence spaces that are the matrix domains of the matrix Δ of the difference operator, or of the matrices of some of the classical methods of summability in spaces such as ℓ_p, c_0, c or ℓ_∞. For instance, some matrix domains of Δ were studied in [12], [16], [26], of the Cesàro matrices in [5], [6], [27], of the Euler matrices in [3, 4], [23], of the Riesz matrices in [2], and of the Nörlund matrices in [28]. All the matrices mentioned are triangles.

In this paper, we study the matrix domains $w_0^p(T) = (w_0^p)_T$, $w^p(T) = (w^p)_T$ and $w_\infty^p(T) = (w_\infty^p)_T$ $(1 \le p < \infty)$ of arbitrary triangles T in the spaces w_0^p, w^p and w_∞^p, determine their β-duals, and characterize matrix transformations on them into the spaces c_0, c and ℓ_∞.

The rest of this paper is organized, as follows:

In Section 2, some required definitions and the characterization of the matrix transformations from the spaces w_0^p, w^p and w_∞^p to the spaces ℓ_∞, c and c_0 are given. Section 3 is devoted to the determination of the β-duals of the spaces $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$. In Section 4, the classes (U_T, V) with $U \in \{w_0^p,\ w_\infty^p\}$ and $V \subset w$, (X_T, Y) with $X \in \{w_0^p,\ w^p,\ w_\infty^p\}$ and $Y \in \{\ell_\infty,\ c,\ c_0\}$ of infinite matrices are characterized. In the final section of the paper, the results are summarized, open problems and further suggestions are recorded.

Corollary 1.2. (a) *The sets $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$ are BK spaces with*

$$\|x\|_{w_\infty^p(T)} = \sup_\nu \left(\frac{1}{2^\nu} \sum\nolimits_\nu |T_k x|^p \right)^{1/p};$$

$w_0^p(T)$ is a closed subspace of $w^p(T)$, and $w^p(T)$ is a closed subspace of $w_\infty^p(T)$.

(b) *We put $c^{(n)} = \{c_k^{(n)}\}_{k=1}^\infty := T^{-1} e^{(n)} = S e^{(n)}$ for $n = 1, 2, \dots,$ that is*

$$c_k^{(n)} = \begin{cases} 0, & (1 \le k \le n-1), \\ s_{kn}, & (k \ge n). \end{cases}$$

Every sequence $z = (z_n)_{n=1}^\infty \in w_0^p(T)$ has a unique representation

$$z = \sum_{n=1}^{\infty} T_n z\, c^{(n)}. \tag{1.2}$$

(c) *We put define the sequence* $c^{(0)} = \{c_k^{(0)}\}_{k=1}^{\infty}$ *by*

$$c_k^{(0)} = \sum_{j=1}^{k} s_{kj} \quad (k = 1, 2, \dots).$$

Every sequence $z = (z_n)_{n=1}^{\infty} \in w^p(T)$ *has a unique representation*

$$w = \xi c^{(0)} + \sum_{n=1}^{\infty} (T_n z - \xi) c^{(n)}, \tag{1.3}$$

where $\xi \in \mathbb{C}$ *is the* strong limit of z in $w^p(T)$, *that is*

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |T_k z - \xi|^p \right) = 0. \tag{1.4}$$

Proof. (a) Part (a) is an immediate consequence of Proposition 1.1 and [29, Theorems 4.3.12 and 4.3.14, pp. 63 and 64].

(b) For every fixed n, we have

$$S_k e^{(n)} = \sum_{j=1}^{k} s_{kj} e^{(n)} = \begin{cases} 0, & (1 \le k \le n-1), \\ s_{kn}, & (k \ge n), \end{cases}$$

hence $c^{(n)} = Se^{(n)}$. Since w_0^p has AK by Proposition 1.1, we have $e^{(n)} \in w_0^p$ for all $n \in \mathbb{N}$, and it follows from $Tc^{(n)} = T(Se^{(n)}) = (TS)e^{(n)} = e^{(n)} \in w_0^p$ that $c^{(n)} \in w_0^p(T)$ for all $n \in \mathbb{N}$. Now let $z = (z_n)_{n=1}^{\infty} \in w_0^p(T)$ be given, that is $x = Tz \in w_0^p$. We obtain for $x^{[m]} = \sum_{k=1}^{m} x_k e^{(k)}$

$$\lim_{m\to\infty} \|x - x^{[m]}\|_{w_\infty^p} = \lim_{m\to\infty} \left\| x - \sum_{[n=1}^{m} x_n e^{(n)} \right\|_{w_\infty^p} = \lim_{m\to\infty} \left\| x - \sum_{n=1}^{m} T_n z\, e^{(n)} \right\|_{w_\infty^p} = 0.$$

We put $z^{\langle m\rangle} = \sum_{n=1}^{m} T_n z\, c^{(n)}$ for all m. Then we have

$$z^{\langle m\rangle} \in w_0^p(T),\ Tz^{\langle m\rangle} = \sum_{n=1}^{m} T_n z\, Tc^{(n)} = \sum_{n=1}^{m} x_n e^{(n)} = x^{[m]}$$

and so by Part (a)

$$\begin{aligned} \lim_{m\to\infty} \|z - z^{\langle m\rangle}\|_{w_\infty^p(T)} &= \lim_{m\to\infty} \|T(z - z^{\langle m\rangle})\|_{w_\infty^p} = \lim_{m\to\infty} \|Tz - Tz^{\langle m\rangle}\|_{w_\infty^p} \\ &= \lim_{m\to\infty} \|x - x^{[m]}\|_{w_\infty^p} = 0, \end{aligned}$$

that is the representation in (1.2) holds, which is obviously unique.

(c) We have $S_k e = \sum_{j=1}^{k} s_{kj} e_j = \sum_{j=1}^{k} s_{kj} = c_k$ for all $k \in \mathbb{N}$, hence $c = Se$ and $Tc = T(Se) = (TS)e = e \in w^p$ implies $c \in w^p(T)$. Let $z = (z_n)_{n=1}^{\infty} \in w^p(T)$ be given. Then $x = Tz \in w^p$ and by Proposition 1.1 there exists a unique complex number ξ that satisfies (1.1). We write $x^{(0)} = x - \xi e$ and put $z^{(0)} = z - \xi c$. Then we have $x^{(0)} \in w_0^p$, and it follows from $Tz^{(0)} = Tz - \xi Tc = x - \xi e = x^{(0)} \in w_0^p$ that $z^{(0)} \in w_0^p(T)$. So $z^{(0)}$ has a unique representation $z^{(0)} = \sum_{n=1}^{\infty} T_n z^{(0)} \, c^{(n)} = \sum_{n=1}^{\infty} (T_n - \xi e) c^{(n)}$ by Part (b), and so

$$z = \xi c + z^{(0)} = \xi c + \sum_{n=1}^{\infty} (T_n - \xi) \, c^{(n)}.$$

This establishes the unique representation in (1.3). □

Example 1.3. Let $\mathcal{U}$ be the set of all sequences $u = (u_k)_{k=1}^{\infty}$ with $u_k \neq 0$ for all $k \in \mathbb{N}$. If $u, v \in \mathcal{U}$ then we write $u/v = (u_k/v_k)_{k=1}^{\infty}$. Let $u, v \in \mathcal{U}$ be given and $T = (t_{nk})_{n,k=1}^{\infty}$ be the factorable matrix with $t_{nk} = u_n v_k$ for $1 \leq k \leq n$ $(n = 1, 2, \dots)$. Then the inverse $S = (s_{nk})_{n,k=1}^{\infty}$ of T is obviously given by

$$s_{nk} = \begin{cases} \dfrac{1}{u_n v_n}, & (k = n), \\ -\dfrac{1}{u_{n-1} v_n}, & (k = n-1), \\ 0, & (\text{otherwise}), \end{cases} \qquad (n = 1, 2, \dots),$$

and so we have

$$c^{(n)} = \frac{1}{u_n} \left[\frac{1}{v_n} e^{(n)} - \frac{1}{v_{n+1}} e^{(n+1)} \right] \text{ for } n = 1, 2, \dots$$

and

$$c^{(0)} = \sum_{j=1}^{k} s_{kj} = \left\{ \frac{1}{v_k} \Delta_k (1/u) \right\}_{k=1}^{\infty}.$$

