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Outer generalized inverses in rings and related idempotents

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Abstract. In this paper we investigate outer generalized inverses of elements in rings, and related idempotents. Among other things, if $a'aa' = a'$ and $b'bb' = b'$, we consider the relations $b'b = a'a + u$ and $bb' = aa' + v$ for a suitable choice of u and v.

1. Introduction and preliminaries

Let $\mathcal{R}$ be ring with the unit 1. We use $\mathcal{R}^{-1}$ and $\mathcal{R}^{\bullet}$, respectively, to denote the set of all invertible elements of $\mathcal{R}$ and the set of all idempotents of $\mathcal{R}$. An element $a \in \mathcal{R}$ is outer generalized invertible, if there exists some $a' \in \mathcal{R}$ satisfying $a' = a'aa'$. Such an a' is called the outer generalized inverse of a. In this case $a'a$ and $1 - aa'$ are idempotents corresponding to a and a'.

For example, the ordinary and generalized Drazin inverse, as well as the Moore–Penrose inverse in rings with involution, are special cases of outer generalized inverses.

Recently, Castro-Gonzalez and Vélez-Cerrada [3] considered generalized Drazin invertible elements $a, b \in \mathcal{R}$, such that the corresponding idempotents $a^{\pi} = 1 - aa^{D}$ and $b^{\pi} = 1 - bb^{D}$ satisfy $1 - (b^{\pi} - a^{\pi})^2 \in \mathcal{R}^{-1}$.

Generalized inverses in rings have been studied in [4], [5], [6] and [8]. Related results concerning the perturbation of the generalized inverse or related idempotents can be found in [1], [2], [3], [9], [10].

In this paper we extend some results from [3] to idempotents related to

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outer generalized inverses of a and b. Particularly, we investigate outer generalized inverses with prescribed idempotents, which are introduced in [4].

2. Idempotents in rings

In this section we prove some statements concerning idempotents in rings. Let $\mathcal{R}$ be a ring with the unit 1. First we investigate the relations between idempotents p and $p+u$ for $u \in \mathcal{R}$ and $1-u^2 \in \mathcal{R}^{-1}$.

Theorem 2.1. *Let $u \in \mathcal{R}$ be such that $1-u^2 \in \mathcal{R}^{-1}$, and let $p \in \mathcal{R}^\bullet$. Then the following conditions are equivalent:*

(i) $p+u \in \mathcal{R}^\bullet$;

(ii) $p+u = (1-u)^{-1}p(1+u) = (1+u)p(1-u)^{-1}$;

(iii) $1-p-u = (1+u)^{-1}(1-p)(1-u) = (1-u)(1-p)(1+u)^{-1}$;

Moreover, if previous conditions hold, and $r = (1+u)p + (1-u)(1-p)$, then $r \in \mathcal{R}^{-1}$, where

$$r^{-1} = p(1-u)^{-1} + (1-p)(1+u)^{-1},$$

and $p+u = rpr^{-1}$.

Proof. (i) $\Longleftrightarrow$ (ii): Since $p \in \mathcal{R}^\bullet$, we have the following:

$$p+u \in \mathcal{R}^\bullet \Leftrightarrow (p+u)^2 = p+u \Leftrightarrow p^2 + pu + up + u^2 = p+u$$
$$\Leftrightarrow p(1+u) = (1-u)(p+u) \Leftrightarrow (1-u)^{-1}p(1+u) = p+u.$$

In the same way the second equality can be proved.

(i) $\Longleftrightarrow$ (iii) Since $p \in \mathcal{R}^\bullet$, $1-p \in \mathcal{R}^\bullet$, we have

$$p+u \in \mathcal{R}^\bullet \iff (1-p)+(-u) \in \mathcal{R}^\bullet.$$

Now we use (i) $\Longleftrightarrow$ (ii) for $1-p$ and $-u$ instead of p and u, respectively, and the result follows.

Now, suppose that (i), (ii) and (iii) hold. Let $r = (1+u)p + (1-u)(1-p)$. If we take

$$r' = p(1-u)^{-1} + (1-p)(1+u)^{-1},$$

then we get

$$rr' = (1+u)p(1-u)^{-1} + (1-u)(1-p)(1+u)^{-1} = (p+u) + (1-p-u) = 1,$$

because of (i) and (ii). From the same reason we have

$$\begin{aligned} r'r &= p(1-u)^{-1}(1+u)p + (1-p)(1+u)^{-1}(1-u)(1-p) \\ &= (1+u)^{-1}(p+u)(p+u)(1-u) + (1-u)^{-1}(1-p-u)(1-p-u)(1+u) \\ &= p + (1-p) = 1. \end{aligned}$$

Consequently $r^{-1} = r'$. Moreover, we have $rpr^{-1} = (1+u)p(1-u)^{-1} + 0 = p+u$, because of (i) and (ii). □

We state two auxiliary results, and prove the second one.

