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Minimal flat Lorentzian surfaces in Lorentzian complex space forms

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Abstract. In this article we study minimal flat Lorentzian surfaces in Lorentzian complex space forms. First we prove that, for minimal flat Lorentzian surfaces in a Lorentzian complex form, the equation of Ricci is a consequence of the equations of Gauss and Codazzi. Then we classify minimal flat Lorentzian surfaces in the Lorentzian complex plane $\mathbf{C}^2_1$. Finally, we classify minimal flat slant surfaces in Lorentzian complex projective plane CP^2_1 and in Lorentzian complex hyperbolic plane CH^2_1.

1. Introduction

Let $\tilde{M}^n_i(4c)$ be an indefinite complex space form of complex dimension n and *complex index* i. The complex index is defined as the complex dimension of the largest complex negative definite subspace of the tangent space. If $i = 1$, we say that $\tilde{M}^n_i(4c)$ is Lorentzian. The curvature tensor $\tilde{R}$ of $\tilde{M}^n_i(4c)$ is given by

$$\begin{aligned} \tilde{R}(X,Y)Z = c\{&\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX \\ &- \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\}. \end{aligned} \tag{1.1}$$

Let $\mathbf{C}^n$ denote the complex number n-space with complex coordinates $z_1,\dots,$ z_n. The $\mathbf{C}^n$ endowed with $g_{i,n}$, i.e., the real part of the Hermitian form

$$b_{i,n}(z,w) = -\sum_{k=1}^{i} \bar{z}_k w_k + \sum_{j=i+1}^{n} \bar{z}_j w_j, \quad z,w \in \mathbf{C}^n,$$

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defines a flat indefinite complex space form with complex index i. We simply denote the pair $(\mathbf{C}^n, g_{i,n})$ by $\mathbf{C}^n_i$. Consider the differentiable manifold:

$$S^{2n+1}_2(c) = \{z \in \mathbf{C}^{n+1}_1;\ b_{1,n+1}(z,z) = c^{-1} > 0\},$$

which is an indefinite real space form of constant sectional curvature c. The Hopf fibration

$$\pi : S^{2n+1}_2(c) \to CP^n_1(4c) : z \mapsto z \cdot \mathbf{C}^*$$

is a submersion and there exists a unique pseudo-Riemannian metric of complex index one on $CP^n_1(4c)$ such that π is a Riemannian submersion. The pseudo-Riemannian manifold $CP^n_1(4c)$ is a Lorentzian complex space form of positive holomorphic sectional curvature $4c$.

Analogously, if $c < 0$, consider

$$H^{2n+1}_2(c) = \{z \in \mathbf{C}^{n+1}_2;\ b_{2,n+1}(z,z) = c^{-1} < 0\},$$

which is an indefinite real space form of constant sectional curvature $c < 0$. The Hopf fibration

$$\pi : H^{2n+1}_2(c) \to CH^n_1(4c) : z \mapsto z \cdot \mathbf{C}^*$$

is a submersion and there exists a unique pseudo-Riemannian metric of complex index 1 on $CH^n_1(4c)$ such that π is a Riemannian submersion. The pseudo-Riemannian manifold $CH^n_1(4c)$ is a Lorentzian complex space form of negative holomorphic sectional curvature $4c$.

A complete simply-connected Lorentzian complex space form $\tilde{M}^n_1(4c)$ is holomorphically isometric to $\mathbf{C}^n_1$, $CP^n_1(4c)$, or $CH^n_1(4c)$, according to $c = 0$, $c > 0$ or $c < 0$, respectively.

Lorentzian surfaces in pseudo-Riemannian spaces of constant curvature with signature $(2,2)$ have been studied by L. Verstraelen and M. Pieters [11], [12] among others. In this article, we study minimal flat Lorentzian surfaces in Lorentzian complex space forms.

In Section 3 of this article, we provide the basic results for Lorentzian surfaces in Lorentzian Kähler surfaces. In particular, we show that each tangent plane of a Lorentzian surface cannot be J-invariant. In Section 4, we prove that the equation of Ricci is a consequence of equations of Gauss and Codazzi for minimal flat Lorentzian surfaces in Lorentzian complex space forms. The complete classification of minimal flat Lorentzian surfaces in Lorentzian complex plane $\mathbf{C}^2_1$ is obtained in Section 5. In Section 6 we show that the only minimal flat slant surfaces in non-flat Lorentzian complex space forms are the Lagrangian ones. In this section, we also classify minimal flat slant surfaces in Lorentzian complex plane CP^2_1. In the last section, we provide the classification of minimal flat Lagrangian surfaces in the Lorentzian complex hyperbolic plane CH^2_1.

2. Preliminaries

Let M be a Lorentzian surface of a Lorentzian Kähler surface $\tilde{M}_1^2$ equipped with an almost complex structure J and metric $\tilde{g}$. Let $\langle\ ,\ \rangle$ denote the inner product associated with $\tilde{g}$. Denote the induced metric on M by g.