So it follows from (1.2) and (1.3) in Corollary 1.2 that every sequence $z = (z_n)_{n=1}^{\infty} \in w_0^p(T)$ has a unique representation

$$z = \sum_{n=1}^{\infty} \left(\sum_{j=1}^{k} v_j z_j \right) \left[\frac{1}{v_n} e^{(n)} - \frac{1}{v_{n+1}} e^{(n+1)} \right],$$

and that every sequence $z = (z_n)_{n=1}^{\infty} \in w^p(T)$ has a unique representation

$$z = \xi \left\{ \frac{1}{v_k} \Delta_k(1/u) \right\}_{k=1}^{\infty} + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{k} v_j z_k - \frac{\xi}{u_n} \right) \left[\frac{1}{v_n} e^{(n)} - \frac{1}{v_{n+1}} e^{(n+1)} \right],$$

with ξ from (1.4).

2. Matrix transformations on w_0^p, w^p and w_∞^p

We will show in Section 3 that the determination of the β-duals of the sets $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$ can be reduced to that of the β-duals of the sets w_0^p, w^p and w_∞^p, and the characterizations of the classes (w_0^p, c_0), (w^p, c) and (w_∞^p, c_0). The β-duals of the sets w_0^p, w^p and w_∞^p are known. In this section, we characterize the classes (X, Y), where X is any of the sets w_0^p, w^p and w_∞^p, and Y is any of the sets c_0, c and ℓ_∞.

Throughout, let $1 \le p < \infty$ and q be the conjugate number of p, that is $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$. If $p = 1$ then we omit the index p, that is we write $w_0 = w_0^1$ etc., for short.

We put

$$\mathcal{M}_p = \left\{ a \in \omega : \|a\|_{\mathcal{M}_p} < \infty \right\}, \quad \text{where}$$

$$\|a\|_{\mathcal{M}_p} = \begin{cases} \displaystyle\sum_{\nu=0}^{\infty} 2^{\nu} \max\nolimits_{\nu} |a_k|, & (p = 1), \\ \displaystyle\sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum\nolimits_{\nu} |a_k|^q \right)^{1/q}, & (1 < p < \infty). \end{cases}$$

Given $a \in \omega$, we write

$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_{k=1}^{\infty} a_k x_k \right|$$

provided the expression on the righthand side is defined and finite which is the case whenever X is a BK space and $a \in X^\beta$ ([29, Theorem 7.2.9, p. 107]).

The next results are known. They give the β-duals of w_0^p, w^p and w_∞^p, the continuous duals of w_0^p and w^p, and the characterization of the class (X, ℓ_∞) for arbitrary BK spaces X.

Proposition 2.1. ([17] and [20, Proposition 3.47, p. 208]) *We have*

(a) $(w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}_p$;

(b) $(w_0^p, \|\cdot\|)^* \equiv \mathcal{M}_p$, *that is* $(w_0^p)^*$ *and* $\mathcal{M}_p$ *are norm isomorphic;*

(c) $f \in (w^p)^*$ *if and only if there exist* $a_0 \in \mathbb{C}$ *and a sequence* $a = (a_k)_{k=1}^\infty \in \mathcal{M}_p$ *such that*

$$f(x) = \xi a_0 + \sum_{k=1}^{\infty} a_k x_k \quad \text{for all } x \in w^p \text{ with } \xi \text{ from (1.1);}$$

moreover

$$\|f\| = |a_0| + \|a\|_{\mathcal{M}_p} \text{ for all } f \in (w^p)^*;$$

(d) $\|a\|_{w_\infty^p}^* = \|a\|_{\mathcal{M}_p}$ *for all* $a \in (w_\infty^p)^\beta$.

Proposition 2.2. ([15, Theorem 1.8]) *Let* X *be a BK space. Then we have* $A \in (X, \ell_\infty)$ *if and only if* $A_n \in X^\beta$, $(n \in \mathbb{N})$ *and* $\|A\|_{(X,\ell_\infty)} = \sup_{n\in\mathbb{N}} \|A_n\|_X^* < \infty$.

We also need the next lemma.

Lemma 2.3. *Let* $B = (b_{nk})_{n,k=1}^\infty$ *be an infinite matrix. If* $\|B_n\|_{\mathcal{M}_p} < \infty$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \|B_n\|_{\mathcal{M}_p} = 0$, *then* $\|B_n\|_{\mathcal{M}_p}$ *converges uniformly in* $n \in \mathbb{N}$.

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} \|B_n\|_{\mathcal{M}_p} = 0$, there exists $N \in \mathbb{N}$ such that $\|B_n\|_{\mathcal{M}_p} < \varepsilon$ for all $n > N$. For $\rho \in \mathbb{N}$ and $\mu \in \mathbb{N} \cup \{\infty\}$ we write

$$\|B_n\|_{\mathcal{M}_p}^{<\rho,\mu>} = \begin{cases} \displaystyle\sum_{\nu=\rho}^{\mu} 2^\nu \max_\nu |b_{nk}|, & (p = 1), \\ \displaystyle\sum_{\nu=\rho}^{\mu} 2^{\nu/p} \left(\sum_\nu |b_{nk}|^q\right)^{1/q}, & (1 < p < \infty). \end{cases}$$

Since $\|B_n\|_{\mathcal{M}_p} < \infty$ for all $n \in \mathbb{N}$, for each n with $1 \le n \le N$, there exists $\nu(n) \in \mathbb{N}_0$ such that $\|B_n\|_{\mathcal{M}_p}^{\langle \nu(n),\infty\rangle} < \varepsilon$. We choose $\rho = \max_{1\le n\le N} \nu(n)$. Then we have

$$\|B_n\|_{\mathcal{M}_p}^{\langle \nu,\infty\rangle} \le \|B_n\|_{\mathcal{M}_p}^{\langle \rho,\infty\rangle} < \varepsilon \quad \text{for all } \nu \ge \rho \text{ and for all } n \in \mathbb{N}. \qquad \square$$

Now we characterize the classes (X, Y) for $X \in \{w_0^p, w^p, w_\infty^p\}$ and $Y \in \{\ell_\infty, c, c_0\}$.

