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Iterative Pexider equation

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Abstract. We consider the Pexider equation $F_{st} = H_s \circ G_t$ for (s,t) belonging to the domain of a binary operation on a groupoid K, where $\{F_t : t \in K\} \subset Z^X$, $\{G_t : t \in K\} \subset Y^X$, $\{H_t : t \in K\} \subset Z^Y$ are unknown families of functions. It is shown that, in the case when there exists a unit element e in K and H_e is an injection and G_e is a surjection, the equation can be reduced to the Cauchy equation. Using the above result we solve the following problem: when does it follow from the equality $F_{st} = H_s \circ G_t$, for (s,t) belonging to a set $L \subsetneq R^2_+$, that $F_{st} = H_s \circ G_t$ for $(s,t) \in R^2_+$? Finally, some other conditions are established under which the equation may be reduced to the Cauchy equation.

Let K be a non-empty set endowed with a binary operation (i.e. a mapping of a subset $D(K)$ of $K \times K$ into K). The set K with the binary operation is called a groupoid (cp. [2]). If $(s,t) \in D(K)$ then we say that st is defined.

The binary operation is said to be associative in case the following implication holds: if in the equation $s(tp) = (st)p$, $s,t,p \in K$, one of its sides or both the products tp and st are defined then both sides of the equation are defined and the equality holds.

An element $e \in K$ will be called a unit if for every $t \in K$ the products te and et are defined and $te = et = t$.

For $t \in K$ we denote by $K_l(t)\,(K_r(t))$ the set of all elements $e \in K$ such that et (te) is defined and $et = t$ $(te = t)$.

Let K be a groupoid and X, Y, Z arbitrary non-empty sets. We shall consider the Pexider functional equation

$$F_{st} = H_s \circ G_t \quad \text{for } (s,t) \in D(K), \tag{1}$$

where $\{F_t : t \in K\} \subset Z^X$, $\{G_t : t \in K\} \subset Y^X$, $\{H_t : t \in K\} \subset Z^Y$ are unknown families of functions. We understand (1) in such a way that if st is defined, then the composition $H_s \circ G_t$ is defined (i.e. $\operatorname{Ran} G_t \subset \operatorname{Dom} H_t$,

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$t \in K$, where $\operatorname{Ran} G_t$ denotes the range of G_t and $\operatorname{Dom} H_t$ denotes the domain of H_t) and equality (1) holds. Similar problems have been studied in [1], [5], [6], [7].

Let us denote by $\operatorname{In}(X,Y)$ ($\operatorname{Sur}(X,Y)$) the set of all injections (surjections) of X into (onto) Y.

Theorem 1. *Let K be a groupoid such that there exists a unit element e in K.*

(i) *If $\{F_t : t \in K\} \subset Z^X$, $\{G_t : t \in K\} \subset Y^X$, $\{H_t : t \in K\} \subset Z^Y$ satisfy* (1) *and $H_e \in \operatorname{In}(Y,Z)$, $G_e \in \operatorname{Sur}(X,Y)$ then there exist functions $A \in \operatorname{In}(Y,Z)$, $B \in \operatorname{Sur}(X,Y)$ and a family of functions $\{T_t : t \in K\} \subset Y^Y$ such that*

$$T_{st} = T_s \circ T_t \quad \text{for } (s,t) \in D(K) \tag{2}$$

and

$$\begin{cases} F_t = A \circ T_t \circ B, \\ G_t = T_t \circ B, \\ H_t = A \circ T_t, \quad t \in K. \end{cases} \tag{3}$$

(ii) *If $A \in Z^Y$, $B \in Y^X$ are arbitrary functions and $\{T_t : t \in K\} \subset Y^Y$ fulfils condition* (2) *then the functions F_t, G_t, H_t given by* (3) *satisfy equation* (1).