Let ∇ and $\tilde{\nabla}$ denote the Levi–Civita connection on M and $\tilde{M}_1^2$, respectively. Then the formulas of Gauss and Weingarten are given respectively by (cf. [2], [3], [6], [9])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to M and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection, respectively.

The shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi\rangle = \langle A_\xi X, Y\rangle \tag{2.3}$$

for X, Y tangent to M and ξ normal to M.

For each normal vector ξ of M at $x \in M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_x M$. However, for Lorentzian surfaces the shape operator A_ξ is not diagonalizable in general.

The mean curvature vector is defined by

$$H = \frac{1}{2}\operatorname{trace} h. \tag{2.4}$$

A Lorentzian surface in $\tilde{M}_1^2$ is called *minimal* if $H = 0$ at each point on M.

For a Lorentzian surface M in a Lorentzian complex space form $\tilde{M}_1^2(4c)$, the equations of Gauss, Codazzi and Ricci are given respectively by

$$\begin{aligned} \langle R(X, Y)Z, W\rangle = {} & \langle \tilde{R}(X, Y)Z, W\rangle + \langle h(X, W), h(Y, Z)\rangle \\ & - \langle h(X, Z), h(Y, W)\rangle, \end{aligned} \tag{2.5}$$

$$(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X h)(Y, Z) - (\bar{\nabla}_Y h)(X, Z), \tag{2.6}$$

$$\langle R^D(X, Y)\xi, \eta\rangle = \langle \tilde{R}(X, Y)\xi, \eta\rangle + \langle [A_\xi, A_\eta]X, Y\rangle, \tag{2.7}$$

where X, Y, Z, W are vector tangent to M, and ∇h is defined by

$$(\bar{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.8}$$

3. Basics on Lorentzian surfaces

Let M be a Lorentzian surface in a Lorentzian Kähler surface $(\tilde{M}^2_1, g, J)$. For each tangent vector X of M, we put

$$JX = PX + FX, \tag{3.1}$$

where PX and FX are the tangential and the normal components of JX.

On the Lorentzian surface M there exists a *pseudo-orthonormal* local frame $\{e_1, e_2\}$ on M such that

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \quad \langle e_1, e_2\rangle = -1. \tag{3.2}$$

For a pseudo-orthonormal frame $\{e_1, e_2\}$ on M satisfying (3.2), it follows from (3.1), (3.2), and $\langle JX, JY\rangle = \langle X, Y\rangle$ that

$$Pe_1 = (\sinh\alpha)e_1, \quad Pe_2 = -(\sinh\alpha)e_2 \tag{3.3}$$

for some function α, which is called the Wirtinger angle.

When the Wirtinger angle α is constant on M, the Lorentzian surface M is called a *slant surface* (cf. [3], [7], [8]). In this case, α is called the *slant angle*; the slant surface is called α-slant. A α-slant surface is Lagrangian if and only if $\alpha = 0$. Obviously, slant surfaces (in particular, Lagrangian surfaces) in a Lorentzian Kähler surface are Lorentzian surfaces.

If we put

$$e_3 = (\operatorname{sech}\alpha)Fe_1, \quad e_4 = (\operatorname{sech}\alpha)Fe_2, \tag{3.4}$$

then we find from (3.1)–(3.4) that

$$Je_1 = \sinh\alpha e_1 + \cosh\alpha e_3, \qquad Je_2 = -\sinh\alpha e_2 + \cosh\alpha e_4, \tag{3.5}$$

$$Je_3 = -\cosh\alpha e_1 - \sinh\alpha e_3, \qquad Je_4 = -\cosh\alpha e_2 + \sinh\alpha e_4, \tag{3.6}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 0, \qquad \langle e_3, e_4\rangle = -1. \tag{3.7}$$

We call such a frame $\{e_1, e_2, e_3, e_4\}$ chosen above an *adapted pseudo-orthonormal frame* for the Lorentzian surface M in $\tilde{M}^2_1$.

From (3.5) we obtain the following.

Proposition 3.1. *Let M be a Lorentzian surface in Lorentzian Kähler surface $(\tilde{M}^2_1, g, J)$. Then every tangent plane of M is not J-invariant.*

We need the following.

Lemma 3.1. *If M is a Lorentzian surface in a Lorentzian Kähler surface $\tilde{M}^2_1$, then with respect to an adapted pseudo-orthonormal frame we have*

$$\nabla_X e_1 = \omega(X)e_1, \qquad \nabla_X e_2 = -\omega(X)e_2, \tag{3.8}$$

$$D_X e_3 = \Phi(X)e_3, \qquad D_X e_4 = -\Phi(X)e_4 \tag{3.9}$$

for some 1-forms ω, Φ on M.