Theorem 2.4. *The necessary and sufficient conditions for* $A \in (X, Y)$ *when* $X \in \{w_0^p, w^p, w_\infty^p\}$ *and* $Y \in \{\ell_\infty, c, c_0\}$ *can be read from the following table:*

From / To	w_∞^p	w_0^p	w^p
ℓ_∞	**1.**	**1.**	**1.**
c_0	**2.**	**3.**	**4.**
c	**5.**	**6.**	**7.**

where

1. (1.1) $\sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{M}_p} < \infty$

2. (2.1) $\lim_{n\to\infty} \|A_n\|_{\mathcal{M}_p} = 0$

3. (1.1) *and* (3.1), *where* (3.1) $\lim_{n\to\infty} a_{nk} = 0$ *for all* $k \in \mathbb{N}$

4. (1.1), (3.1) *and* (4.1), *where* (4.1) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 0$

5. (5.1), (5.2) *and* (5.3), *where* (5.1) $\alpha_k = \lim_{n\to\infty} a_{nk}$ *exists for all* $k \in \mathbb{N}$
(5.2) $(\alpha_k)_{k=1}^{\infty}, A_n \in \mathcal{M}_p$ *for all* $n \in \mathbb{N}$
(5.3) $\lim_{n\to\infty} \|A_n - (\alpha_k)_{k=1}^{\infty}\|_{\mathcal{M}_p} = 0$

6. (1.1) *and* (5.1)

7. (1.1), (5.1) *and* (7.1), *where* (7.1) $\alpha = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk}$ *exists.*

We remark that the conditions for $A \in (w_\infty^p, c_0)$ *and* $A \in (w_\infty^p, c)$ *can be replaced by the conditions*

2'. (2.1') *and* (3.1), *where* (2.1') $\|A_n\|_{\mathcal{M}_p}$ *converges uniformly in* $n \in \mathbb{N}$

5'. (2.1') *and* (5.1).

Proof. **1.** Condition (1.1) for $A \in (w_0^p, \ell_\infty)$ follows from Propositions 2.2 and 2.1 (b) and (d). Then (1.1) for $A \in (w^p, \ell_\infty)$ and $A \in (w_\infty^p, \ell_\infty)$ follows from the fact that $(w_\infty^p, \ell_\infty) \supset (w^p, \ell_\infty) \supset (w_0^p, \ell_\infty)$ by Proposition 1.1.

3. and **6.** Since w_0^p is a BK space with AK by Proposition 1.1, and c_0 and c are closed subspaces of ℓ_∞ the conditions follow from the characterization of (w_0^p, ℓ_∞) and [29, 8.3.6, p. 123].

4. and **7.** The conditions follow from those in **3.** and **6.** and [29, 8.3.7, p. 123].

5. and **5'.** First we show that the conditions in **5.** imply those in **5'.**. We assume that the conditions in **5.** are satisfied, and define the matrix $B = (b_{nk})_{n,k=1}^{\infty}$ by $b_{nk} = a_{nk} - \alpha_k$ for all $n, k \in \mathbb{N}$. It follows from (5.2) and (5.3) that $B_n \in \mathcal{M}_p$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|B_n\|_{\mathcal{M}_p} = 0$. Hence $\|B_n\|_{\mathcal{M}_p}$ is uniformly convergent in $n \in \mathbb{N}$ by Lemma 2.3, and so $\|A_n\|_{\mathcal{M}_p} = \|B_n + (\alpha_k)_{k=1}^{\infty}\|_{\mathcal{M}_p}$ is uniformly convergent in n. This shows that the conditions in **5.** imply those in **5'.**.

Now we show the sufficiency of the conditions in **5'.** for $A \in (w_\infty^p, c)$. We assume that (2.1') and (5.1) are satisfied. It follows from (2.1') and (5.1) that there exists $\rho \in \mathbb{N}_0$ such that $\|A_n\|_{\mathcal{M}_p}^{\langle \rho+1,\infty\rangle} < 1$ for all $n \in \mathbb{N}$, and $(a_{nk})_{n=1}^{\infty} \in c \subset \ell_\infty$ for every $k \in \mathbb{N}$, hence for every $k \in \mathbb{N}$, there exists a constant $M_k > 0$ such that $|a_{nk}| \le M_k$ for all $n \in \mathbb{N}$. We put $M = 1 + \|(M_k)_{k=1}^{\infty}\|_{\mathcal{M}_p}^{\langle 0,\rho\rangle}$. Then we have $\|A_n\|_{\mathcal{M}_p} = \|A_n\|_{\mathcal{M}_p}^{\langle 0,\rho\rangle} + \|A_n\|_{\mathcal{M}_p}^{\langle \rho+1,\infty\rangle} \le \|(M_k)_{k=1}^{\infty}\|_{\mathcal{M}_p}^{\langle 0,\rho\rangle} + 1 = M$ for all $n \in \mathbb{N}$, hence (1.1) holds.

Now we show that (1.1) and (5.1) together imply $(\alpha_k)_{k=1}^\infty \in \mathcal{M}_p$. Let $\mu \in \mathbb{N}_0$ be given. Then we have

$$\begin{aligned}\|(\alpha_k)_{k=1}^\infty\|_{\mathcal{M}_p}^{\langle 0,\mu\rangle} &\leq \|A_n - (\alpha_k)_{k=1}^\infty\|_{\mathcal{M}_p}^{\langle 0,\mu\rangle} + \|A_n\|_{\mathcal{M}_p}^{\langle 0,\mu\rangle} \\ &\leq \|A_n - (\alpha_k)_{k=1}^\infty\|_{\mathcal{M}_p}^{\langle 0,\mu\rangle} + \sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{M}_p} \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

Since the first term on the right side of the inequality converges to zero for $n \to \infty$, we obtain from (5.1) and (1.1)

$$\|(\alpha_k)_{k=1}^\infty\|_{\mathcal{M}_p}^{\langle 0,\mu\rangle} \leq \sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{M}_p} < \infty.$$

Since $\mu \in \mathbb{N}_0$ was arbitrary, it follows that $\|(\alpha_k)_{k=1}^\infty\|_{\mathcal{M}_p} < \infty$, hence $(\alpha_k)_{k=1}^\infty \in \mathcal{M}_p$, and so $(\alpha_k)_{k=1}^\infty \in (w_\infty^p)^\beta$ by Proposition 2.1 (a). Furthermore, (1.1) and (2.1') imply that $A_n x$ is absolutely and uniformly convergent in n for each $x \in w_\infty^p$, since $\sum_{k=1}^\infty |a_{nk}x_k| \leq \sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{M}_p} \|x\|_{w_\infty^p}$. This implies

$$\lim_{n\to\infty} A_n x = \sum_{k=1}^\infty \left(\lim_{n\to\infty} a_{nk}\right) x_k = \sum_{k=1}^\infty \alpha_k x_k \quad \text{for each } x \in w_\infty^p,$$

that is $Ax \in c$ for all $x \in w_\infty^p$, hence $A \in (\ell_\infty, c)$. Thus we have proved the sufficiency of conditions (2.1') and (5.1).

