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Extension theory and the Ψ^∞ operator

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Abstract. We are going to define for each simplicial complex K, an operator Ψ^∞ on the subcomplexes of K. If one is given a collection of spaces, closed subspaces of them, and maps of the closed subspaces to a subpolyhedron of $|K|$ that extend to maps into $|K|$, then we are going to use the Ψ^∞ operator to help determine a subcomplex of minimal cardinality into which the maps can be extended simultaneously.

The question (raised by A. Dranishnikov and J. Dydak) of whether the extension dimension, $\operatorname{extdim}_{(\mathcal{C},\mathcal{T})} X$, has a countable representative when X is compact and metrizable, $\mathcal{C}$ is the class of compact metrizable spaces, and $\mathcal{T}$ is the class of CW-complexes is an unsolved problem. We shall define an "anti-basis" for a CW-complex and use this along with the Ψ^∞ operator to allow one to view this problem from another perspective.

1. Introduction

Extension theory, which was first introduced by A. Dranishnikov in 1994, is based on the following notion. If K is a CW-complex and X is a space, then one says that K is an *absolute extensor* for X, $K \in \mathrm{AE}(X)$, or X is an *absolute co-extensor* for K, $X\tau K$, if for each closed subset A of X and map (i.e. continuous function) $f : A \to K$, there exists a map $F : X \to K$ such that F is an extension of f. For example, if X is a normal space and $K = I = [0, 1]$, then Tietze's extension theorem yields that $I \in \mathrm{AE}(X)$, or $X\tau I$.

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Key words and phrases: absolute co-extensor, absolute extensor, cardinality of complex, cohomological dimension, CW-complex, compactification, covering dimension, extension theory, finitely-bounded, Hausdorff σ-compactum, λ-bounded, σ-compactum, Stone-Čech compactification, universal compactum, weight.

Suppose that X is either a metrizable space or a compact Hausdorff space. A classical result from the theory of covering dimension is that $\dim X \leq n$ if and only if $X \tau S^n$. For cohomological dimension $\dim_G$ over an abelian group G, a similar fact is true: $\dim_G X \leq n$ if and only if $X \tau K$ where K is an Eilenberg–Mac Lane CW-complex in the class $K(G, n)$. For these and other reasons, A. Dranishnikov [2] defined the notions of extension theory and extension dimension. Given a class $\mathcal{C}$ of spaces and a class $\mathcal{T}$ of CW-complexes, one defines (see Section 5) an equivalence relation $\sim_{(\mathcal{C},\mathcal{T})}$ on the CW-complexes. For a given space X, not necessarily in $\mathcal{C}$, its *extension dimension*, $\operatorname{extdim}_{(\mathcal{C},\mathcal{T})} X$, may exist. The latter, when it exists, is a uniquely determined equivalence class under $\sim_{(\mathcal{C},\mathcal{T})}$. He and J. Dydak asked in [4] (Problem 5.4 below) whether with respect to the classes $\mathcal{C}$ of compact Hausdorff spaces and $\mathcal{T}$ of CW-complexes, the extension dimension of every metrizable compactum has a countable representative. We shall show, Proposition 5.5, that for certain universal compacta the answer is yes.

Whenever X is a Tychonoff space, then by βX we shall mean the Stone-Čech compactification of X. Let $X = \sum\{X_s \mid s \in S\}$ be a topological sum of compact Hausdorff spaces, K be a CW-complex, and assume that $X_s \tau K$ for each $s \in S$. Suppose that one is given a collection $\{A_s \mid s \in S\}$ of closed subsets A_s of X_s along with maps $f_s : A_s \to K$. Under what conditions can these maps f_s be extended to maps $F_s : X_s \to K$ so that for some finite subcomplex K_0 of K, $F_s(X_s) \subset K_0$ for all $s \in S$? Definitions needed to describe such a problem and others more general than it along with some rudimentary results, e.g., Corollary 2.9, about such extensions can be found in Section 2 below. A situation like this was encountered in [16] where the author (see Proposition 2.4 of that citation) determined a relationship between that kind of extension problem and whether $\beta X \tau K$.

To deal simultaneously with the problems outlined above, we shall introduce in Section 3, for each simplicial complex K, the operator Ψ^∞ on the subcomplexes of K. It will be true that Ψ^∞ is idempotent and that if X is a space, $X \tau |K|$, and L is a subcomplex of K, then $X \tau |\Psi^\infty(L)|$. Moreover, if L is of infinite cardinality, then the cardinality of $\Psi^\infty(L)$ equals the cardinality of L. We apply the Ψ^∞ operator in Section 4. Our main result in that section is Theorem 4.7 which covers as a special case metrizable σ-compacta. One might also view Corollary 4.5 to see one of the fundamental properties of the Ψ^∞ operator.

In Section 5 we introduce the concept of an *anti-basis* for a polyhedron $|K|$. Roughly speaking, it consists of a set of subcomplexes of K that detect when a space Y is not an absolute co-extensor for K. Theorem 5.10 states that for certain classes of spaces the existence of a countable anti-basis consisting of finite subcomplexes implies the existence of a countable representative of extension dimension as in the question of Dranishnikov and Dydak.

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2. λ-bounded collections of maps

If K is a simplicial complex, then $|K|$ will be endowed with the weak topology and we shall treat $|K|$ as a CW-complex whose CW-structure is determined in the usual way by the triangulation K. By $\operatorname{card} K$ we of course mean the cardinal number of the set of simplexes of K. If K is a CW-complex, then by $\operatorname{card} K$ we mean the cardinal number of the set of cells of K. If X is a space, then $\operatorname{wt} X$ will designate the weight of X. The next lemma will be used implicitly below.

Lemma 2.1. *Let $f : K \to L$ be a map of* CW*-complexes and K_0 a subcomplex of K:*

(1) *if K_0 is finite, then $f(K_0)$ is contained in a finite subcomplex of L;*

(2) *if K_0 is infinite, then $f(K_0)$ is contained in a subcomplex M of L with* $\operatorname{card} M \leq \operatorname{card} K_0$. □

Let us recall Definition 2.1 of [16]; we use a slightly different terminology in order to conform to the needs of this paper.