Proof. Let us put

$$\nabla_X e_1 = \omega_1^1(X)e_1 + \omega_1^2(X)e_2, \quad \nabla_X e_2 = \omega_2^1(X)e_1 + \omega_2^2(X)e_2. \tag{3.10}$$

Then we obtain from (3.2) that $\omega_1^2 = \omega_2^1 = 0$ and $\omega_2^2 = -\omega_1^1$. Thus, if we put $\omega = \omega_1^1$, then we get (3.8). Similarly, if we put

$$D_X e_3 = \omega_3^3(X)e_3 + \omega_3^4(X)e_4, \quad D_X e_4 = \omega_4^3(X)e_3 + \omega_4^4(X)e_4, \tag{3.11}$$

then it follows from (3.7) that $\omega_3^4 = \omega_4^3 = 0$ and $\omega_3^3 = -\omega_4^4$. So, after putting $\Phi = \omega_3^3$, we get (3.9). $\square$

For a Lorentzian surface M in $\tilde{M}^2_1$ with second fundamental form h, we put

$$h(e_i, e_j) = h_{ij}^3 e_3 + h_{ij}^4 e_4, \tag{3.12}$$

where e_1, e_2, e_3, e_4 is an adapted pseudo-orthonormal frame.

Lemma 3.2. *If M is a Lorentzian surface in a Lorentzian Kähler surface $\tilde{M}^2_1$, then with respect to an adapted pseudo-orthonormal frame $\{e_1, e_2, e_3, e_4\}$ we have*

$$\begin{cases} A_{e_3} e_j = h_{j2}^4 e_1 + h_{1j}^4 e_2, \\ A_{e_4} e_j = h_{j2}^3 e_1 + h_{1j}^3 e_2, \end{cases} \tag{3.13}$$

$$e_j\alpha = (\omega_j - \Phi_j)\coth\alpha - 2h_{1j}^3, \tag{3.14}$$

$$e_1\alpha = h_{12}^4 - h_{11}^3, \ e_2\alpha = h_{22}^4 - h_{12}^3, \tag{3.15}$$

$$\omega_j - \Phi_j = (h_{1j}^3 + h_{j2}^4)\tanh\alpha, \tag{3.16}$$

for $j = 1, 2$, where $\omega_j = \omega(e_j)$ and $\Phi_j = \Phi(e_j)$.

Proof. This is done by direct computation using $\tilde{\nabla}_X(JY) = J\tilde{\nabla}_X Y$ together with (3.5)–(3.7), and Lemma 3.2. $\square$

4. Fundamental equations of minimal flat Lorentzian surfaces

In general, the three fundamental equations of Gauss, Codazzi and Ricci are independent. However, for minimal flat Lorentzian surfaces in Lorentzian complex space forms we have the following.

Theorem 4.1. *The equation of Ricci is a consequence of the equations of Gauss and Codazzi for minimal flat Lorentzian surfaces in a Lorentzian complex space form* $\tilde{M}_1^2(4c)$.

Proof. Let M be a minimal flat Lorentzian surface in a Lorentzian complex space form $\tilde{M}_1^2(4c)$. Since M is flat, we may assume that M is an open connected subset of $\mathbb{E}_1^2$ equipped with the Lorentzian metric tensor:

$$g_o = -dx \otimes dy - dy \otimes dx. \tag{4.1}$$

Put $e_1 = \partial/\partial x, e_2 = \partial/\partial y$. Then $\{e_1, e_2\}$ is a pseudo-orthonormal frame on M such that $\nabla e_1 = \nabla e_2 = 0$. Thus, we have $\omega = 0$.

Let e_3, e_4 be the normal vector fields as (3.4). Then $\{e_1, e_2, e_3, e_4\}$ is an adapted pseudo-orthonormal frame. Since M is minimal and Lorentzian, it follows from (2.4) and (3.2) that

$$h(e_1, e_1) = \beta e_3 + \gamma e_4, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \lambda e_3 + \mu e_4 \tag{4.2}$$

for some functions β, γ, λ, μ.

After applying Lemma 3.2 we find from (4.2) that

$$\begin{aligned} &(\bar{\nabla}_{e_1} h)(e_1, e_2) = (\bar{\nabla}_{e_2} h)(e_1, e_2) = 0, \\ &(\bar{\nabla}_{e_2} h)(e_1, e_1) = (\beta_y + \beta\Phi_2)e_3 + (\gamma_y - \gamma\Phi_2)e_4, \\ &(\bar{\nabla}_{e_1} h)(e_2, e_2) = (\lambda_x + \lambda\Phi_1)e_3 + (\mu_x - \mu\Phi_1)e_4. \end{aligned} \tag{4.3}$$

On the other hand, it follows from (1.1) and (3.5) that

$$\begin{aligned} &(\tilde{R}(e_1, e_2)e_1)^{\perp} = 3c \sinh\alpha \cosh\alpha e_3, \\ &(\tilde{R}(e_1, e_2)e_2)^{\perp} = 3c \sinh\alpha \cosh\alpha e_4. \end{aligned} \tag{4.4}$$