Proof. Put $F(t) := F_t$, $G(t) := G_t$, $H(t) := H_t$. Setting in (1) $t = e$ and then $s = e$ we get

$$F(s) = H(s) \circ G(e), \quad s \in K, \tag{4}$$

$$F(t) = H(e) \circ G(t), \quad t \in K. \tag{5}$$

Comparing the right hand sides of (4) and (5) for $s = t$ we obtain

$$H(t) \circ G(e) = H(e) \circ G(t), \quad t \in K. \tag{6}$$

By (6) and the relation $G(e) \in \operatorname{Sur}(X,Y)$ we infer that

$$\operatorname{Ran} H(t) \subset \operatorname{Ran} H(e) \quad \text{for } t \in K. \tag{7}$$

Note that in view of (4) and the fact that $G(e) \in \operatorname{Sur}(X,Y)$

$$\operatorname{Ran} H(t) = \operatorname{Ran} F(t), \quad t \in K. \tag{8}$$

From (1) we have

$$F(st) = F(e(st)) = H(e) \circ G(st) \quad \text{for } (s,t) \in D(K). \tag{9}$$

Hence

$$\operatorname{Ran} F(st) \subset \operatorname{Ran} H(e) \quad \text{for } (s,t) \in D(K). \tag{10}$$

Thus from (7), (8), (10) we have the following relations

$$\operatorname{Dom} H(e)^{-1} \supset \operatorname{Ran} H(t) = \operatorname{Ran} F(t), \quad t \in K,$$
$$\operatorname{Dom} H(e)^{-1} \supset \operatorname{Ran} F(st), \quad (s,t) \in D(K).$$

We introduce on X an equivalence relation $\sim$ putting

$$x \sim y \quad \text{iff} \quad G(e)(x) = G(e)(y).$$

Denote $\widetilde{X} := X/\sim$ and let g be an invertible mapping such that $g([x]) \in [x]$. Thus the function $G(e) \circ g : \widetilde{X} \to Y$ is a bijection. From (4) we obtain $F(t) \circ g = H(t) \circ G(e) \circ g$ whence

$$H(t) = F(t) \circ g \circ (G(e) \circ g)^{-1} \quad \text{for } t \in K. \tag{11}$$

Hence (1) may be written as follows:

$$F(st) = F(s) \circ g \circ (G(e) \circ g)^{-1} \circ G(t) \quad \text{for } (s,t) \in D(K). \tag{12}$$

(5) yields

$$G(t) = H(e)^{-1} \circ F(t), \quad t \in K. \tag{13}$$

Next (6) implies

$$G(t) = H(e)^{-1} \circ H(t) \circ G(e), \quad t \in K. \tag{14}$$

Putting (13) into (12) we obtain

$$F(st) = F(s) \circ g \circ (G(e) \circ g)^{-1} \circ H(e)^{-1} \circ F(t), \quad (s,t) \in D(K). \tag{15}$$

Define $T(t) := H(e)^{-1} \circ F(t) \circ g \circ (G(e) \circ g)^{-1}, \quad t \in K$. Hence by (12) and (13) we can write

$$\begin{aligned} T(st) &= H(e)^{-1} \circ F(st) \circ g \circ (G(e) \circ g)^{-1} = \\ &= H(e)^{-1} \circ F(s) \circ g \circ (G(e) \circ g)^{-1} \circ G(t) \circ g \circ (G(e) \circ g)^{-1} = \\ &= H(e)^{-1} \circ F(s) \circ g \circ (G(e) \circ g)^{-1} \circ H(e)^{-1} \circ F(t) \circ g \circ (G(e) \circ g)^{-1} = \\ &= T(s) \circ T(t). \end{aligned}$$

Then (2) holds, where $T_t := T(t)$, $t \in K$. By (11) we have

$$H(t) = H(e) \circ T(t), \quad t \in K, \tag{16}$$

and from (16), (14)

$$G(t) = T(t) \circ G(e), \quad t \in K. \tag{17}$$

Setting $s = e$ in (15) and (12) we get

$$F(t) = F(e) \circ g \circ (G(e) \circ g)^{-1} \circ H(e)^{-1} \circ F(t), \quad t \in K, \tag{18}$$

and

$$F(t) = F(e) \circ g \circ (G(e) \circ g)^{-1} \circ G(t), \quad t \in K$$

respectively. Hence by (17), the definition of T, and (18) we obtain

$$F(t) = H(e) \circ T(t) \circ G(e), \quad t \in K. \tag{19}$$

Putting $A := H(e)$, $B := G(e)$ we get from (19), (17) and (16) the formulas (3).