Definition 2.2. Let K be a CW-complex and $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ a collection of pairs of spaces. Suppose that for each $s \in S$, A_s is closed in X_s and a map $f_s : A_s \to K$ has been given. We shall say that $\{f_s \mid s \in S\}$ is $\mathcal{X}$ *finitely-bounded* in K if there exists a finite subcomplex K_0 of K such that each map f_s can be extended to a map of X_s into K_0. If $A_s = A$, $X_s = X$, and $f_s = f$ for all $s \in S$, then we shall refer to the map f as being (X, A) finitely-bounded in K with the obvious meaning.

Let us first note:

Lemma 2.3. *Let $f : K \to L$ be a map between* CW*-complexes K and L. Suppose that $\{X_s \mid s \in S\}$ is a set of spaces. Let $\{A_s \mid s \in S\}$ and $\{f_s \mid s \in S\}$ be collections such that for each $s \in S$, A_s is a closed subspace of X_s and $f_s : A_s \to K$ is a map. Put $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$.*

(1) *If $\{f_s \mid s \in S\}$ is $\mathcal{X}$ finitely-bounded in K, then $\{f \circ f_s \mid s \in S\}$ is $\mathcal{X}$ finitely-bounded in L.*

(2) *Suppose that $g : L \to K$ is a map, $g \circ f \simeq 1_K$, and for each $s \in S$, X_s has the homotopy extension property with respect to* CW*-complexes. Assume also that there is a finite subcomplex M of K such that $f_s(A_s) \subset M$ for all $s \in S$ and that $\{f \circ f_s \mid s \in S\}$ is $\mathcal{X}$ finitely-bounded in L. Then $\{f_s \mid s \in S\}$ is $\mathcal{X}$ finitely-bounded in K.*

Proof. (1) Let K_0 be a finite subcomplex of K, and $\{F_s \mid s \in S\}$ a collection such that for each $s \in S$, $F_s : X_s \to K_0$ is a map having the property that $F_s \mid A_s = f_s$. There exists a finite subcomplex L_0 of L (see Lemma 2.1(1)) such that $f(K_0) \subset L_0$. Then $\{f \circ F_s \mid s \in S\}$ witnesses the fact that $\{f \circ f_s \mid s \in S\}$ is $\mathcal{X}$ finitely-bounded in L.

(2) Let L_0 be a finite subcomplex of L and $\{G_s \mid s \in S\}$ a collection of maps $G_s : X_s \to L_0$ such that $G_s \mid A_s = f \circ f_s$ for all $s \in S$. Again applying Lemma 2.1(1), choose a finite subcomplex K^* of K such that $g(L_0) \subset K^*$. We may assume that $M \subset K^*$. Hence, $\{g \circ G_s \mid s \in S\}$ is a collection such that $g \circ G_s : X_s \to K^*$ is a map for each $s \in S$.

Let $F : K \times [0,1] \to K$ be a homotopy such that $F(x,0) = x$ and $F(x,1) = g \circ f(x)$ for all $x \in K$. Put $K' = F(K^* \times [0,1])$. Then K' is contained in a finite subcomplex K_0 of K. Moreover, $F(K^* \times \{0\}) = K^*$, so $K^* \subset K_0$. Putting $F^* = F \mid (K^* \times [0,1]) : K^* \times [0,1] \to K_0$ one gets a deformation F^* of K^* in K_0 having the property that $F^*(x,1) = g \circ f(x)$ for all $x \in K^*$.

Notice that if $s \in S$ and $a \in A_s$, then $f_s(a) \in M \subset K^*$. So there is a homotopy $Q_s : A_s \times I \to K_0$ given by $Q_s(a,t) = F^*(f_s(a),t)$. We see that $Q_s(a,0) = F^*(f_s(a),0) = f_s(a)$ and $Q_s(a,1) = F^*(f_s(a),1) = g \circ f \circ f_s(a)$. But $g \circ G_s \mid A_s = g \circ f \circ f_s$ and $g \circ G_s : X_s \to K^* \subset K_0$. The homotopy extension property shows that f_s extends to a map of X_s into K_0. Since K_0 is finite and is independent of the choice of $s \in S$, our proof of (2) is complete. □

We now extend Definition 2.2 to consider maps to CW-complexes whose images must land in subcomplexes of infinite cardinalities.

Definition 2.4. Let K be a CW-complex, λ be an infinite cardinal, and $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ a collection of pairs of spaces. Suppose that for each $s \in S$, A_s is closed in X_s and a map $f_s : A_s \to K$ has been given. We shall say that $\{f_s \mid s \in S\}$ is $\mathcal{X}$ λ-*bounded* in K if there exists a subcomplex K_0 of K such that $\operatorname{card} K_0 \le \lambda$ and each map f_s can be extended to a map of X_s into K_0. If $A_s = A$, $X_s = X$, and $f_s = f$ for all $s \in S$, then we shall refer to the map f as being (X,A) λ-bounded in K with the obvious meaning.

Lemma 2.5. *Let Y be a space, A a closed subspace of Y, and K a* CW-*complex. Let X be a space, $g : X \to Y$ a map, B a closed subspace of $g^{-1}(A)$, and $f : A \to K$ a map.*

(1) *If f is (Y, A) finitely-bounded, then $f \circ (g \mid B)$ is (X, B) finitely bounded.*

(2) *If λ is an infinite cardinal and f is (Y, A) λ-bounded in K, then $f \circ (g \mid B)$ is (X, B) λ-bounded in K.* $\square$

As pointed out in [16], nontrivial examples of CW-complexes K along with a non-finitely-bounded collection of maps in K can be extrapolated from the proof of Theorem 1.5 of [12]. In that proof, the author produces a countably infinite set $\mathcal{T}$ and a collection, $\{X^T \mid T \in \mathcal{T}\}$ of metrizable compacta. Each X^T has a specified closed subspace S_T homeomorphic to S^2. It is true that $\dim_G X^T \leq 2$ for every abelian group G. In the last paragraph of the proof, select K (designated P there) to be $K(G, 2)$ for any nontrivial abelian group G. Then for each $T \in \mathcal{T}$, let $f_T : S_T \to K$ be a map such that $f_T(S_T) = f_{T'}(S_{T'})$ and $(f_T)_*(H_2(S_T)) = (f_{T'})_*(H_2(S_{T'})) \neq 0$ for each T, $T' \in \mathcal{T}$. With this and an examination of the finale of the proof of Theorem 1.5 in [12], we have,

Proposition 2.6. *For every nontrivial abelian group G and $K = K(G, 2)$, there exist a countably infinite set S, collections $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$, and $\{f_s \mid s \in S\}$ where for each $s \in S$, X_s is a compact metrizable space with $X_s \tau K$, A_s is a closed subspace of X_s, and $f_s : A_s \to K$ is a map whose image lies in a fixed finite subcomplex of K, chosen in such a manner that, $\{f_s \mid s \in S\}$ is $\mathcal{X}$ $\aleph_0$-bounded in K but not $\mathcal{X}$ finitely-bounded in K.* $\square$

The next lemma can be proved using the same techniques found in our proof of Lemma 2.3.