Thus, by using (4.3), (4.4), we obtain from the equation of Codazzi that

$$\beta_y = -\beta\Phi_2 - 3c \sinh\alpha \cosh\alpha, \tag{4.5}$$

$$\gamma_y = \gamma\Phi_2, \quad \lambda_x = -\lambda\Phi_1, \tag{4.6}$$

$$\mu_x = \mu\Phi_1 + 3c \sinh\alpha \cosh\alpha. \tag{4.7}$$

Also, it follows from (4.2), $\omega = 0$ and Lemma 3.2 that

$$A_{e_3}e_1 = \gamma e_2, \qquad A_{e_3}e_2 = \mu e_1, \quad A_{e_4}e_1 = \beta e_2, \quad A_{e_4}e_2 = \lambda e_1, \tag{4.8}$$

$$\beta = -\alpha_x, \qquad \mu = \alpha_y, \tag{4.9}$$

$$\Phi_1 = \alpha_x \tanh\alpha, \qquad \Phi_2 = -\alpha_y \tanh\alpha. \tag{4.10}$$

Substituting (4.9) and (4.10) into (4.5) and (4.7) gives

$$\alpha_{xy} = \alpha_x\alpha_y \tanh\alpha + 3c\sinh\alpha\cosh\alpha. \tag{4.11}$$

In views of (1.1), (3.5)–(3.7), (4.2), and (4.8), the equation of Gauss becomes

$$\gamma\lambda = \alpha_x\alpha_y + c(3\sinh^2\alpha - 1). \tag{4.12}$$

On the other hand, by applying (3.5) and (3.6), we have

$$\langle \tilde R(e_1, e_2)e_3, e_4\rangle = c(3\sinh^2\alpha + 1). \tag{4.13}$$

Using $\omega = 0$, Lemma 3.1 and (4.8)–(4.10), we find

$$\langle R^D(e_1, e_2)e_3, e_4\rangle = e_2\Phi_1 - e_1\Phi_2 = 2\alpha_{xy}\tanh\alpha + 2\alpha_x\alpha_y \operatorname{sech}^2\alpha, \tag{4.14}$$

$$\langle [A_{e_3}, A_{e_4}]e_1, e_2\rangle = \gamma\lambda + \alpha_x\alpha_y. \tag{4.15}$$

Hence, in view of (4.9), (4.13), (4.14) and (4.15), the equation of Ricci becomes

$$2\alpha_{xy}\tanh\alpha + 2\alpha_x\alpha_y \operatorname{sech}^2\alpha = \gamma\lambda = \alpha_x\alpha_y + c(3\sinh^2\alpha - 1). \tag{4.16}$$

After applying (4.11), the equation (4.16) of Ricci can be simplified exactly as the equation (4.12) of Gauss. □

5. Classification of minimal flat Lorentzian surfaces in $\mathbf{C}^2_1$

Minimal flat Lagrangian surfaces in the Lorentzian complex plane $\mathbf{C}^2_1$ have been classified by B. Y. Chen and L. Vrancken in [9]. Clearly, Lagrangian surfaces in $\mathbf{C}^2_1$ are Lorentzian surfaces automatically. In this section we completely classify minimal flat Lorentzian surfaces in the Lorentzian complex plane $\mathbf{C}^2_1$.

Theorem 5.1. *Let $\alpha(y)$ and $f(y)$ be two arbitrary differentiable functions of single variable defined on an open interval $I \ni 0$. Then*

$$\psi(x,y) = \Big(x + \mathrm{i}\, f(y) + \frac{1}{2}\int_0^y \cosh^2 \alpha dy - \int_0^y f'(y) \sinh \alpha dy, x - y + \mathrm{i}\, f(y) + \frac{1}{2}\int_0^y \cosh^2 \alpha dy - \int_0^y f'(y) \sinh \alpha dy - \mathrm{i}\int_0^y \sinh \alpha dy\Big)$$

defines a minimal flat Lorentzian surface in the Lorentzian complex plane $\mathbf{C}^2_1$ with α as its Wirtinger angle.

Conversely, every minimal flat Lorentzian surface in $\mathbf{C}^2_1$ is either an open portion of a totally geodesic Lorentzian plane or congruent to the Lorentzian surface described above.

Proof. It is straight-forward to show that the mapping ψ defined in the theorem gives rise to a minimal flat Lorentzian surface in $\mathbf{C}^2_1$.

Conversely, assume that M is a minimal flat Lorentzian surface in $\mathbf{C}^2_1$. If the second fundamental form vanishes identically, then M is an open portion of a totally geodesic Lorentzian plane. So, we assume from now on that M is a non-totally geodesic minimal flat Lorentzian surface in $\mathbf{C}^2_1$.