Now we show the necessity of the conditions in **5.** and **5'.**. We assume $A \in (w_\infty^p, c)$. Since $e^{(k)} \in w_\infty^p$ for every $k \in \mathbb{N}$, it follows that $Ae^{(k)} = (a_{nk})_{n=1}^\infty \in c$, hence (5.1) holds. Also $w^p \subset w_\infty^p$ implies $(w_\infty^p, c) \subset (w^p, c)$, hence (1.1) holds by **1.**. Obviously (1.1) implies $A_n \in \mathcal{M}_p$ for all $n \in \mathbb{N}$, and as in the sufficiency part of the proof, (1.1) and (5.1) imply $(\alpha_k)_{k=1}^\infty \in \mathcal{M}_p$, so the conditions in (5.2) hold. Now $A \in (w_\infty^p, c)$ and $(\alpha_k)_{k=1}^\infty \in \mathcal{M}_p = (w_\infty^p)^\beta$ trivially imply $B \in (w_\infty^p, c)$, where the matrix B is defined as above. We show that this implies

$$\lim_{n\to\infty} \|B_n\|_{\mathcal{M}_p} = 0, \tag{I}$$

that is (5.3). Then it will follow from Lemma 2.3 that $\|B_n\|_{\mathcal{M}_p}$ converges uniformly in n, whence $\|A_n\|_{\mathcal{M}_p} = \|B_n + (\alpha_k)_{k=1}^\infty\|_{\mathcal{M}_p}$ converges uniformly in n, that is (2.1'). Thus all the conditions in **5.** and **5'.** hold.

To show that (I) is necessary, we assume that it is not satisfied and construct a sequence $x \in w_\infty^p$ with $Bx \notin c$, which is a contradiction to $B \in (w_\infty^p, c)$. If $\|B_n\|_{\mathcal{M}_p} \not\to 0$ $(n \to \infty)$ then there exists a real $c > 0$ such that $\limsup_{n\to\infty} \|B_n\|_{\mathcal{M}_p} = c$, hence $\lim_{j\to\infty} \|B_{n_j}\|_{\mathcal{M}_p} = c$ for some subsequence (n_j). We omit the indices j, that is we assume without loss of generality

$$\lim_{n\to\infty} \|B_n\|_{\mathcal{M}_p} = c. \tag{II}$$

It follows from (5.1) that

$$\lim_{n\to\infty} b_{nk} = 0 \text{ for every } k \in \mathbb{N}. \tag{III}$$

By (II) and (III), there exists an integer $n(1)$ such that

$$\left|\, \|B_{n(1)}\|_{\mathcal{M}_p} - c \right| < \frac{c}{10} \quad \text{and} \quad |b_{n(1),1}| < \frac{c}{10}.$$

Since $\|B_{n(1)}\|_{\mathcal{M}_p} < \infty$, we can choose an integer $\nu(2) > 0$ such that

$$\|B_{n(1)}\|_{\mathcal{M}_p}^{\langle \nu(2),\infty\rangle} < \frac{c}{10},$$

and it follows that

$$\left|\, \|B_{n(1)}\|_{\mathcal{M}_p}^{\langle 0,\nu(2)\rangle} - c \right| \le \left|\, \|B_{n(1)}\|_{\mathcal{M}_p} - c \right| + \|B_{n(1)}\|_{\mathcal{M}_p}^{\langle \nu(2)+1,\infty\rangle} + |b_{n(1),1}| < \frac{3c}{10}.$$

Now we choose an integer $n(2) > n(1)$ such that

$$\|B_{n(2)}\|_{\mathcal{M}_p}^{\langle 0,\nu(2)\rangle} < \frac{c}{10} \quad \text{and} \quad \left|\, \|B_{n(2)}\|_{\mathcal{M}_p} - c \right| < \frac{c}{10},$$

and an integer $\nu(3) > \nu(2)$ such that $\|B_{n(2)}\|_{\mathcal{M}_p}^{\langle \nu(3)+1,\infty\rangle} < c/10$. Again it follows that

$$\left|\, \|B_{n(2)}\|_{\mathcal{M}_p}^{\langle \nu(2)+1,\nu(3)\rangle} - c \right| < \frac{3c}{10}.$$

Continuing in this way, we can determine sequences $\{n(r)\}_{r=1}^{\infty}$ and $\{\nu(r)\}_{r=1}^{\infty}$ of integers $n(1) < n(2) < \dots$ and $0 = \nu(1) < \nu(2) < \dots$ such that for all $r = 1, 2, \dots$

$$\|B_{n(r)}\|_{\mathcal{M}_p}^{\langle 0,\nu(r)\rangle} < \frac{c}{10}, \quad \|B_{n(r)}\|_{\mathcal{M}_p}^{\langle \nu(r+1)+1,\infty\rangle} < \frac{c}{10}$$
$$\text{and} \quad \left|\, \|B_{n(r)}\|_{\mathcal{M}_p}^{\langle \nu(r)+1,\nu(r+1)\rangle} - c \right| < \frac{3c}{10}.$$

If $p = 1$, we define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0, & (k = 1), \\ (-1)^r 2^\nu \operatorname{sgn}(b_{n(r),k(\nu)}), & (k = k(\nu)), \text{ where } k(\nu) \in [2^\nu, 2^{\nu+1} - 1] \\ & \qquad \text{is the smallest integer with} \\ & \qquad |b_{n(r),k(\nu)}| = \max_\nu |b_{n(r),k}|, \\ 0, & (k \ne k(\nu)), \\ & (\nu(r) + 1 \le \nu \le \nu(r+1);\ r = 1, 2, \dots). \end{cases}$$

Then we obviously have $x \in w_\infty$ and $\|x\|_{w_\infty} \le 1$, and

$$\begin{aligned}
&\left|B_{n(r)}x - (-1)^r c\right| \le \sum_{\nu=0}^{\nu(r)} \sum\nolimits_\nu |b_{n(r),k}|\,|x_k| + \sum_{\nu=\nu(r+1)+1}^{\infty} \sum\nolimits_\nu |b_{n(r),k}|\,|x_k| \\
&\quad + \left|\sum_{\nu=\nu(r)+1}^{\nu(r+1)} \sum\nolimits_\nu b_{n(r),k}x_k - c\right| \le \sum_{\nu=0}^{\nu(r)} 2^\nu \max\nolimits_\nu |b_{n(r),k}| + \sum_{\nu=\nu(r+1)+1}^{\infty} 2^\nu \max\nolimits_\nu |b_{n(r),k}| \\
&\quad + \left|(-1)^r\left(\sum_{\nu=\nu(r)+1}^{\nu(r+1)} 2^\nu \max\nolimits_\nu |b_{n(r),k}| - c\right)\right| = \|B_{n(r)}\|_{\mathcal{M}_1}^{\langle 0,\nu(r)\rangle} + \|B_{n(r)}\|_{\mathcal{M}_1}^{\langle \nu(r+1)+1,\infty\rangle} \\
&\quad + \left|\,\|B_{n(r)}\|_{\mathcal{M}_1}^{\langle \nu(r)+1,\nu(r+1)\rangle} - c\right| < \frac{c}{10} + \frac{c}{10} + \frac{3c}{10} = \frac{c}{2} \quad \text{for all } r \in \mathbb{N}.
\end{aligned}$$

Consequently $(B_n x)_{n=1}^\infty$ is not a Cauchy sequence, hence not convergent.