Lemma 2.7. *Let $f : K \to L$ be a map between* CW-*complexes K and L. Suppose that $\{X_s \mid s \in S\}$ is a set of spaces. Let $\{A_s \mid s \in S\}$ and $\{f_s \mid s \in S\}$ be collections such that for each $s \in S$, A_s is a closed subspace of X_s and $f_s : A_s \to K$ is a map. Put $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$ and suppose that λ is an infinite cardinal.*

(1) *If $\{f_s \mid s \in S\}$ is $\mathcal{X}$ λ-bounded in K, then $\{f \circ f_s \mid s \in S\}$ is $\mathcal{X}$ λ-bounded in L.*

(2) *Suppose that $g : L \to K$ is a map, $g \circ f \simeq 1_K$, and for each $s \in S$, X_s has the homotopy extension property with respect to* CW-*complexes. Assume also that there is a subcomplex M of K with* $\operatorname{card} M \leq \lambda$ *such that $f_s(A_s) \subset M$ for all $s \in S$ and that $\{f \circ f_s \mid s \in S\}$ is $\mathcal{X}$ λ-bounded in L. Then $\{f_s \mid s \in S\}$ is $\mathcal{X}$ λ-bounded in K.*

A proof similar to that of Lemma 3.1 of [6], which applies to polyhedra, can be used to obtain the following stronger result applying to CW-complexes:

Lemma 2.8. *Let X be a space with* $\operatorname{wt} X \leq \lambda$ *for some infinite cardinal λ. Suppose $f : X \to K$ is a map where K is a* CW*-complex. Then $f(X) \subset L$ for some subcomplex L of K where* $\operatorname{card} L \leq \lambda$. □

Corollary 2.9. *Let K be a* CW*-complex and λ an infinite cardinal. Suppose that S is a set with* $\operatorname{card} S \leq \lambda$, *$\{X_s \mid s \in S\}$, $\{A_s \mid s \in S\}$ are collections of spaces with $X_s \tau K$,* $\operatorname{wt} X_s \leq \lambda$, *and A_s is a closed subset of X_s for each $s \in S$. Put $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$. Then every collection $\{f_s \mid s \in S\}$ of maps $f_s : A_s \to K$ is $\mathcal{X}$ λ-bounded in K.* □

This leads to a result showing that "gluing" together such a collection of spaces does not change the "λ-bounded in K" condition.

Corollary 2.10. *Let λ be an infinite cardinal, S a set, $\{X_s \mid s \in S\}$, $\{A_s \mid s \in S\}$ collections of spaces, and A_s a closed subset of X_s for each $s \in S$. Put $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$. Assume that there is a space A and for each $s \in S$, a homeomorphism $h_s : A_s \to A$. Let X be the quotient set formed from $\sum\{X_s \mid s \in S\}$ by gluing the sets X_s to A via the homeomorphisms h_s and let q be the quotient function. Let X be given a topology such that A is closed in X and $q \mid X_s : X_s \to X$ is a map for each $s \in S$. Then for each* CW*-complex K the following are true:*

(1) *If $f : A \to K$ is a map that is (X, A) λ-bounded, then $\{f \circ (q|A_s)|s \in S\}$ is $\mathcal{X}$ λ-bounded in K.*
(2) *If* $\operatorname{card} S \leq \lambda$ *and for each $s \in S$,* $\operatorname{wt} X_s \leq \lambda$ *and $X_s \tau K$, then every map $f : A \to K$ is (X, A) λ-bounded in K.* □

An example of a space X as in Corollary 2.10 could be obtained as follows. Suppose that $\{X_s \mid s \in S\}$ is a collection of Hausdorff spaces each containing a closed subspace A_s homeomorphic to say S^n. Then form X by gluing these spaces together along S^n and applying the weak topology to X.

3. Ψ Operators

For each simplicial complex K, denote by $\mathcal{F}_K$ the set of nonempty finite subcomplexes of K. Fix a simplicial complex K. Suppose that $M \in \mathcal{F}_K$; let $\mathcal{D}_{(M,K)}$ be the set of $D \in \mathcal{F}_K$ such that $M \subset D$. Define a relation $\sim_{(M,K)}$ on $\mathcal{D}_{(M,K)}$ by declaring that if D, $C \in \mathcal{D}_{(M,K)}$, then $D \sim_{(M,K)} C$ if there exists a simplicial isomorphism of D to C which is the identity on M. Plainly $\sim_{(M,K)}$ is an equivalence relation on $\mathcal{D}_{(M,K)}$, and we shall write the equivalence class of an element D of $\mathcal{D}_{(M,K)}$ as $[D]_{(M,K)}$. The equivalence class $[M]_{(M,K)}$ is just $\{M\}$.

Let $\mathcal{E}_{(M,K)}$ be the set of equivalence classes of $\mathcal{D}_{(M,K)}$ under the relation $\sim_{(M,K)}$ and $q_{(M,K)} : \mathcal{D}_{(M,K)} \to \mathcal{E}_{(M,K)}$ the quotient function. The set $\mathcal{E}_{(M,K)}$ is countable. Using the axiom of choice, fix once and for all a function $\theta_{(M,K)} : \mathcal{E}_{(M,K)} \to \mathcal{D}_{(M,K)}$ such that $\theta_{(M,K)}([D]_{(M,K)}) \in [D]_{(M,K)}$ for each $D \in \mathcal{D}_{(M,K)}$, i.e., $\theta_{(M,K)}(E) \in \overset{-1}{q}_{(M,K)}(E)$ for each $E \in \mathcal{E}_{(M,K)}$. We point out that $M \subset \theta_{(M,K)}(E)$. Assume that the preceding construction has been applied to each $M \in \mathcal{F}_K$.