Since M is flat, we may assume that as before that M is an open connected subset of $\mathbb{E}^2_1$ equipped with the Lorentzian metric tensor:

$$g_o = -dx \otimes dy - dy \otimes dx. \tag{5.1}$$

Put $e_1 = \partial/\partial x, e_2 = \partial/\partial y$. Then $\{e_1, e_2\}$ is a pseudo-orthonormal frame on M such that $\nabla e_1 = \nabla e_2 = 0$. Thus, we have $\omega = 0$.

Let e_3, e_4 be the normal vector fields defined by (3.4). Since M is a minimal Lorentzian surface, we have

$$h(e_1,e_1) = \beta e_3 + \gamma e_4, \quad h(e_1,e_2) = 0, \;\; h(e_2,e_2) = \lambda e_3 + \mu e_4, \tag{5.2}$$

for some functions $\beta, \gamma, \lambda, \mu$. By applying (3.7), (5.2) and the equation of Gauss, we find

$$\gamma\lambda = -\beta\mu. \tag{5.3}$$

Case (A): $\beta = 0$ *on* M. From (5.3), we get $\gamma\lambda = 0$.

Case (A.1): $\gamma = 0$ *on* M. In this case, (5.2) reduces to

$$h(e_1,e_1) = h(e_1,e_2) = 0, \quad h(e_2,e_2) = \lambda e_3 + \mu e_4. \tag{5.4}$$

Since M is not totally geodesic, at least one of λ, μ is a nonzero function. Now, by applying the equation of Codazzi, we find from (5.4) that

$$\lambda_x = -\lambda \Phi_1, \quad \mu_x = \mu \Phi_1. \tag{5.5}$$

On the other hand, it follows from $\omega = 0$, (5.4), and Lemma 3.2 that

$$\alpha_x = 0, \quad \alpha_y = \mu = -\Phi_2 \coth \alpha, \quad \Phi_1 = 0. \tag{5.6}$$

From the first two equations in (5.6), we get $\alpha = \alpha(y)$ and $\mu = \alpha'(y)$. Also, from (5.5) and the last equation in (5.6), we have $\lambda = \lambda(y)$ and $\mu = \mu(y)$. Therefore, after applying (3.5), (5.4) and the formula of Gauss, we know that the immersion of the surface in $\mathbf{C}^2_1$ satisfies

$$\begin{aligned} &\psi_{xx} = \psi_{xy} = 0, \\ &\psi_{yy} = \lambda(y)(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh \alpha)\psi_x + \alpha''(y)(\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh \alpha)\psi_y. \end{aligned} \tag{5.7}$$

Solving the first two equations of (5.7) shows that the immersion is given by

$$\psi = c_1 x + B(y) \tag{5.8}$$

for some vector $c_1 \in \mathbf{C}^2_1$ and $\mathbf{C}^2_1$-valued function $B(y)$. Thus, by applying (5.1) and $\langle \mathrm{i}\,\psi_x, \psi_y \rangle = -\sinh \alpha$, we may find from (3.2) and (3.5) that

$$\langle c_1, c_1 \rangle = 0, \quad \langle c_1, B' \rangle = -1, \quad \langle \mathrm{i}\,c_1, B' \rangle = -\sinh \alpha, \tag{5.9}$$

$$\langle B', B' \rangle = 0. \tag{5.10}$$

Without loss of generality, we may put

$$c_1 = (1,1), \quad B(y) = (k(y) + \mathrm{i}\,f(y), u(y) + \mathrm{i}\,v(y)). \tag{5.11}$$

Now, by applying conditions in (5.9) and (5.11), we obtain

$$u = k - y + a_1, \quad v = f - \int_0^y \sinh \alpha dy + a_2 \tag{5.12}$$

for some real numbers a_1, a_2. From (5.10) and (5.10), we find

$$k = \frac{1}{2}\int_0^y \cosh^2 \alpha dy - \int_0^y f'(y) \sinh \alpha dy + a_3 \tag{5.13}$$

for some real number a_3.

By combining (5.8), (5.11), (5.12) and (5.13) we know that the immersion is congruent to the one described in the theorem.

Case (A.2): $\lambda = 0$ *and* $\gamma \neq 0$ *on some open subset* $U \subset M$. Let us work on U. From (5.2) we have

$$h(e_1, e_1) = \gamma e_4, \quad h(e_1, e_2) = 0, \quad h(e_2, e_2) = \mu e_4. \tag{5.14}$$

Thus, the equation of Codazzi yields

$$\gamma_y = \gamma \Phi_2, \quad \mu_x = \mu \Phi_1. \tag{5.15}$$

When $\mu = 0$, this reduces to case (A.1) after interchanging x and y. So, we assume that $\mu \neq 0$. Hence (5.15) gives

$$(\ln \gamma)_y = \Phi_2, \quad (\ln \mu)_x = \Phi_1. \tag{5.16}$$

It follows from (3.9) of Lemma 3.2 that the normal curvature tensor R^D satisfies

$$\begin{aligned} \langle R^D(e_1, e_2)e_3, e_4 \rangle &= \langle D_{e_1}(\Phi_2 e_3) - D_{e_2}(\Phi_1 e_3), e_4 \rangle \\ &= e_2 \Phi_1 - e_1 \Phi_2 = (\ln \mu)_{xy} - (\ln \gamma)_{xy}. \end{aligned} \tag{5.17}$$