The proof of (ii) is easy.

Now we present an application of the above result.
Let $D(R_+) := \{(s,t) \in R_+ : s \cdot t = 0 \text{ or } s = c \cdot t\}$ where R_+ denotes the set of all non-negative real numbers and $c \in R_+$.

Assume that $\{F_t : t \in R_+\}$, $\{G_t : t \in R_+\}$, $\{H_t : t \in R_+\}$ are one-parameter families of functions mapping a real interval $I := \langle a, b\rangle$ into itself. We consider the following problem: when does the equality $F_{t+s} = H_s \circ G_t$ for $(s,t) \in D(R_+)$ imply that $F_{t+s} = H_s \circ G_t$ for $(s,t) \in R_+^2$? The analogous problem for the Cauchy equation

$$F_{t+s} = F_t \circ F_s, \quad (s,t) \in D(R_+) \tag{20}$$

has been considered by M.C. Zdun in [8] and M. Sablik in [3]. In the latter paper there has been proved the following

Theorem 2. *If the limit* $\lim\limits_{t\to 0} \frac{F_t(x)-x}{t} =: d(x) \neq 0$ *exists in* (a,b), d *is a continuous function,* $F(x,t) = F_t(x)$ *is continuous (as a function of two variables) and* (20) *holds, then* $\{F_t : t \in R_+\}$ *is an iteration semigroup (i.e.* $F_{t+s} = F_t \circ F_s$ *for* $(s,t) \in R_+^2$*, cp. [4]).*

Using Theorems 1(i) and 2 we shall prove the following

Proposition 1. *Suppose that* $H_0 \in \mathrm{In}(I)$, $G_0 \in \mathrm{Sur}(I)$*, the functions* $H(x,t) := H_t(x)$, H_0 *are continuous and* $F_{t+s} = H_s \circ G_t$ *for* $(s,t) \in D(R_+)$*. If the limit* $\lim\limits_{t\to 0} \frac{(H_0^{-1}\circ H_t)(x)-x}{t} =: d(x) \neq 0$ *exists in* (a,b) *and* d *is continuous then* $F_{t+s} = H_s \circ G_t$ *for* $(s,t) \in R_+^2$.

Proof. By the proof of Theorem 1(i) we have the formulas

$$\begin{cases} F_t = H_0 \circ T_t \circ G_0, \\ G_t = T_t \circ G_0, \\ H_t = H_0 \circ T_t, \quad t \in R_+, \end{cases} \tag{21}$$

where $\{T_t : t \in R_+\} \subset I^I$ is a family of functions such that $T_{t+s} = T_t \circ T_s$ for $(s,t) \in D(R_+)$. On account of (21) we have

$$\lim_{t\to 0} \frac{(H_0^{-1} \circ H_t)(x) - x}{t} = \lim_{t\to 0} \frac{T_t(x) - x}{t}.$$

Hence, by Theorem 2, we infer that $T_{t+s} = T_t \circ T_s$ for $(s,t) \in R_+^2$ and consequently, from formulas (21), we get the Proposition.

In the associative case we have the following general Lemma which will be used in the proof of the next Theorem:

Lemma. (i) *Let K be a groupoid such that the binary operation is associative and $K_l(t) \neq \emptyset$, $K_r(t) \neq \emptyset$ for every $t \in K$. Suppose that $l \in \underset{t\in K}{\times} K_l(t)$, $r \in \underset{t\in K}{\times} K_r(t)$ and*

$$\{F_t : t \in K\} \subset Z^X, \{G_t : t \in K\} \subset Y^X, \{H_t : t \in K\} \subset Z^Y \textit{satisfy } (1).$$