For $M \in \mathcal{F}_K$ and $E \in \mathcal{E}_{(M,K)}$, $\theta_{(M,K)}(E)$ is a subcomplex of K. Thus,

(i) for all $M \in \mathcal{F}_K$, $\bigcup \theta_{(M,K)}(\mathcal{E}_{(M,K)})$ is a subcomplex of K containing the subcomplex M.

We now define the function Ψ from the set of subcomplexes L of K to the set of subcomplexes of K by,

$$\Psi(L) = \bigcup \Big\{ \bigcup \theta_{(M,K)}(\mathcal{E}_{(M,K)}) \bigm| M \in \mathcal{F}_L \Big\}.$$

An application of (i) shows that for each pair $L \subset L'$ of subcomplexes of K,

(ii) $\Psi(L)$ is a subcomplex of K, and $L \subset \Psi(L)$, and

(iii) $\Psi(L) \subset \Psi(L')$.

Let us denote $\Psi^0(L) = L$; inductively for each $k \in \mathbb{N}$ if $\Psi^{k-1}(L)$ has been defined, then by $\Psi^k(L)$ we mean $\Psi(\Psi^{k-1}(L))$. Put,

$$\Psi^\infty(L) = \bigcup \big\{ \Psi^k(L) \mid k \in \mathbb{N} \big\}.$$

Of course $\Psi^\infty(L)$ is a subcomplex of K. We shall show that Ψ^∞ is an idempotent operator on the set of subcomplexes of a given simplicial complex K.

Lemma 3.1. *Let K be a simplicial complex and L a subcomplex of K. Then $\Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$, and hence $\Psi^\infty(\Psi^\infty(L)) = \Psi^\infty(L)$.*

Proof. Suppose that M is a finite subcomplex of $\Psi^\infty(L)$. Then for some $k \in \mathbb{N}$, $M \subset \Psi^k(L)$, so by (iii), $\Psi(M) \subset \Psi(\Psi^k(L)) = \Psi^{k+1}(L) \subset \Psi^\infty(L)$. □

From the construction of Ψ, it is not difficult to see that,

(iv) in case L is infinite, then $\mathrm{card}(\Psi^1(L)) = \mathrm{card}(L)$.

We therefore may state the following lemma.

Lemma 3.2. *Let K be a simplicial complex and L a subcomplex of K. If*

(1) $L = \emptyset$, *then* $\Psi^1(L) = \emptyset$,

(2) L *is finite, then* $\operatorname{card}(\Psi^1(L)) \leq \aleph_0$,

(1) L *is infinite and* $k \in \mathbb{N}$, *then we may conclude that* $\operatorname{card}(\Psi^k(L)) = \operatorname{card}(\Psi^\infty(L)) = \operatorname{card}(L)$. □

We now provide an example to illustrate the operator Ψ^1. Let K be an infinite wedge of one-simplexes with vertex v and L the subcomplex consisting of the vertex v. The finite subcomplexes of K that contain L consist of finite wedges of 1-simplexes along with a finite set of vertices, including v, not in that wedge. With this in mind, one may choose $\theta_{(L,K)}$ in such a manner that $\Psi^1(L)$ is a countable wedge of 1-simplexes. Simply fix in advance a subcomplex M of K that is a countably infinite wedge of 1-simplexes. We may write $M = \bigcup\{M_n \mid n \in \mathbb{N}\}$ where for each $n \in \mathbb{N}$, M_n is a wedge of n 1-simplexes and $M_n \subset M_{n+1}$. Make all choices of values of $\theta_{(L,K)}$ to be subcomplexes of M and so that for each $n \in \mathbb{N}$, $\theta_{(L,K)}([M_n]) = M_n$. Then $\Psi^1(L) = M$.

On the other hand, we may also choose $\theta_{(L,K)}$ in a way that $\Psi^1(L)$ consists of a countable wedge of 1-simplexes along with a nonintersecting discrete, nonempty, finite or countably infinite set of vertices. Indeed, for any choice of $\theta_{(L,K)}$, $\Psi^1(L)$ will always consist of a countable wedge of 1-simplexes along with a nonintersecting countable set of vertices.

The situation with $\Psi^2(L)$ will again depend on $\theta_{(L,K)}$. If $\Psi^1(L) = M$ as above, then $\Psi^2(L)$ consists of M along with a countable (possibly empty) set of vertices outside M. If $\Psi^1(L)$ contains some discrete nonempty set of vertices, then $\Psi^2(L)$ could consist of $\Psi^1(L)$ along with some additional 1-simplexes and perhaps an additional countable discrete set of vertices.

When a homotopy $F : |M| \times I \to |M|$ is treated then we in addition define the Ψ_F^∞ operator, derive its properties (see Lemma 3.3), and use it to give a short proof of Proposition 3.4.

Let M be a simplicial complex and $F : |M| \times I \to |M|$ a homotopy having the property that if $x \in |M|$, $\sigma \in M$, and $x \in \operatorname{int}\sigma$, then $F(x, 0) \in \sigma$. For each finite subcomplex Q of M, note that $F(|Q| \times \{0\}) \subset |Q|$. Let S_Q be the smallest subcomplex of M such that $F(|Q \mid \times I) \subset |S_Q|$. Then Q is a subcomplex of S_Q and S_Q is finite. For any subcomplex L of M, put $\Psi_F(L) = \bigcup\{S_Q \mid Q \in \mathcal{F}_L\}$. Then $\Psi_F(L)$ is a subcomplex of M. Let $\Psi_F^0(L) = L$, and for each $k \in \mathbb{N}$, if $\Psi_F^{k-1}(L)$ has been defined, then we let $\Psi_F^k(L) = \Psi_F(\Psi_F^{k-1}(L))$. Finally, let $\Psi_F^\infty(L) = \bigcup\{\Psi_F^k(L) \mid k \in \mathbb{N}\}$. Then it is easy to check the next result.