If $1 < p < \infty$, we define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 0, & (k=1), \\ 2^{\nu/p}(-1)^r \operatorname{sgn}(b_{n(r),k})|b_{n(r),k}|^{q-1}\left(\sum_\nu |b_{n(r),k}|^q\right)^{-1/p}, & (2^\nu \le k \le 2^{\nu+1}-1) \\ & (\nu(r)+1 \le \nu \le \nu(r+1);\ r = 1,2,\dots). \end{cases}$$

Let $\nu \in \mathbb{N}_0$ be given. Then there exists r such that $\nu(r)+1 \le \nu \le \nu(r+1)$ and

$$\frac{1}{2^\nu}\sum\nolimits_\nu |x_k|^p = \frac{1}{2^\nu}\sum\nolimits_\nu 2^\nu |b_{n(r),k}|^{pq-p}\left(\sum\nolimits_\nu |b_{n(r),k}|^q\right)^{-1} = 1,$$

that is $x \in w_\infty^p$ and $\|x\|_{w_\infty^p} \le 1$. We also have by Hölder's inequality

$$\begin{aligned}
&\left|B_{n(r)}x - (-1)^r c\right| \le \sum_{\nu=0}^{\nu(r)} 2^{\nu/p}\left(\sum\nolimits_\nu |b_{n(r),k}|^q\right)^{1/q} + \sum_{\nu=\nu(r+1)+1}^{\infty} 2^{\nu/p}\left(\sum\nolimits_\nu |b_{n(r),k}|^q\right)^{1/q} \\
&\quad + \left|(-1)^r\left(\sum_{\nu=\nu(r)+1}^{\nu(r+1)} 2^{\nu/p}\left(\sum\nolimits_\nu |b_{n(r),k}|^q\right)^{1/q} - c\right)\right| \\
&\quad = \|B_{n(r)}\|_{\mathcal{M}_p}^{\langle 0,\nu(r)\rangle} + \|B_{n(r)}\|_{\mathcal{M}_p}^{\langle \nu(r+1)+1,\infty\rangle} + \left|\,\|B_{n(r)}\|_{\mathcal{M}_p}^{\langle \nu(r)+1,\nu(r+1)\rangle} - c\right| \\
&\quad < \frac{c}{10} + \frac{c}{10} + \frac{3c}{10} = \frac{c}{2} \quad \text{for all } r \in \mathbb{N}.
\end{aligned}$$

Consequently $(B_n x)_{n=1}^\infty$ is not a Cauchy sequence, hence not convergent. So we have $x \in w_\infty^p$, but $Bx \notin c$ in both cases, which is a contradiction to $B \in (w_\infty^p, c)$. This completes the proof of **5.** and **5'.**.

2. and **2'.** are proved in the same way as **5.** and **5'.** with $\alpha_k = 0$ for all $k \in \mathbb{N}$.

□

3. The β-duals of $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$

Now we determine the β-duals of $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$.

We write $\Sigma = (\sigma_{nk})_{n,k=1}^\infty$ for the triangle with $\sigma_{nk} = 1$ for $1 \le k \le n$ $(n = 1, 2, \dots)$, $cs = c_\Sigma$ for the set of all convergent series, and $R = S^t$ for the transpose of the inverse S of the triangle T.

The following results are helpful.

Lemma 3.1. *We have*

(a) $a \in \{w_0^p(T)\}^\beta$ *if and only if* $a \in (\mathcal{M}_p)_R$ *and* $W \in (w_0^p, c_0)$, *where* $W = (w_{mk})_{m,k=1}^\infty$ *is the triangle with* $w_{mk} = \sum_{j=m}^\infty s_{jk}a_j$ *for* $1 \le k \le m$ $(m = 1, 2, \dots)$;

(b) $a \in \{w^p(T)\}^\beta$ *if and only if* $a \in (\mathcal{M}_p)_R$ *and* $W \in (w^p, c)$;

(c) $a \in \{w_\infty^p(T)\}^\beta$ *if and only if* $a \in (\mathcal{M}_p)_R$ *and* $W \in (w_\infty^p, c_0)$.

Proof. Let $a = (a_n)_{n=1}^\infty \in \omega$. We define the triangles $B = (b_{nk})_{n,k=1}^\infty$ and $C = (c_{nk})_{n,k=1}^\infty$ by $b_{nk} = a_n s_{nk}$ and $c_{nk} = \sum_{j=k}^n a_j s_{jk}$ for $1 \le k \le n$ $(n = 1, 2, \dots)$, hence $C = \Sigma B$. Let X be any of the sets w_0^p, w^p or w_∞^p. Since $x \in X$ if and only if $z = Sx \in X_T$, and since $a_n z_n = a_n S_n x = a_n \sum_{k=1}^n s_{nk}x_k = \sum_{k=1}^n a_n s_{nk} x_k = B_n x$ for all $n \in \mathbb{N}$, we observe that $a \in (X_T)^\beta$ if and only if $B \in (X, cs)$, and this is the case by [20, Theorem 3.8, p. 180] if and only if $C \in (X, c)$.

(a) First we assume $a \in \{w_0^p(T)\}^\beta$. Then $C \in (w_0^p, c)$ which is the case by **6.** in Theorem 2.4 if and only if

$$R_k a = \lim_{n\to\infty} c_{nk} = \sum_{j=k}^\infty a_j s_{jk} \quad \text{exists for every } k \in \mathbb{N} \tag{3.1}$$

and

$$\|C\|_{\mathcal{M}_p} = \sup_n \|C_n\|_{\mathcal{M}_p} < \infty. \tag{3.2}$$

We show that (3.1) and (3.2) imply

$$Ra = (R_k a)_{k=1}^\infty \in \mathcal{M}_p = (w_0^p)^\beta. \tag{3.3}$$

Let $\mu \in \mathbb{N}_0$ be given. It follows from (3.2) that there exists a constant $M > 0$ such that $\|(\sum_{j=k}^n a_j s_{jk})_{k=1}^\infty\|_{\mathcal{M}_p}^{\langle 0,\mu\rangle} \le M$. Letting $n \to \infty$ and using (3.1), we obtain $\|Ra\|_{\mathcal{M}_p}^{\langle 0,\mu\rangle} \le M$, and (3.3) follows, since μ was arbitrary.

We note that the matrix W is defined in view of (3.1). Furthermore also have for all $z \in \omega$ and for all $m \in \mathbb{N}$

$$\sum_{k=1}^m (R_k a)(T_k z) - W_m(Tz) = \sum_{k=1}^m \left(\sum_{j=k}^\infty a_j s_{jk}\right) T_k z - \sum_{k=1}^m \left(\sum_{j=m}^\infty a_j s_{jk}\right) T_k z$$

$$= \sum_{k=1}^{m}\left(\sum_{j=k}^{m-1} a_j s_{jk}\right) T_k z = \sum_{k=1}^{m-1}\left(\sum_{j=k}^{m-1} a_j s_{jk}\right) T_k z$$
$$= \sum_{j=1}^{m-1} a_j \sum_{k=1}^{j} s_{jk} T_k z = \sum_{j=1}^{m-1} a_j S_j(Tz) = \sum_{j=1}^{m-1} a_j z_j,$$

that is

$$\sum_{k=1}^{m-1} a_k z_k = \sum_{k=1}^{m} (R_k a)(T_k z) - W_m(Tz) \quad \text{for all } m \in \mathbb{N}. \tag{3.4}$$

It follows from (3.4), $a \in \{w_0^p(T)\}^\beta$ and (3.3) that $\{W_m(Tz)\}_{m=1}^\infty \in c$ for all $z \in X_T$, which is equivalent to $W \in (w_0^p, c)$. Now (3.1) implies

$$\lim_{m\to\infty} w_{mk} = \lim_{m\to\infty} \sum_{j=m}^{\infty} a_j s_{jk} = 0 \quad \text{for all } k \in \mathbb{N} \tag{3.5}$$

and $W \in (w_0^p, c)$ and (3.5) imply $W \in (w_0^p, c_0)$ by **3.** and **6.** in Theorem 2.4.