If

$$\text{(C)} \qquad H_{l(t)} \in \mathrm{In}(Y,Z),\ G_{r(t)} \in \mathrm{Sur}(X,Y) \quad \textit{for } t \in K$$

then there exist $\{M_t\}_{t\in K} \subset \mathrm{In}(Y,Z)$, $\{N_t\}_{t\in K} \subset \mathrm{Sur}(X,Y)$ and $\{T_t\}_{t\in K} \subset Y^Y$ such that

$$\text{(H)} \qquad T_{st} = T_s \circ T_t, \quad M_{st} \circ T_{st} = M_s \circ T_{st}, \quad T_{st} \circ N_{st} = T_{st} \circ N_t$$

for $(s,t) \in D(K)$, and

$$\text{(22)} \qquad \begin{cases} F_t = M_t \circ T_t \circ N_t, \\ G_t = T_t \circ N_t, \\ H_t = M_t \circ T_t, \quad t \in K \end{cases}$$

(ii) *Conversely, if $\{M_t\}_{t\in K} \subset Z^Y$, $\{N_t\}_{t\in K} \subset Y^X$, $\{T_t\}_{t\in K} \subset Y^Y$ satisfy (H) then the functions F_t, G_t, H_t given by* (22) *fulfil equation* (1).

Proof. Suppose that $\{F_t\}_{t\in K}$, $\{G_t\}_{t\in K}$, $\{H_t\}_{t\in K}$ satisfy (1) and condition (C). Put $F(t) := F_t$, $G(t) := G_t$, $H(t) := H_t$, $l_t := l(t)$, $r_t := r(t)$.

From equation (1) we directly obtain

$$\begin{aligned} &\text{(23)} && F(st) = F(l_{st}(st)) = H(l_{st}) \circ G(st), \\ &\text{(24)} && F(st) = F((st)r_{st}) = H(st) \circ G(r_{st}), \end{aligned} \qquad (s,t) \in D(K).$$

In view of associativity we have

$$\begin{aligned} &\text{(25)} && F(st) = F((l_s s)t) = F(l_s(st)) = H(l_s) \circ G(st), \\ &\text{(26)} && F(st) = F(s(tr_t)) = F((st)r_t) = H(st) \circ G(r_t), \end{aligned} \qquad (s,t) \in D(K).$$

Setting in (1) $t = r_s$ and then $s = l_t$ we get

$$F(s) = H(s) \circ G(r_s), \quad s \in K \tag{27}$$

$$F(t) = H(l_t) \circ G(t), \quad t \in K. \tag{28}$$

Comparing the right hand sides of (27) and (28) for $s = t$ we obtain

$$H(t) \circ G(r_t) = H(l_t) \circ G(t), \quad t \in K. \tag{29}$$

Hence, using (29) and the relation $G(r_t) \in \mathrm{Sur}(X, Y)$, we infer that

$$\mathrm{Ran}\, H(t) \subset \mathrm{Ran}\, H(l_t) \quad \text{for all } t \in K. \tag{30}$$

Moreover, from (25) it follows that

$$\mathrm{Ran}\, F(st) \subset \mathrm{Ran}\, H(l_s) \quad \text{for } (s,t) \in D(K). \tag{31}$$

Note that, in view of (27) and the fact that $G(r_t) \in \mathrm{Sur}(X, Y)$,

$$\mathrm{Ran}\, H(t) = \mathrm{Ran}\, F(t), \quad t \in K. \tag{32}$$

Thus from (30), (31), (32) we have the following relations

$$\mathrm{Dom}\, H(l_t)^{-1} \supset \mathrm{Ran}\, H(t) = \mathrm{Ran}\, F(t), \quad t \in K,$$
$$\mathrm{Dom}\, H(l_s)^{-1} \supset \mathrm{Ran}\, F(st), \quad (s,t) \in D(K).$$

Now, on account of (23) and (25) we get

$$H(l_{st})^{-1} \circ F(st) = H(l_s)^{-1} \circ F(st) \quad \text{for } (s,t) \in D(K). \tag{33}$$

Fix a $t \in K$ and introduce an equivalence relation $\sim_t$ on X putting $x \sim_t y$ iff $G(r_t)(x) = G(r_t)(y)$. Denote

$$\overset{\sim_t}{X} := X/\sim_t .$$

Fix an invertible mapping $g_t : \overset{\sim_t}{X} \to X$ such that

$$g_t([x]) \in [x].$$

Then for every $t \in K$ the mapping $G(r_t) \circ g_t : \overset{\sim_t}{X} \to Y$ is a bijection.