On the other hand, from (3.13) of Lemma 3.2 and (5.14) we get $A_{e_4} = 0$. Thus, by combining these with the equation of Ricci, we obtain $(\ln \gamma)_{xy} = (\ln \mu)_{xy}$. Consequently, we have

$$\gamma = (f(x) + k(y))\mu \tag{5.18}$$

for some real-valued functions $f(x)$, $k(y)$. Therefore, after applying (3.5), (5.14) and the formula of Gauss, we know that the immersion satisfies

$$\begin{aligned} \psi_{xx} &= (f(x) + k(y))\mu(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh \alpha)\psi_y, \\ \psi_{xy} &= 0, \\ \psi_{yy} &= \mu(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh \alpha)\psi_y. \end{aligned} \tag{5.19}$$

It follows from $(\psi_{yy})_x = (\psi_{xy})_y = 0$ that

$$\mu_x = -\,\mathrm{i}\,\mu \alpha_x \,\mathrm{sech}\,\alpha.$$

Hence, we have

$$\mu = \phi(y) e^{-2\,\mathrm{i} \tan^{-1}(\tanh \alpha/2)} \tag{5.20}$$

for some nonzero real-valued function $\phi(y)$. Substituting this into (5.18) gives

$$\begin{aligned} &\psi_{xx} = \mathrm{i}\,\phi(y)(f(x)+k(y))\psi_y, \\ &\psi_{xy} = 0, \quad \psi_{yy} = \mathrm{i}\,\phi(y)\psi_y. \end{aligned} \tag{5.21}$$

Now, it follows from $(\psi_{xx})_y = (\psi_{xy})_x = 0$ and (5.21) that

$$\mathrm{i}[(\phi(y)k'(y) + (f(x)+k(y))\phi'(y)] = (f(x)+k(y))\phi^2(y). \tag{5.22}$$

Since ϕ, f, k are real-valued, (5.22) implies that $(f(x)+k(y))\phi(y)$=0. But this is impossible, since γ and μ are nonzero functions. Thus, this case cannot occur.

Case (B): $\gamma = 0$ *and* $\beta \neq 0$ *on some open subset* $V \subset M$. Let us work on V. It follows from (5.3) that $\mu = 0$. Hence, (5.2) reduces to

$$h(e_1,e_1) = \beta e_3, \quad h(e_1,e_2) = 0, \quad h(e_2,e_2) = \lambda e_3. \tag{5.23}$$

But this case is also impossible after applying a similar argument as case (A.2).

Case (C): β, γ, λ, *and* μ *are nonzero on some open subset* $W \subset M$. Let us work on W. It follows from (5.2), $\omega = 0$, Lemma 3.1, and the equation of Codazzi that

$$(\ln\beta)_y = -\Phi_2, \quad (\ln\gamma)_y = \Phi_2, \quad (\ln\lambda)_x = -\Phi_1, \quad (\ln\mu)_x = \Phi_1, \tag{5.24}$$

which imply that

$$\beta\gamma = \varphi(x), \ \ \lambda\mu = \eta(y) \tag{5.25}$$

for some nonzero real-valued functions $\varphi(x), \eta(y)$. Hence, (5.2) becomes

$$\begin{aligned} &h(e_1,e_1) = \beta e_3 + \frac{\varphi(x)}{\beta} e_4, \\ &h(e_1,e_2) = 0, \\ &h(e_2,e_2) = \frac{\eta(y)}{\mu} e_3 + \mu e_4. \end{aligned} \tag{5.26}$$

Since the surface is flat, (5.26) and the equation of Gauss gives

$$\beta^2\mu^2 = -\varphi(x)\eta(y). \tag{5.27}$$

By applying (3.5), (5.26) and the formula of Gauss, we know that the immersion satisfies

$$\begin{aligned}
&\psi_{xx} = \beta(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\psi_x + \frac{\varphi(x)}{\beta}(\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)\psi_y, \\
&\psi_{xy} = 0, \\
&\psi_{yy} = \frac{\eta(y)}{\mu}(\mathrm{i}\,\mathrm{sech}\,\alpha - \tanh\alpha)\psi_x + \mu(\mathrm{i}\,\mathrm{sech}\,\alpha + \tanh\alpha)\psi_y.
\end{aligned} \tag{5.28}$$