Conversely if $a \in (\mathcal{M}_p)_R$ and $W \in (w_0^p, c_0)$, then $Ra \in \mathcal{M}_p = (w_0^p)^\beta$ by Proposition 2.1 (a), and then $a \in \{w_0^p(T)\}^\beta$ follows from (3.4).

(b) and (c) Let $X = w^p$ or $X = w_\infty^p$. First we assume $a \in (X_T)^\beta$. Since $w_0^p(T) \subset X_T$ by Corollary 1.2 (a), we have $a \in \{w_0^p(T)\}^\beta$, and $Ra \in \mathcal{M}_p = X^\beta$ follows from Part (a). Now (3.4) implies $W \in (X, c)$. Again $Ra \in \mathcal{M}_p$ implies (3.5), and $W \in (w_\infty^p, c)$ and (3.5) imply $W \in (w_\infty^p, c_0)$ by **2.** and **5.** in Theorem 2.4.

The proof of the converse part is analogous to that of the converse part of (a). □

Theorem 3.2. *For every $m \in \mathbb{N}$, let $\nu(m)$ be the uniquely defined number with $2^{\nu(m)} \le m \le 2^{\nu(m)+1} - 1$. We have*

$$\|W_m\|_{\mathcal{M}_p} = \begin{cases} \displaystyle\sum_{\nu=0}^{\nu(m)-1} 2^\nu \max\nolimits_\nu \left|\sum_{j=m}^{\infty} a_j s_{jk}\right| \\ \displaystyle\quad + 2^{\nu(m)} \max_{2^{\nu(m)} \le k \le m} \left|\sum_{j=m}^{\infty} a_j s_{jk}\right| < \infty, & (p = 1), \\ \displaystyle\sum_{\nu=0}^{\nu(m)-1} 2^{\nu/p} \left(\sum\nolimits_\nu \left|\sum_{j=m}^{\infty} a_j s_{jk}\right|^q\right)^{1/q} \\ \displaystyle\quad + 2^{\nu(m)/p} \left(\sum_{k=2^{\nu(m)}}^{m} \left|\sum_{j=m}^{\infty} a_j s_{jk}\right|^q\right)^{1/q} < \infty, & (1 < p < \infty), \end{cases} \tag{3.6}$$

and

(a) $a \in \{w_0^p(T)\}^\beta$ *if and only if*

$$\|Ra\|_{\mathcal{M}_p} = \begin{cases} \displaystyle\sum_{\nu=0}^{\infty} 2^{\nu} \max\nolimits_{\nu} \left| \sum_{j=k}^{\infty} a_j s_{jk} \right| < \infty, & (p=1), \\ \displaystyle\sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum\nolimits_{\nu} \left| \sum_{j=k}^{\infty} a_j s_{jk} \right|^q \right)^{1/q} < \infty, & (1 < p < \infty), \end{cases} \tag{3.7}$$

and

$$\sup_{m \in \mathbb{N}} \|W_m\|_{\mathcal{M}_p} < \infty; \tag{3.8}$$

(b) $a \in \{w^p(T)\}^\beta$ *if and only if* (3.7) *and* (3.8) *hold and*

$$\eta = \lim_{m\to\infty} \sum_{k=1}^{m} \sum_{j=m}^{\infty} a_j s_{jk} \quad \textit{exists;} \tag{3.9}$$

(c) $a \in \{w_\infty^p(T)\}^\beta$ *if and only if* (3.7) *holds and*

$$\lim_{m\to\infty} \|W_m\|_{\mathcal{M}_p} = 0. \tag{3.10}$$

(d) *Let* $X = w_0^p$ *or* $X = w_\infty^p$. *If* $a \in (X_T)^\beta$ *then*

$$\sum_{k=1}^{\infty} a_k z_k = \sum_{k=1}^{\infty} (R_k a)(T_k z) \textit{ for all } z \in X_T; \quad \textit{also } \|a\|^*_{X_T} = \|Ra\|_{\mathcal{M}_p}. \tag{3.11}$$

If $a \in \{w^p(T)\}^\beta$ *then*

$$\sum_{k=1}^{\infty} a_k z_k = \sum_{k=1}^{\infty} (R_k a)(T_k z) - \xi\eta \quad \textit{for all } z \in w^p(T), \\ \textit{where } \xi \textit{ and } \eta \textit{ are from (1.1) and (3.10);} \tag{3.12}$$

also

$$\|a\|^*_{w^p(T)} = |\eta| + \|Ra\|_{\mathcal{M}_p} \quad \textit{for all } a \in \{w^p(T)\}^\beta. \tag{3.13}$$

Proof. We apply Lemma 3.1 and Theorem 2.4.

Condition (3.7) is $Ra \in \mathcal{M}_p = (w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta$ by Proposition 2.1 (a).

Condition (3.8) comes from $W \in (w_0^p, c_0)$ and $W \in (w^p, c)$ and is (1.1) in Theorem 2.4 **3.** and **7.**; the conditions $\lim_{m\to\infty} w_{mk} = 0$ and $\lim_{m\to\infty} w_{mk} = \beta_k$, which are (3.1) and (5.1) in **3.** and **7.**, are redundant. Condition (3.9) for $W \in (w^p, c)$ comes from (7.1) in Theorem 2.4 **7.**.

Condition (3.10) comes from $W \in (w^p_\infty, c_0)$ and is (2.1) in Theorem 2.4 **2.**.

Thus we have shown Parts (a), (b) and (c).

(d) The first condition in (3.11) follows from (3.4) and the fact that $W \in (X, c_0)$; the second condition follows from Proposition 2.1 (b) and (d).

Now let $a \in \{w^p(T)\}^\beta$ and $z \in w^p(T)$. Then $x = Tz \in w^p$ and ξ from (1.1) exists, hence there exists $x^{(0)} \in w^p_0$ such that $x = x^{(0)} + \xi e$. We put $z^{(0)} = Sx^{(0)}$. Then it follows that $z^{(0)} \in w^p_0(T)$ and $z = Sx = S(x^{(0)} + \xi e) = z^{(0)} + \xi Se$, and we obtain as in (3.4) for all $m \in \mathbb{N}$

$$\begin{aligned}\sum_{k=1}^{m-1} a_k z_k &= \sum_{k=1}^{m} (R_k a)(T_k z) - W_m[T(z^{(0)} + \xi Se)] \\ &= \sum_{k=1}^{m} (R_k a)(T_k z) - W_m[Tz^{(0)}] - \xi W_m e.\end{aligned}$$

The first term on the righthand side of the last equation converges since $Ra \in \mathcal{M}_p$. The second term tends to 0, since $a \in \{w^p(T)\}^\beta \subset \{w^p_0(T)\}^\beta$ implies $W \in (w^p_0, c_0)$. Furthermore, since $W \in (w^p, c)$ implies that $\eta = \lim_{m\to\infty} W_m e$ exists by (7.1) in Theorem 2.4 **7.**, the identity in (3.12) follows. Finally, (3.13) follows from Proposition 2.1 (c). □

We apply Theorem 3.2 to the matrix T of Example 1.3.