Lemma 3.3. *Let* M *be a simplicial complex and* $F : |M| \times I \to |M|$ *a homotopy having the property that if* $x \in |M|$, $\sigma \in M$, *and* $x \in \operatorname{int}\sigma$, *then*

$F(x,0) \in \sigma$. *Then for each subcomplex* L *of* M, $\Psi^1_F(\Psi^\infty_F(L)) = \Psi^\infty_F(L)$, *and hence* Ψ^∞_F *is an idempotent operator on the set of subcomplexes of* M. *Moreover,*

(1) *if* $\operatorname{card} L$ *is finite, then* $\operatorname{card}(\Psi^\infty_F(L)) \leq \aleph_0$,

(2) *if* $\operatorname{card} L$ *is infinite, then* $\operatorname{card}(\Psi^\infty_F(L)) = \operatorname{card}(L)$, *and*

(3) $F(|\Psi^\infty_F(L)| \times I) \subset |\Psi^\infty_F(L)|$. □

In Proposition 3.4, for completeness we state (1) without proof since this is a standard fact in the theory of CW-complexes, and our current techniques are useful only for proving (2).

Proposition 3.4. *Let* K *be a* CW*-complex of cardinality* α.

(1) *If* α *is finite, then there exists a finite simplicial complex* T *and a homotopy equivalence between* K *and* $|T|$.

(2) *If* α *is infinite, then there exists a simplicial complex* T *of cardinality* $\leq \alpha$ *and a homotopy equivalence between* K *and* $|T|$.

Proof. As mentioned above, we only prove (2). There exists a simplicial complex M and a homotopy equivalence $h : K \to |M|$. Let $f : |M| \to K$ be a homotopy inverse of h and $F : |M| \times I \to |M|$ a homotopy from the identity of $|M|$ to the map $h \circ f$. Choose a subcomplex L of M with $\operatorname{card} L \leq \alpha$ such that $h(K) \subset |L|$. Let $T = \Psi^\infty_F(L) \subset M$ and $f^* = f \mid |T| : |T| \to K$. Apply Lemma 3.3(3) to see that $h \circ f^*$ is homotopic to the identity on $|T|$. It is routine to check that $f^* \circ h$ is homotopic to the identity on K. Apply Lemma 3.3(1,2) to see that $\operatorname{card} T \leq \alpha$. □

4. X-connectedness and λ-boundedness

By a pair (U,V) of spaces we mean a space U along with a subspace V of U. Next is Definition 6.1 of [16]. As mentioned there, this should be compared with similar ones given in [9], [10], and [11].

Definition 4.1. Let X be a space and (U,V) a pair of spaces. We shall say that (U,V) is X*-connected* if for each closed subset A of X and map $f : A \to V$, there exists a map $F : X \to U$ that extends f.

The term σ-compactum usually refers to a metrizable space that can be written as a countable union of compact subspaces of itself. Such a space is obviously normal and Hausdorff; moreover, every CW-complex is an absolute neighborhood extensor for it. Let us generalize that definition.

Definition 4.2. Let X be a space. Then we shall say that X is a *Hausdorff σ-compactum* if X is a normal Hausdorff space, every CW-complex is an absolute neighborhood extensor for X, and X can be written as a countable union of compact Hausdorff subspaces.

Proposition 4.3. *Let K be a simplicial complex and L a subcomplex of K. Suppose that $\{X_s \mid s \in S\}$ is a collection of Hausdorff σ-compacta and that for each $s \in S$, $X_s \tau |K|$. The following are true.*

(1) *The pair $(|\Psi^\infty(L)|, |\Psi^\infty(L)|)$ is X_s-connected for each $s \in S$.*

(2) *If $s \in S$ and X_s is compact Hausdorff, then $(|\Psi^{n+1}(L)|, |\Psi^n(L)|)$ is X_s-connected.*

(3) *If λ is an infinite cardinal, $\operatorname{card} L \le \lambda$, for each $s \in S$, A_s is a closed subset of X_s, $f_s : A_s \to |L|$ is a map, and $\mathcal{X} = \{(X_s, A_s) \mid s \in S\}$, then $\{f_s \mid s \in S\}$ is $\mathcal{X}$ λ-bounded in $|K|$.*

Proof. Statement (3) of this proposition will follow from Statement (1) along with an application of Lemma 3.2(2,3). We proceed with a proof of (1).

Consider $s \in S$ and a map $f_s : A_s \to |\Psi^\infty(L)|$. Write $X_s = \bigcup\{Z_i \mid \in \mathbb{N}\}$ with $Z_1 \subset Z_2 \subset \dots$, and for each $i \in \mathbb{N}$, Z_i is a compact Hausdorff space. We shall proceed with an induction argument.

Since $X_s \tau |K|$, then $Z_1 \tau |K|$. Let $g_1 : A_s \cup Z_1 \to |K|$ be a map such that $g_1 \mid A_s = f_s$. There exists $M \in \mathcal{F}_{\Psi^\infty(L)}$ such that $g_1(Z_1 \cap A_s) \subset |M|$. Now $g_1(Z_1) \subset |M'|$ for some finite subcomplex M' of K, where $M \subset M'$. By the definition of $\Psi^1(\Psi^\infty(L))$, we may as well assume that $M' \subset \Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$. Hence we may treat $g_1 : A_s \cup Z_1 \to |\Psi^\infty(L)|$.

Using the ANE property of $|\Psi^\infty(L)|$, there exists a closed neighborhood D_1 of $A_s \cup Z_1$ in X_s and a map $h_1 : D_1 \to |\Psi^\infty(L)|$ that extends g_1.

Suppose that $k \in \mathbb{N}$ and we have found $D_1 \subset \cdots \subset D_k$, and $h_1, \dots, h_k$ such that for $1 \le i \le k$:

(a) D_i is a closed neighborhood of $A_s \cup Z_i$ in X_s,

(b) h_i is a map of D_i to $|\Psi^\infty(L)|$,

(c) $h_i \mid A_s = f_s$, and

(d) if $1 \le i < j \le k$, then $D_i \subset D_j$ and $h_j \mid D_i = h_i$.

Choose a map $g_{k+1} : D_k \cup Z_{k+1} \to |K|$ such that $g_{k+1} \mid D_k = h_k$. Now $h_k(Z_k \cup (Z_{k+1} \cap D_k)) = g_{k+1}(Z_k \cup (Z_{k+1} \cap D_k)) \subset |\Psi^\infty(L)|$. There exists $N \in \mathcal{F}_{\Psi^\infty(L)}$ such that $g_{k+1}(Z_k \cup (Z_{k+1} \cap D_k)) \subset |N|$.