The compatibility conditions of system (5.28) are given by

$$\mu\beta\beta_y = -\varphi(x)\eta(y)\tanh\alpha, \tag{5.29}$$

$$\mu = \alpha_y, \tag{5.30}$$

$$\beta^2\mu\alpha_y = -\varphi(x)\eta(y), \tag{5.31}$$

$$\beta_y = \beta\mu\tanh\alpha, \tag{5.32}$$

$$(\sinh 2\alpha)\mu_x - 2\mu\alpha_x = (3 - \cosh 2\alpha)\beta\mu, \tag{5.33}$$

$$\mu_x + \mu\alpha_x\tanh\alpha = -2\beta\mu\tanh\alpha, \tag{5.34}$$

$$\beta\mu^2\alpha_x = \varphi(x)\eta(y), \tag{5.35}$$

$$\beta\mu\mu_x = \varphi(x)\eta(y)\tanh\alpha. \tag{5.36}$$

Form (5.30), (5.31), and (5.35), we get

$$\beta = -\alpha_x, \quad \mu = \alpha_y. \tag{5.37}$$

Thus, (5.31), (5.34) and (5.37) imply that

$$\alpha_x^2\alpha_y^2 = -\varphi(x)\eta(y), \tag{5.38}$$

$$\alpha_{xy} = \alpha_x\alpha_y\tanh\alpha. \tag{5.39}$$

Solving (5.39) yields

$$\alpha = 2\tanh^{-1}(\tan(f(x) + k(y))) \tag{5.40}$$

for some functions $f(x)$, $k(y)$. Since $\varphi(x)\eta(y) \neq 0$, (5.38) shows that α is a non-constant function. Hence, $f(x) + k(y)$ is also non-constant.

Substituting (5.40) into (5.38) gives

$$16\left(\frac{f'(x)^2}{\varphi(x)}\right)\left(\frac{k'(y)^2}{\eta(y)}\right)=-\cos^4(\tan(2f(x)+2k(y))). \tag{5.41}$$

It follows from (5.41) that at least one of $f(x)$, $k(y)$ is a constant function. But this is impossible, since it leads to

$$\cos^4(\tan(2f(x)+2k(y)))=0.$$

Consequently, this case also cannot occur. $\square$

The following result is a special case of Theorem 5.1.

Corollary 5.1. *Every minimal flat θ-slant surface in $\mathbf{C}^2_1$ is either an open portion of a totally geodesic slant plane or congruent to the surface defined by*

$$\begin{aligned}\psi(x,y)=\Big(&x+\frac{y}{2}\cosh^2\theta+(\mathrm{i}-\sinh\theta)f(y),x-y\\ &+\frac{y}{2}\cosh^2\theta+(\mathrm{i}-\sinh\theta)f(y)-\mathrm{i}\,y\sinh\theta\Big)\end{aligned}$$

for some function $f(y)$.

Remark 5.1. When $\theta=0$, Corollary 5.1 reduces to a result of [9].

Remark 5.2. If $\alpha(y)$ and $f(y)$ are functions defined on the entire real line, then the minimal flat Lorentzian surface defined in Theorem 5.1 is a complete surface. Consequently, there exist infinitely many complete minimal flat Lorentzian surfaces in $\mathbf{C}^2_1$. Moreover, Corollary 5.1 shows that there exist infinitely many complete minimal flat slant surfaces in $\mathbf{C}^2_1$.

6. Classification of minimal flat slant surfaces in $CP^2_1(4)$

The following lemma follows easily from the proof of Theorem 4.1.

Lemma 6.1. *The only minimal flat slant surfaces in a Lorentzian complex space form $\tilde{M}^2_1(4c)$ with $c\neq 0$ are the Lagrangian ones.*

Proof. Let M be a minimal flat Lorentzian slant surface in $\tilde{M}^2_1(4c)$ with $c\neq 0$. Then α is constant. Thus (4.11) implies that $\sinh\alpha\cosh\alpha=0$, which is impossible unless $\alpha=0$, i.e., M is Lagrangian. $\square$

The following theorem completely classifies minimal flat slant surfaces in $CP^2_1(4)$.

Theorem 6.1. *If $L : M \to CP^2_1(4)$ is a minimal flat slant surface in the Lorentzian complex projective plane $CP^2_1(4)$, then L is Lagrangian. Moreover, the immersion is congruent to $\pi \circ \tilde{L}$, where*

$$\begin{aligned}\tilde{L}(x,y) = \frac{1}{\sqrt{3}}\Bigg(&\sqrt{2}e^{\frac{\mathrm{i}}{2a}(x-a^2y)}\cosh\left(\frac{\sqrt{3}}{2a}(x+a^2y)\right), e^{\frac{\mathrm{i}}{a}(a^2y-x)},\\ &\sqrt{2}e^{\frac{\mathrm{i}}{2a}(x-a^2y)}\sinh\left(\frac{\sqrt{3}}{2a}(x+a^2y)\right)\Bigg),\end{aligned} \tag{6.1}$$

a is a nonzero real number and $\pi : S^5_2(1) \to CP^2_1(4)$ is the Hopf fibration.

Proof. Let $L : M \to CP^2_1(4)$ be a minimal flat slant surface in $CP^2_1(4)$. Then L is Lagrangian according to Lemma 6.1.