Example 3.3. Let T be the matrix of Example 1.3. Then it is easy to see that

(a) $a \in \{w^p_0(T)\}^\beta$ if and only if

$$\|Ra\|_{\mathcal{M}_p} = \begin{cases} \displaystyle\sum_{\nu=0}^{\infty} 2^\nu \max\nolimits_\nu \left| \frac{a_k}{u_k v_k} - \frac{a_{k+1}}{u_k v_{k+1}} \right| < \infty, & (p = 1), \\ \displaystyle\sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum\nolimits_\nu \left| \frac{a_k}{u_k v_k} - \frac{a_{k+1}}{u_k v_{k+1}} \right|^q \right)^{1/q} < \infty, & (1 < p < \infty) \end{cases} \tag{3.14}$$

and

$$\sup_{m\in\mathbb{N}} \left(2^{\nu(m)/p} \left| \frac{a_{m+1}}{u_m v_{m+1}} \right| \right) < \infty; \tag{3.15}$$

(b) $a \in \{w^p(T)\}^\beta$ if and only if (3.14), (3.15) and

$$\eta = \lim_{m\to\infty} \left(\frac{a_m}{u_{m-1} v_m} + \frac{a_m}{u_m v_m} - \frac{a_{m+1}}{u_m v_{m+1}} \right) \quad \text{exists;}$$

(c) $a \in \{w_\infty^p(T)\}^\beta$ if and only if (3.14) and

$$\lim_{m\to\infty}\left(2^{\nu(m)/p}\,\frac{a_{m+1}}{u_m v_{m+1}}\right)=0.$$

4. Matrix transformations on the spaces $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$

Now we characterize the classes (X,Y) where X is any of the spaces $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$, and Y is any of the spaces c_0, c and ℓ_∞.

The following results are useful.

Lemma 4.1. (a) *Let $X = w_0^p$ or $X = w_\infty^p$, and Y be an arbitrary subset of ω. Then we have $A \in (X_T, Y)$ if and only if $\hat{A} \in (X,Y)$ and $W^{(n)} \in (X, c_0)$ for all $n = 1, 2, \ldots$, where the matrix $\hat{A} = (\hat{a}_{nk})_{n,k=1}^\infty$ and the triangles $W^{(n)} = \{w_{mk}^{(n)}\}_{m,k=1}^\infty$ are defined by*

$$\hat{a}_{nk} = \sum_{j=k}^{\infty} a_{nj}s_{jk} \text{ for all } n,k\in\mathbb{N} \quad \text{and} \quad w_{mk}^{(n)} = \sum_{j=m}^{\infty} a_{nj}s_{jk} \text{ for } 1\le k\le m.$$

Moreover, we also have if $A \in (X_T, Y)$ then

$$Az = \hat{A}(Tz) \quad \text{for all } z \in X_T. \tag{4.1}$$

(b) *Let Y be an arbitrary linear subspace of ω. Then we have $A \in (w^p(T), Y)$ if and only if*

$$\hat{A} \in (w_0^p, Y), \tag{4.2}$$

$$W^{(n)} \in (w^p, c) \quad \text{for all } n \in \mathbb{N} \tag{4.3}$$

and

$$\hat{A}e - \{\rho^{(n)}\}_{n=1}^\infty \in Y, \text{ where } \rho^{(n)} = \lim_{m\to\infty}\sum_{k=1}^{m} w_{mk}^{(n)} \text{ for all } n\in\mathbb{N}. \tag{4.4}$$

Moreover, if $A \in (w^p(T), Y)$ then we also have

$$Az = \hat{A}(Tz) - \xi\{\rho^{(n)}\}_{n=1}^\infty \quad \text{for all } z \in w^p(T), \text{ where } \xi \text{ is from (1.4).} \tag{4.5}$$

Proof. (a) First we assume $A \in (X_T, Y)$. Then it follows that $A_n \in (X_T)^\beta$ for all $n \in \mathbb{N}$, hence $\hat{A}_n \in X^\beta$ and $W^{(n)} \in (X, c_0)$ for all $n \in \mathbb{N}$ by Lemma 3.1 (a) and (c). Let $x \in X$ be given, hence $z = Sx \in X_T$. Since $A_n \in (X_T)^\beta$

implies $A_n z = \hat{A}_n(Tz) = \hat{A}_n x$ for all $n \in \mathbb{N}$ by (3.11) in Theorem 3.2 (d), that is $\hat{A}x = Az$, $Az \in Y$ for all $z \in X_T$ implies $Ax \in Y$, that is (4.1) holds, and we have $\hat{A} \in (X, Y)$ since $x \in X$ was arbitrary.

Conversely, we assume $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$ for all $n \in \mathbb{N}$. Then we have $\hat{A}_n \in X^\beta$, and this and $W^{(n)} \in (X, c_0)$ together imply $A_n \in X_T^\beta$ by Lemma 3.1 (a) and (c). Now let $z \in X_T$ be given, hence $x = Tz \in X$. Again we have $A_n z = \hat{A}_n x$ for all $n \in \mathbb{N}$ by (3.11) in Theorem 3.2 (d), hence $Az = \hat{A}x \in Y$, and $\hat{A}x \in Y$ for all $x \in X$ implies $Az \in Y$. Thus we have $\hat{A} \in (X, Y)$, since $z \in X_T$ was arbitrary.

(b) First we assume that $A \in (w^p(T), Y)$. Then it follows that $A \in (w_0^p(T), Y)$ and so $\hat{A} \in (w_0^p, Y)$ by Part (a). Also $A_n \in \{w^p(T)\}^\beta$ implies $W^{(n)} \in (w^p, c)$ by Lemma 3.1 (b), and also (3.12) by Theorem 3.2 (d). Since $z = Se \in w^p(T)$, hence $Az \in Y$, and $\xi = 1$, we obtain (4.4) from (3.12).