From (27) we obtain

$$F(t) \circ g_t = H(t) \circ G(r_t) \circ g_t, \quad \text{whence}$$

$$H(t) = F(t) \circ g_t \circ (G(r_t) \circ g_t)^{-1}. \tag{34}$$

Using (34), (1) may be written as follows:

$$F(st) = F(s) \circ g_s \circ (G(r_s) \circ g_s)^{-1} \circ G(t) \quad \text{for } (s,t) \in D(K). \tag{35}$$

Note that the relations (24) and (26) imply the equalities

$$\begin{aligned} F(st)\circ g_{st} &= H(st)\circ G(r_{st})\circ g_{st}, \\ F(st)\circ g_t &= H(st)\circ G(r_t)\circ g_t, \end{aligned} \qquad (s,t)\in D(K).$$

Hence we get

$$F(st)\circ g_{st}\circ (G(r_{st})\circ g_{st})^{-1} = F(st)\circ g_t\circ (G(r_t)\circ g_t)^{-1} \tag{36}$$

for $(s,t)\in D(K)$. By (28) we have

$$G(t) = H(l_t)^{-1}\circ F(t), \quad t\in K. \tag{37}$$

Next, (29) implies

$$G(t) = H(l_t)^{-1}\circ H(t)\circ G(r_t), \quad t\in K. \tag{38}$$

Putting (37) into (35) we obtain

$$F(st) = F(s)\circ g_s\circ (G(r_s)\circ g_s)^{-1}\circ H(l_t)^{-1}\circ F(t) \tag{39}$$

for $(s,t)\in D(K)$. Define

$$T(t) := H(l_t)^{-1}\circ F(t)\circ g_t\circ (G(r_t)\circ g_t)^{-1}, \quad t\in K.$$

Hence by (33), (36), and (39) we can write

$$\begin{aligned} T(st) &= H(l_{st})^{-1}\circ F(st)\circ g_{st}\circ (G(r_{st})\circ g_{st})^{-1} = \\ &= H(l_s)^{-1}\circ F(st)\circ g_{st}\circ (G(r_{st})\circ g_{st})^{-1} = \\ &= H(l_s)^{-1}\circ F(st)\circ g_t\circ (G(r_t)\circ g_t)^{-1} = \\ &= H(l_s)^{-1}\circ F(s)\circ g_s\circ (G(r_s)\circ g_s)^{-1}\circ H(l_t)^{-1}\circ \\ &\quad \circ F(t)\circ g_t\circ (G(r_t)\circ g_t)^{-1} = T(s)\circ T(t). \end{aligned}$$

Thus $T(st) = T(s)\circ T(t)$ for $(s,t)\in D(K)$.

From (34) we have

$$H(t) = H(l_t)\circ T(t), \quad t\in K, \tag{40}$$

and from (40), (38)

$$G(t) = T(t)\circ G(r_t), \quad t\in K. \tag{41}$$

Setting in (39) and then in (35) $s = l_t$ we get

$$F(t) = F(l_t)\circ g_{l_t}\circ (G(r_{l_t})\circ g_{l_t})^{-1}\circ H(l_t)^{-1}\circ F(t), \quad t\in K \tag{42}$$

and

$$F(t) = F(l_t)\circ g_{l_t}\circ (G(r_{l_t})\circ g_{l_t})^{-1}\circ G(t), \quad t\in K,$$

respectively. Hence, using (41), the definition of the function T and (42), we can write

$$\begin{aligned} F(t) &= F(l_t) \circ g_{l_t} \circ (G(r_{l_t}) \circ g_{l_t})^{-1} \circ T(t) \circ G(r_t) = \\ &= F(l_t) \circ g_{l_t} \circ (G(r_{l_t}) \circ g_{l_t})^{-1} \circ H(l_t)^{-1} \circ F(t) \circ g_t \circ \\ &\quad \circ (G(r_t) \circ g_t)^{-1} \circ G(r_t) = \\ &= F(t) \circ g_t \circ (G(r_t) \circ g_t)^{-1} \circ G(r_t) = H(l_t) \circ T(t) \circ G(r_t). \end{aligned}$$

Thus, the following equality holds:

$$F(t) = H(l_t) \circ T(t) \circ G(r_t), \quad t \in K. \tag{43}$$

Define

$$M(t) := H(l_t),\ N(t) := G(r_t) \quad \text{for } t \in K. \tag{44}$$

It is clear that $M(t) \in \mathrm{In}(Y, Z)$ and $N(t) \in \mathrm{Sur}(X, Y)$ for $t \in K$.