As in the proof of Theorem 4.1, we may assume that M is an open connected subset of $\mathbb{E}^2_1$ with

$$g_o = -dx \otimes dy - dy \otimes dx. \tag{6.2}$$

Let e_1, e_2, e_3, e_4 be as in the proof of Theorem 4.1. Then we have

$$\beta = \mu = \omega = \Phi = 0.$$

Thus, we see from (4.5), (4.7), (4.9) and (4.10) that γ and λ are nonzero real numbers satisfying $\gamma\lambda = -1$. Hence, if we put $\lambda = -a^3$, then (4.2) reduces to

$$h(e_1,e_1) = \frac{Je_2}{a^3}, \quad h(e_1,e_2) = 0, \quad h(e_2,e_2) = -a^3 Je_1. \tag{6.3}$$

Therefore, if $\tilde{L} : M \to S^5_2(1)$ is a horizontal lift of L (cf. [10]), then we have

$$\tilde{L}_{xx} = \frac{\mathrm{i}}{a^3}L_y, \quad \tilde{L}_{xy} = \tilde{L}, \quad \tilde{L}_{yy} = -\mathrm{i}\, a^3 L_x. \tag{6.4}$$

It follows from the first two equations in (6.4) that

$$a^3 \tilde{L}_{xxx} = \mathrm{i}\, \tilde{L}.$$

Solving this equation gives

$$\tilde{L} = e^{\mathrm{i}\, x/(2a)}B(y)\left(\cosh\left(\frac{\sqrt{3}x}{2a}\right)B(y) + \sinh\left(\frac{\sqrt{3}x}{2a}\right)C(y)\right) + e^{-\mathrm{i}\, x/a}A(y) \tag{6.5}$$

for some functions $A(y)$, $B(y)$, $C(y)$. Substituting this into the first equation in (6.4) gives

$$A'(y) = \mathrm{i}\, aA(y), \tag{6.6}$$

$$2B'(y) + \mathrm{i}\, aB(y) = \sqrt{3}aC(y), \tag{6.7}$$

$$2C'(y) + \mathrm{i}\, aC(y) = \sqrt{3}aB(y). \tag{6.8}$$

After solving these differential equations we have

$$A(y) = c_1 e^{iay}, \tag{6.9}$$

$$B(y) = (b_2 e^{\sqrt{3}ay} + b_3)e^{-\frac{1}{2}(\mathrm{i}+\sqrt{3})ay}, \tag{6.10}$$

$$C(y) = (b_2 e^{\sqrt{3}ay} - b_3)e^{-\frac{1}{2}(\mathrm{i}+\sqrt{3})ay} \tag{6.11}$$

for some constant vectors c_1, b_2, b_3. Combining these with (6.5) gives

$$\begin{aligned}\tilde{L}(x,y) = e^{\frac{\mathrm{i}}{2a}(x-a^2y)} & \left\{ c_2 \cosh\left(\frac{\sqrt{3}}{2a}(x+a^2y)\right) + c_3 \sinh\left(\frac{\sqrt{3}}{2a}(x+a^2y)\right)\right\} \\ & + c_1 e^{\mathrm{i}(ay-\frac{x}{a})},\end{aligned}$$

where c_1, c_2, c_3 are vectors in $\mathbf{C}^3_1$. Consequently, after choosing suitable initial conditions we obtain the immersion (6.1). $\square$

7. Minimal flat slant surfaces in $CH^2_1(-4)$

Similarly, we have the following classification of minimal flat Lagrangian surfaces in $CH^2_1(-4)$.

Theorem 7.1. *If $L : M \to CH^2_1(-4)$ is a minimal flat slant surfaces in the Lorentzian complex projective plane $CH^2_1(-4)$, then L is Lagrangian. Moreover, it is congruent to $\pi \circ \tilde{L}$, where*

$$\begin{aligned}\tilde{L}(x,y) = \frac{1}{\sqrt{3}} \Bigg(& \sqrt{2}e^{-\frac{\mathrm{i}}{2a}(x+a^2y)} \cosh\left(\frac{\sqrt{3}}{2a}(x-a^2y)\right), e^{\mathrm{i}(ay+\frac{x}{a})}, \\ & \sqrt{2}e^{-\frac{\mathrm{i}}{2a}(x+a^2y)} \sinh\left(\frac{\sqrt{3}}{2a}(x-a^2y)\right)\Bigg),\end{aligned} \tag{7.1}$$

a is a nonzero real number and $\pi : H^5_2(-1) \to CH^2_1(-4)$ is the Hopf fibration.

Proof. This can be proved in a way similar to the proof of Theorem 6.1. So, we omit the details. □

Remark 7.1. The surfaces defined by (6.1) and (7.1) are also complete.

Remark 7.2. Further results on minimal Lorentzian surfaces in Lorentzian complex space forms have been later obtained in [5] (added on May 8, 2008).

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