Now $g_{k+1}(Z_{k+1}) \subset |N'|$ for some finite subcomplex N' of K, where $N \subset N'$. By the definition of $\Psi^1(\Psi^\infty(L)) = \Psi^\infty(L)$, we may as well assume that $M' \subset \Psi^\infty(L)$.

There exists a closed neighborhood D_{k+1} of $D_k \cup Z_{k+1}$ in X_s and a map $h_{k+1} : D_{k+1} \to |\Psi^\infty(L)|$ that extends g_{k+1}.

This completes the induction. Observe that $\bigcup\{\operatorname{int}_{X_s} D_k \mid k \in \mathbb{N}\} = X_s$. Define a function $F_s : X_s \to |\Psi^\infty(L)|$ to be $\bigcup\{h_k \mid k \in \mathbb{N}\}$. Clearly F_s is a map, and $F_s \mid A_s = f_s$. Hence for all $s \in S$, there exists a map of X_s into $|\Psi^\infty(L)| \subset |K|$ that extends f_s. This completes our proof of (1).

In the special case that X_s is a compact Hausdorff space, start with a map $f_s : A_s \to |\Psi^n(L)|)$. Just apply the first step of the above inductive argument and see that $F_s(X_s) \subset |\Psi^1(\Psi^n(L))| = |\Psi^{n+1}(L)|$, so (2) is true. □

Corollary 4.4. *Let K be a simplicial complex, λ an infinite cardinal, and L a subcomplex of K with* $\operatorname{card} L \leq \lambda$. *Suppose that X is a Hausdorff σ-compactum such that $X\tau|K|$; then for each closed subspace A of X every map $f : A \to |L|$ is (X, A) λ-bounded in $|K|$.* □

Corollary 4.5. *Let K be a simplicial complex and X a Hausdorff σ-compactum with $X\tau|K|$. Then for every subcomplex L of K, $X\tau|\Psi^\infty(L)|$.*

We shall use Proposition 4.6 in our proof of Theorem 4.7. It appears as Proposition 3.1 of [8] where we explain that this result differs from E. Michael's Proposition 3.6(a) of [14], but is an improved version based on Lemma 1 of [15].

Proposition 4.6. *Let X be a paracompact space and $\mathcal{G}$ a collection of subsets of X. Suppose that the following are true:*

(1) *$\mathcal{G}$ contains an open cover of X,*

(2) *if $U \in \mathcal{G}$ and W is open in U, then $W \in \mathcal{G}$,*

(3) *if U, Q are open elements of $\mathcal{G}$, then $U \cup Q \in \mathcal{G}$, and*

(4) *if $\mathcal{K} \subset \mathcal{G}$ is a discrete collection of open subsets of X, then $\bigcup \mathcal{K} \in \mathcal{G}$.*

Then the entire space X is in $\mathcal{G}$. □

Theorem 4.7. *Let K be a simplicial complex, λ an infinite cardinal, and L a subcomplex of K with* $\operatorname{card} L \leq \lambda$. *Suppose that X is a paracompact space, $\{X_s \mid s \in S\}$ a locally finite cover of X consisting of closed subspaces that are Hausdorff σ-compacta, and $X_s\tau|K|$ for each $s \in S$. Then $(|\Psi^\infty(L)|, |\Psi^\infty(L)|)$ is X-connected, and hence every map $f : A \to |\Psi^\infty(L)|$ of a closed subset A of X is λ-bounded in $|K|$. If for each $s \in S$, X_s is compact and Hausdorff, then we may state additionally that for all $n \in \mathbb{N}$, $(|\Psi^{n+1}(L)|, |\Psi^n(L)|)$ is X-connected.*

Proof. Let $\mathcal{G}$ be the collection of open subsets G of X such that if A is a closed subset of $\operatorname{cl}_X G$, and $f : A \to |\Psi^\infty(L)|$ is a map, then f extends to a map of $\mathrm{cl}_X G$ to $|\Psi^\infty(L)|$. We will show that $\mathcal{G}$ satisfies conditions (1)–(4) of Proposition 4.6. Then we will be assured that $X \in \mathcal{G}$. The proof of the first part will be concluded by referring to Lemma 3.2(2,3).

Let $\mathcal{U}$ be an open cover of X with the property that if $U \in \mathcal{U}$, then $\mathrm{cl}_X(U)$ intersects X_s for only finitely many $s \in S$. Fix $U \in \mathcal{U}$, let A be a closed subset of $\mathrm{cl}_X U$, and $f : A \to |\Psi^\infty(L)|$ a map.

Let $T \subset S$ be the finite subset having the property that $X_s \cap \mathrm{cl}_X U \neq \emptyset$ if and only if $s \in T$. Define T_1 to be the subset of T such that if $s \in T$ and X_s is a compact Hausdorff space, then $s \in T_1$. Let $T_2 = T \setminus T_1$.

Put $Y = \bigcup\{X_s \mid s \in T_1\}$. Then Y is a compact Hausdorff space. By Proposition 4.3, there exists a map $h : Y \to |\Psi^1(\Psi^\infty(L)|) = |\Psi^\infty(L)|$ that extends $f|(A \cap Y) : A \cap Y \to |\Psi^\infty(L)|$. Let $h^* : A \cup Y \to |\Psi^\infty(L)|$ be the map such that $h^*|A = f$ and $h^*|Y = h$.

Let $s \in T_2$. By Proposition 4.3, there exists a map $f_s : X_s \to |\Psi^\infty(\Psi^\infty(L))| = |\Psi^\infty(L)|$ that extends $h^* \mid ((A \cup Y) \cap X_s) : ((A \cup Y) \cap X_s) \to |\Psi^\infty(L)|$. Put $f_1^* : A \cup Y \cup X_s \to |\Psi^\infty(L)|$ such that $f_1^* \mid (A \cup Y) = h^*$ and $f_1^* \mid X_s = f_s$. Using the fact from Lemma 3.1 that $\Psi^\infty(\Psi^\infty(L)) = \Psi^\infty(L)$, one may, step by step, add the remaining σ-compacta indexed by T_2 to end up with a map of $\mathrm{cl}_X U$ to $|\Psi^\infty(L)|$ that extends f. This shows that $\mathcal{G}$ contains an open cover of X. Part (2) of Proposition 4.6 is easily seen to be true.