Now we show that the functions M, N satisfy condition (H). Using (33) it is easy to check that

$$M(st) \circ T(st) = M(s) \circ T(st) \quad \text{for } (s,t) \in D(K).$$

Note that (24) and (26) yield

$$H(st) \circ G(r_{st}) = H(st) \circ G(r_t) \quad \text{for } (s,t) \in D(K),$$

hence by (40)

$$H(l_{st}) \circ T(st) \circ G(r_{st}) = H(l_{st}) \circ T(st) \circ G(r_t), \quad (s,t) \in D(K),$$

and consequently

$$T(st) \circ N(st) = T(st) \circ N(t), \quad (s,t) \in D(K).$$

Finally, formulas (22) result directly from (43), (41) and (40).

The proof of (ii) is easy.

Theorem 3. (i) *Let K be a groupoid such that the binary operation is associative and $K_l(t) \neq \emptyset$, $K_r(t) \neq \emptyset$ for $t \in K$. Assume that there exist functions $l : K \ni t \to l(t) \in K_l(t)$, $r : K \ni t \to r(t) \in K_r(t)$ such that*

$$l(st) = l(s),\ r(st) = r(t) \quad \text{for } (s,t) \in D(K) \tag{45}$$

and $\{F_t : t \in K\} \subset Z^X$, $\{G_t : t \in K\} \subset Y^X$, $\{H_t : t \in K\} \subset Z^Y$ satisfy (1). *If $H_{l(t)} \in \mathrm{In}(Y, Z)$, $G_{r(t)} \in \mathrm{Sur}(X, Y)$, $t \in K$ then there exist $\{M_t : t \in K\} \subset \mathrm{In}(Y, Z)$, $\{N_t : t \in K\} \subset \mathrm{Sur}(X, Y)$, $\{T_t : t \in K\} \subset Y^Y$ satisfying the condition*

$$M_{st} = M_s,\ N_{st} = N_t,\ T_{st} = T_s \circ T_t \quad \text{for } (s,t) \in D(K) \tag{G}$$

and

$$(46)\qquad \begin{cases} F_t &= M_t \circ T_t \circ N_t, \\ G_t &= T_t \circ N_t, \\ H_t &= M_t \circ T_t, \quad t \in K. \end{cases}$$

(ii) *Conversely, if* $\{M_t : t \in K\} \subset Z^Y$, $\{N_t : t \in K\} \subset Y^X$, $\{T_t : t \in K\} \subset Y^Y$ *satisfy (G) then the functions* F_t, G_t, H_t *given by* (46) *fulfil equation* (1).

Proof. According to the Lemma there exist families of functions $\{M_t : t \in K\} \subset \mathrm{In}(Y, Z)$, $\{N_t : t \in K\} \subset \mathrm{Sur}(X, Y)$, $\{T_t : t \in K\} \subset Y^Y$ satisfying condition (H) and such that formulas (46) hold. It is easy to see, using (45), that the families $\{M_t : t \in K\}$, $\{N_t : t \in K\}$ defined by (44) satisfy condition (G). So, the proof of (i) is finished.

The proof of (ii) is trivial.

The following example gives an application of Theorem 3.