Conversely, we assume that the conditions in (4.2), (4.3) and (4.4) are satisfied. First $\hat{A}_n \in (w_0^p)^\beta = \mathcal{M}_p$ and $W^n \in (w^p, c)$ together imply $A_n \in \{w^p(T)\}^\beta$ by Lemma 3.1 (b), and again (3.12) follows by Theorem 3.2 (d). Let $z \in w^p(T)$ be given. Then we have $x = Tz \in w^p$. We put $x^{(0)} = x - \xi e$, where ξ is from $\lim_{n\to\infty}(1/n)\sum_{k=1}^n |x_k - \xi|^p = \lim_{n\to\infty}(1/n)\sum_{k=1}^n |T_k z - \xi|^p = 0$, that is from (1.4). Then we have $x^{(0)} \in w_0^p$ and it follows from (3.12) that $Az = \hat{A}(Tz) - \xi\{\rho^{(n)}\}_{n=1}^\infty = \hat{A}(x^{(0)}) + \xi[\hat{A}e - \{\rho^{(n)}\}_{n=1}^\infty] \in Y$, since $\hat{A} \in (w_0^p, Y)$, $\hat{A}e - \{\rho^{(n)}\}_{n=1}^\infty \in Y$ and Y is a linear space. □

Theorem 4.2. *The necessary and sufficient conditions for the entries of $A \in (X_T, Y)$ when $X \in \{w_0^p,\ w^p,\ w_\infty^p\}$ and $Y \in \{\ell_\infty, c_0, c\}$ can be read from the following table:*

From / *To*	$w_\infty^p(T)$	$w_0^p(T)$	$w^p(T)$
ℓ_∞	**1.**	**2.**	**3.**
c_0	**4.**	**5.**	**6.**
c	**7.**	**8.**	**9.**

where

1. (1.1) $\sup_{n\in\mathbb{N}} \|\hat{A}_n\|_{\mathcal{M}_p} < \infty$ (1.2) $\lim_{m\to\infty} \|W_m^{(n)}\|_{\mathcal{M}_p} = 0$ *for all* $n \in \mathbb{N}$ *with* $\|\hat{A}\|_{\mathcal{M}_p}$ *and* $\|W_m^{(n)}\|_{\mathcal{M}_p}$ *defined as in* (3.7) *and* (3.6) *with a_j replaced by a_{nj}*

2. (1.1) *and* (2.1), *where* (2.1) $\sup_m \|W_m^{(n)}\|_{\mathcal{M}_p} < \infty$ *for all* $n \in \mathbb{N}$

3. (1.1), (2.1) (3.1) *and* (3.2), *where*

(3.1) $\rho^{(n)} = \lim_{m\to\infty} \sum_{k=1}^{m} \sum_{j=m}^{\infty} a_{nj} s_{jk}$ *exists for all* $n \in \mathbb{N}$
(3.2) $\sup_{n\in\mathbb{N}} |\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} a_{nj} s_{jk} - \rho^{(n)}| < \infty$

4. (4.1) *and* (1.2), *where* (4.1) $\lim_{n\to\infty} \|\hat{A}_n\|_{\mathcal{M}_p} = 0$

5. (1.1), (5.1) *and* (2.1), *where* (5.1) $\lim_{n\to\infty} \hat{a}_{nk} = 0$ *for all* $k \in \mathbb{N}$

6. (1.1), (5.1), (2.1), (3.1) *and* (6.1) *where*
(6.1) $\lim_{n\to\infty} (\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} a_{nj} s_{jk} - \rho^{(n)}) = 0$

7. (1.2), (7.1), (7.2) *and* (7.3) *where* (7.1) $\hat{\alpha}_k = \lim_{n\to\infty} \hat{a}_{nk}$ *exists for all* $k \in \mathbb{N}$
(7.2) $(\hat{\alpha})_{k=1}^{\infty}$, $\hat{A}_n \in \mathcal{M}_p$ *for all* $n \in \mathbb{N}$ (7.3) $\lim_{n\to\infty} \|\hat{A}_n - (\hat{\alpha}_k)_{k=1}^{\infty}\|_{\mathcal{M}_p} = 0$

8. (1.1), (7.1) *and* (2.1)

9. (1.1), (7.1), (2.1), (3.1) *and* (9.2), *where*
(9.2) $\beta = \lim_{n\to\infty} (\sum_{k=1}^{\infty} \sum_{j=k}^{\infty} a_{nj} s_{jk} - \rho^{(n)})$ *exists.*

PROOF. We apply Lemma 4.1 and Theorems 3.2 and 2.4.

If $X = w_\infty^p$ or $X = w_0^p$ then we apply Lemma 4.1 (a), that is we get the conditions for $\hat{A} \in (X, Y)$ and $W^{(n)} \in (X, c_0)$. Applying Theorem 2.4 **1.**, **2.**, **3.**, **5.** and **6.**, we obtain (1.1) in **1.** and **2.**, (4.1) in **4.**, (1.1) and (5.1) in **5.**, (7.1), (7.2) and (7.3) in **7.**, and (1.1) and (7.1) in **8.** for $\hat{A} \in (X, Y)$. We obtain the conditions (1.2) in **1.**, **4.** and **7.** for $W^{(n)} \in (w_\infty^p, c_0)$ and (2.1) in **2.**, **5.** and **8.** from Theorem 2.4 **2.** and **3.**, taking into account that the condition $\lim_{m\to\infty} w_{mk}^{(n)} = 0$ for all $k \in \mathbb{N}$ is redundant as in the proof of Theorem 3.2.

If $X = w^p$, we apply Lemma 4.1 (b), that is we get the conditions for $\hat{A} \in (w_0^p, Y)$, $W^{(n)} \in (w^p, c)$ for all n and $\hat{A}e - \{\rho^{(n)}\}_{n=1}^{\infty} \in Y$. Applying Theorem 2.4 **1.**, **3.** and **6.** for $\hat{A} \in (w_0^p, Y)$, we obtain (1.1) in **3.**, (1.1) and (5.1) in **6.** and (1.1) and (7.1) in **9.**. We obtain the conditions (2.1) and (3.1) in **3.**, **6.** and **9.** for $W^{(n)} \in (w^p, c)$; again the condition $\lim_{m\to\infty} w_{mk}^{(n)}$ exists for all $k \in \mathbb{N}$ is redundant. Finally, $\hat{A}e - \{\rho^{(n)}\}_{n=1}^{\infty} \in Y$ yields (3.2) in **3.**, (6.1) in **6.** and (9.2) in **9.**. □

5. Conclusion

Let X denotes the anyone of the spaces w_0^p , w^p or w_∞^p. We have introduced the sequence space $X(T)$ which is the domain of a triangle matrix $T = (t_{nk})$ in the sequence space X. We have essentially concerned with two subjects: Determination of the β-dual of the space $X(T)$ and the characterization of the certain matrix transformations defined on the sequence space $X(T)$.

Although the domain of summability matrices in the classical spaces ℓ_∞, c, c_0 and ℓ_p of sequences were studied by several authors (see ALTAY [1], ALTAY and

Başar [2], [3], [4], Aydin and Başar [5], [6], Başar and Altay [7], Çolak and Et [9], Çolak, Et and Malkowsky [10], Et [11], Et and Çolak [13], Et and Başarir [14], Malkowsky, Mursaleen and Suantai [18], Malkowsky and Parashar [19], Malkowsky and Savaş [21], Mursaleen [22], Mursaleen, Başar, Altay [23], Ng and Lee [24], Polat and Başar [25], Şengönül and Başar [27], Wang [28]), the matrix domain of the spaces w_0^p , w^p and w_∞^p have not been examined. The present work fills up this gap in the existing literature. It is obvious that the α- and γ-dulas of the new spaces $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$ are still open. Besides this, one can try to characterize the classes of infinite matrices from the spaces $w_0^p(T)$, $w^p(T)$ or $w_\infty^p(T)$ to a sequence space Y which is different than that of Section 4. We should note from now on that a new paper can be based on the extension of the new spaces $w_0^p(T)$, $w^p(T)$ and $w_\infty^p(T)$ to the paranormed case.

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