Since for open elements U and Q of $\mathcal{G}$, $\mathrm{cl}_X(U \cup Q) = \mathrm{cl}_X U \cup \mathrm{cl}_X Q$, the reader can see how to prove (3) of Proposition 4.6 by using the same techniques we just employed above. That $\mathcal{G}$ satisfies Part (4) of Proposition 4.6 is obvious.

In case X_s is compact and Hausdorff for each $s \in S$, then one may simply change the definition of $\mathcal{G}$ to require that f extends to a map of $\mathrm{cl}_X G$ to $|\Psi^{n+1}(L)|$. □

5. Extension Dimension and Anti-Bases

We shall now recall the notion of extension dimension. Let $\mathcal{C}$ be a class of spaces, $\mathcal{T}$ a class of CW-complexes, and $K, K' \in \mathcal{T}$. If it is true that for all $X \in \mathcal{C}$, $X\tau K$ implies that $X\tau K'$, then we write $K \leq_{(\mathcal{C},\mathcal{T})} K'$. This defines a preorder on $\mathcal{T}$ (see [4] or [7]). One specifies $K \sim_{(\mathcal{C},\mathcal{T})} K'$ if and only if $K \leq_{(\mathcal{C},\mathcal{T})} K'$ and $K' \leq_{(\mathcal{C},\mathcal{T})} K$; then $\sim_{(\mathcal{C},\mathcal{T})}$ is an equivalence relation on $\mathcal{T}$. An equivalence class $[K]_{(\mathcal{C},\mathcal{T})}$ under this relation is called an *extension type* relative to $(\mathcal{C}, \mathcal{T})$. For any space X, we write $X\tau[K]_{(\mathcal{C},\mathcal{T})}$ to mean that $X\tau K'$ for all $K' \in [K]_{(\mathcal{C},\mathcal{T})}$.

The relation $\leq_{(\mathcal{C},\mathcal{T})}$ induces a partial order, also denoted $\leq_{(\mathcal{C},\mathcal{T})}$, on the extension types. We write that $[K]_{(\mathcal{C},\mathcal{T})} \leq_{(\mathcal{C},\mathcal{T})} [K']_{(\mathcal{C},\mathcal{T})}$ if it is true that $L \leq_{(\mathcal{C},\mathcal{T})} L'$ for all $L \in [K]_{(\mathcal{C},\mathcal{T})}$ and $L' \in [K']_{(\mathcal{C},\mathcal{T})}$. One may check that $[K]_{(\mathcal{C},\mathcal{T})} \leq_{(\mathcal{C},\mathcal{T})} [K']_{(\mathcal{C},\mathcal{T})}$ if and only if $L \leq_{(\mathcal{C},\mathcal{T})} L'$ for some $L \in [K]_{(\mathcal{C},\mathcal{T})}$ and some $L' \in [K']_{(\mathcal{C},\mathcal{T})}$.

For Lemma 5.1 to follow, one might take $\mathcal{C}$ to be any class of compact Hausdorff spaces or metrizable spaces.

Lemma 5.1. *Let $\mathcal{T}$ be a class of* CW*-complexes and $\mathcal{C}$ be a class spaces X having the homotopy extension property with respect to K for any element K of $\mathcal{T}$. Whenever K, L are homotopy equivalent elements of $\mathcal{T}$, then $[K]_{(\mathcal{C},\mathcal{T})} = [L]_{(\mathcal{C},\mathcal{T})}$.* □

Let X be a space. Consider $S = \{[K]_{(\mathcal{C},\mathcal{T})} \mid X\tau[K]_{(\mathcal{C},\mathcal{T})}\}$. If S has an initial element[1] with respect to the relation $\leq_{(\mathcal{C},\mathcal{T})}$, then that element is called the *extension dimension* of X relative to $(\mathcal{C},\mathcal{T})$, written $\operatorname{extdim}_{(\mathcal{C},\mathcal{T})} X$.

In the sequel we shall use,

$$\begin{aligned}
\mathcal{T}_{\mathrm{CW}} &= \text{the class of CW-complexes},\\
\mathcal{T}_{\mathrm{POL}} &= \text{the class of polyhedra},\\
\mathcal{K} &= \text{the class of compact Hausdorff spaces},\\
\mathcal{K}_{\mathrm{m}} &= \text{the class of compact metrizable spaces}.
\end{aligned}$$

Theorem 11 of [3] along with Lemma 1.1 of [6] can be used to obtain the next information.

Theorem 5.2. *For each L, $K \in \mathcal{T}_{\mathrm{CW}}$, it is true that $L \leq_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})} K$ if and only if $L \leq_{(\mathcal{K}_{\mathrm{m}},\mathcal{T}_{\mathrm{CW}})} K$. Hence, $[K]_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})} = [K]_{(\mathcal{K}_{\mathrm{m}},\mathcal{T}_{\mathrm{CW}})}$. Similarly, if $K \in \mathcal{T}_{\mathrm{POL}}$, then $[K]_{(\mathcal{K},\mathcal{T}_{\mathrm{POL}})} = [K]_{(\mathcal{K}_{\mathrm{m}},\mathcal{T}_{\mathrm{POL}})}$.* □

It is remarked in Theorem 5.5 of [6] (see also [5]) that for any compact Hausdorff space X, $\operatorname{extdim}_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})} X$ exists. This extension dimension has a special type of representative. Let us cite Theorem 13 of [3].

Theorem 5.3. *For each $X \in \mathcal{K}$, there exists $L = \bigvee\{L_a \mid a \in A\}$ where* card $A \leq 2^{\aleph_0}$*, for each $a \in A$, $L_a \in \mathcal{T}_{\mathrm{CW}}$, L_a is countable, and,*

$$\operatorname{extdim}_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})} X = [L]_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})}. \qquad \square$$

Now we state Problem 2.19.2 of [4], noting that it has also been posed as Problem 2 of [3] and Problem 2.1 of [1].