Example 1. Let us consider the following functional equation

$$(47)\qquad F_{\min\{s,t\}} = H_s \circ G_t \quad \text{for } (s,t) \in D$$

where $D := \{(s,t) \in R^2 : t \le s \le c\}$, c is a fixed real number and $\{F_t : t \le c\} \subset Z^X$, $\{G_t : t \le c\} \subset Y^X$, $\{H_t : t \le c\} \subset Z^Y$ are unknown families of functions. Putting $l(s) := c$ and $r(s) := s$ for $s \in R$, $s \le c$, it is easy to check that (45) holds. Analysing the proof of the Lemma, it is easy to see that the associativity assumption in Theorem 3 can be omitted. Thus, assuming that $H_c \in \mathrm{In}(Y, Z)$, $G_s \in \mathrm{Sur}(X, Y)$ for $s \le c$, we may use Theorem 3 to get a solution of equation (47). Namely, according to (G), we have $M_t = M_{\min\{c,t\}} = M_c =: A$ for $t \le c$. So, every solution has the form

$$\begin{cases} F_t &= A \circ T_t \circ N_t \\ G_t &= T_t \circ N_t, \\ H_t &= A \circ T_t \quad t \le c, \end{cases}$$

for some $A \in Z^Y$, $\{N_t : t \le c\} \subset Y^X$ and $\{T_t : t \le c\} \subset Y^Y$ such that $T_{\min\{s,t\}} = T_s \circ T_t$ for $(s,t) \in D$.

The next Proposition gives a condition under which a groupoid K has the choice functions l, r satisfying condition (45). To precise the formulation of the Proposition let us denote by K° the set of all elements e from a groupoid K such that the following condition holds for all $t \in K$:

$$(48)\qquad \begin{cases} \text{if } et \text{ is defined then } et = t, \\ \text{if } te \text{ is defined then } te = t. \end{cases}$$

Define

$$K_l^\circ(t) := \{e \in K^\circ : et \text{ is defined}\},$$
$$K_r^\circ(t) := \{e \in K^\circ : te \text{ is defined}\}, \quad t \in K.$$

Proposition 2. *Let K be a groupoid such that the binary operation is associative. Suppose that the set $K_l^\circ(t)$ ($K_r^\circ(t)$) is nonempty for every $t \in K$. Then there exists a function $l : K \ni t \to l(t) \in K_l(t)$ ($r : K \to K_r(t)$) such that $l(st) = l(s)$ ($r(st) = r(t)$) for $(s,t) \in D(K)$.*

Proof. Let $l^\circ : K \ni t \to l^\circ(t) =: l_t^\circ \in K_l^\circ(t)$. From associativity we obtain

$$l_{st}^\circ(st) = (l_{st}^\circ s)t, \quad l_s^\circ(st) = (l_s^\circ s)t \quad \text{for } (s,t) \in D(K).$$

Consequently, the products $l_{st}^\circ s$, $l_s^\circ(st)$ are defined for $(s,t) \in D(K)$. Moreover, in virtue of associativity, we may write

$$l_{st}^\circ s = l_{st}^\circ(l_s^\circ s) = (l_{st}^\circ l_s^\circ)s, \quad l_{st}^\circ(st) = l_s^\circ(l_{st}^\circ st) = (l_s^\circ l_{st}^\circ)st$$

for $(s,t) \in D(K)$. Hence the products $l_{st}^\circ l_s^\circ$, $l_s^\circ l_{st}^\circ$ are defined and, on account of (48), we get $l_{st}^\circ = l_s^\circ$ for $(s,t) \in D(K)$. Putting $l(s) := l^\circ(s)$ for $s \in K$ we get the Proposition. In the case when $K_r^\circ(t)$ is a nonempty set one can proceed in an analogous way.

Example 2. Let $\{X_i : i \in W\}$ be a family of disjoint sets and let S_{ij} be the family of all mappings $f : X_i \to X_j$ for $i,j \in W$. It is easy to check that, for the groupoid $S := \bigcup_{i,j} S_{ij}$ (with composition of functions as a binary operation), the sets $K_l^\circ(f)$, $K_r^\circ(f)$ are nonempty for every $f \in S$. So, under suitable assumptions, we may use Proposition 2 to reduce the equation

$$F_{f \circ g} = H_f \circ G_g, \quad f,g \in S,$$

where $\{F_f : f \in S\} \subset Z^Y$, $\{G_f : f \in S\} \subset Y^X$, $\{H_f : f \in S\} \subset Z^Y$ are unknown families of functions, to the Cauchy equation.

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