[1]By an *initial element* of S, we mean $s_0 \in S$ having the property that $s_0 \leq_{(\mathcal{C},\mathcal{T})} s$ for all $s \in S$. If such s_0 exists, it is unique.

Problem 5.4. Determine whether for each compact metrizable space X, there is a countable CW-complex M such that $\operatorname{extdim}_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})} X = [M]_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})}$.

The next fact is immediate from Corollary 1.3 of [7]

Proposition 5.5. *Let K be a countable* CW-*complex and α an infinite ordinal. Suppose that X is a compact Hausdorff space with* $\operatorname{wt} X \leq \alpha$ *having the property that $X\tau K$ and each compact Hausdorff space Y with $Y\tau K$ and* $\operatorname{wt} Y \leq \alpha$ *embeds in X. Then* $\operatorname{extdim}_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})} X = [K]_{(\mathcal{K},\mathcal{T}_{\mathrm{CW}})}$.

This provides many examples of compact Hausdorff spaces with "countable" extension dimension, since by Corollary 1.9 of [13], every finite CW-complex admits a universal Hausdorff compactum of a given weight.

Lemma 5.6. *Let K be a* CW-*complex and X a Hausdorff σ-compactum. Suppose that K is not an absolute extensor for X. Then there exists a compact subset A of X and a map $f : A \to K$ that does not extend to a map of X to K.*

Proof. There exists a closed subspace B of X and a map $g : B \to K$ that does not extend to a map of X to K. Write $X = \bigcup\{X_i \mid i \in \mathbb{N}\}$ where for each $i \in \mathbb{N}$, X_i is a compact Hausdorff space.

If $g \mid (B \cap X_1) : B \cap X_1 \to K$ does not extend to a map of $X_1 \to K$, then define $A = B \cap X_1$ and $f = g \mid A : A \to K$. Otherwise, choose a map $h_1 : B \cup X_1 \to K$ that extends g. We may as well assume that the domain of h_1 is a closed neighborhood N_1 of $B \cup X_1$. Suppose that $k \in \mathbb{N}$ and we have found closed subsets $N_1 \subset \cdots \subset N_k$, of X, and maps $h_i : N_i \to K$, $1 \leq i \leq k$, such that for $1 \leq i \leq j \leq k$,

(i) $h_j \mid N_i = h_i$,

(ii) $X_i \subset \operatorname{int}_X N_i$, and

(iii) $h_i \mid B = g$.

If $h_k \mid (N_k \cap X_{k+1})$ does not extend to a map of X_{k+1} to K, then choose $A = N_k \cap X_{k+1}$ and $f = h_k \mid A : A \to K$. If it does extend, there exists a closed neighborhood N_{k+1} of $N_k \cup X_{k+1}$ and a map $h_{k+1} : N_{k+1} \to K$ such that $h_{k+1} \mid N_k = h_k : N_k \to K$.

If this recursive process ends after finitely many steps, then our proof is complete. If it does not end, then put $G = \bigcup\{h_i \mid i \in \mathbb{N}\} : X \to K$. Then G is a map that extends g, and we have reached a contradiction. $\square$

We have the following statement in case K is a CW-complex, Y a space, and $K \notin \mathrm{AE}(Y)$.

Lemma 5.7. *Let K be a* CW*-complex, Y a space, A a closed subspace of Y, L a subcomplex of K, and $f : A \to L$ a map that does not extend to a map of Y to K. Then for any subcomplex M of K with $L \subset M$, the map $f : A \to M$ does not extend to a map of Y to M.* □

This motivates us to define the notion of an "anti-basis" and show how this is related to Problem 5.4.

Definition 5.8. Let $\mathcal{K}^*$ be a class of spaces, K be a simplicial complex, and $\mathcal{F}$ a collection of subcomplexes of K having the property that whenever $Y \in \mathcal{K}^*$ and $|K|$ is not an absolute extensor for Y, then there exist a closed subspace A of Y, $F \in \mathcal{F}$, and map $f : A \to |F|$ that does not extend to a map of Y into $|K|$. Then we shall call $\mathcal{F}$ an *anti-basis* for K relative to $\mathcal{K}^*$.

An application of Lemma 5.6 shows the following.

Example 5.9. Let $\mathcal{K}^*$ be a class of Hausdorff σ-compacta and K a simplicial complex. Then $\mathcal{F}_K$ is an anti-basis for K relative to $\mathcal{K}^*$.

Now we have the following theorem.

Theorem 5.10. *Let $\mathcal{K}^*$ be a class of Hausdorff σ-compacta, $X \in \mathcal{K}^*$, and K a simplicial complex. Suppose that $\mathrm{extdim}_{(\mathcal{K}^*,\mathcal{T}_{\mathrm{CW}})} X$ exists and equals $[|K|]_{(\mathcal{K}^*,\mathcal{T}_{\mathrm{CW}})}$. If K has a countable anti-basis $\mathcal{F}$ relative to $\mathcal{K}^*$ such that $\mathcal{F}$ consists of finite subcomplexes of K, then there is a countable representative of $\mathrm{extdim}_{(\mathcal{K}^*,\mathcal{T}_{\mathrm{CW}})} X$. Indeed, $M = \Psi^\infty(\bigcup \mathcal{F})$ is a countable subcomplex of K and $|M|$ represents $\mathrm{extdim}_{(\mathcal{K}^*,\mathcal{T}_{\mathrm{CW}})} X$.*

Proof. Put $L = \bigcup \mathcal{F}$. Then L is a countable subcomplex of K. Let $M = \Psi^\infty(L)$. By Lemma 3.2(3), M is a countable subcomplex of K. Moreover since $X\tau|K|$, by Corollary 4.5, $X\tau|M|$. We know that $|K| \le_{(\mathcal{K}^*,\mathcal{T}_{\mathrm{CW}})} |M|$. It remains to prove the opposite inequality.

Suppose that $Y \in \mathcal{K}^*$, $Y\tau|M|$, and $Y\tau|K|$ is false. By Definition 5.8, there is an element L of $\mathcal{F}$, a closed subspace A of Y, and a map $f : A \to |L|$ that does not extend to a map of Y to K. But $|L| \subset |M|$ and $Y\tau|M|$, so f extends to a map of Y to $|M|$. This contradicts Lemma 5.7. □

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