Lemma 2.2. *If $p, p+u \in \mathcal{R}^\bullet$ hold, then $pup = -pu^2 = -u^2p$. If $m, m-u \in \mathcal{R}^\bullet$ is satisfied, then $mum = mu^2 = u^2m$.*

Theorem 2.3. *Let $u \in \mathcal{R}$ be such that $1-u^2 \in \mathcal{R}^{-1}$, and let $p, m, p+u \in \mathcal{R}^\bullet$. Then the following conditions are equivalent:*

(i) $m = p + u$;

(ii) $p(1+u)(1-m) = (1-p)(1-u)m$;

(iii) $m(1-u)(1-p) = (1-m)(1+u)p$.

Proof. (i) $\Longrightarrow$ (ii) and (iii): Suppose that (i) holds. Using the results from Theorem 2.1 we have $p(1+u)(1-m) = (1-u)(p+u)(1-(p+u)) = 0$ and $(1-p)(1-u)m = (1+u)(1-(p+u))(p+u) = 0$. Similarly, it is straightforward to prove that (iii) holds.

(ii) $\Longrightarrow$ (i): Now, let $p(1+u)(1-m) = (1-p)(1-u)m$. Multiplying this equality from the right side with m and $(1-m)$ respectively, we get

$$(1-p)(1-u)m = 0 \text{ and } p(1+u)(1-m) = 0.$$

Using Theorem 2.1 again, it follows that

$$m = (p+u)m \text{ and } p+u = (p+u)m$$

so (i) follows.

In the same manner (iii) $\Longrightarrow$ (i) can be proven, taking $p, m-u$ and $-u$ instead of $m, m+u$ and u, respectively. □

3. Outer generalized inverses and idempotents

In this section we prove some results concerning the outer generalized inverses with prescribed idempotents.

Definition 3.1. Let $a \in \mathcal{R}$ and $p, q \in \mathcal{R}^{\bullet}$. An element $a' \in \mathcal{R}$ satisfying

$$a'aa' = a', \quad a'a = p, \quad 1 - aa' = q$$

is called a (p,q)-outer generalized inverse of a, denoted by $a' = a^{(2)}_{p,q}$. It is proved in [4] that if $a^{(2)}_{p,q}$ exists, then it is unique. The set of all (p,q)-outer invertible elements of $\mathcal{R}$ is denoted by $\mathcal{R}^{(2)}_{p,q}$.

Now, as our main result, we characterize elements a and b such that $b'b = a'a + u$ and $bb' = aa' + v$ such that $1 - u^2 \in \mathcal{R}^{-1}$ and $1 - v^2 \in \mathcal{R}^{-1}$.

Theorem 3.2. *Let $a, b, u, v \in \mathcal{R}$ such that a and b are outer invertible and $1 - u^2, 1 - v^2 \in \mathcal{R}^{-1}$. Then the following conditions are equivalent*

(i) $b'b = a'a + u$ *and* $bb' = aa' + v$;

(ii) $ub' + a'v = b' - a' - a'(a - b)b'$ *and* $au + vb = bb'b - aa'a - a(a' - b')b$;

(iii) $ua' + b'v = b' - a' - b'(a - b)a'$ *and* $bu + va = bb'b - aa'a - b(a' - b')a$.

Proof. From the fact that a and b are outer invertible it follows that there exist $p, q, m, n \in \mathcal{R}^{\bullet}$ and there exist $a', b' \in R$ such that $a' = a^{(2)}_{p,q}$ and $b' = b^{(2)}_{m,n}$. That is $a'a = p$, $1 - aa' = q$, $b'b = m$ and $1 - bb' = n$.

(i) $\Longrightarrow$ (ii): Using direct computations from $u = b'b - a'a$ and $v = bb' - aa'$ we have that (ii) is satisfied.

(ii) $\Longrightarrow$ (i): Suppose that (ii) holds. If we multiply the first equality with $1 - a'a$ from the left side we get

$$(1 - a'a)(1 - u)b' = 0, \tag{3.1}$$

and then multiplying the last equality by b from the right side we get

$$(1 - a'a)(1 - u)b'b = 0$$

that is

$$(1 - p)(1 - u)m = 0 \tag{3.2}$$

holds.

In the same manner, if we multiply the first equality in (ii) with $1 - bb'$ from the right side we get

$$a'(1 + v)(1 - bb') = 0, \tag{3.3}$$

and than multiplying the last equality with a from the left side we have

$$aa'(1+v)(1-bb')=0,$$

that is

$$(1-q)(1+v)n=0 \tag{3.4}$$

holds.

Similarly, if we multiply second equality in (ii) with $1-b'b$ from the right side we get

$$au(1-b'b)+vb(1-b'b)=-aa'(a+b)(1-b'b),$$

which is the same as

$$au(1-b'b)+aa'a(1-b'b)=-v(1-bb')b-aa'(1-bb')b.$$

Multiplying the last equality with a' from the left side we get

$$a'a(1+u)(1-b'b)=-a'(1+v)(1-bb')b. \tag{3.5}$$

The right-hand side of (3.5) is equal to zero because of (3.3). So, we have $a'a(1+u)(1-b'b)=0$ or

$$p(1+u)(1-m)=0. \tag{3.6}$$

Now, using Theorem 2.3 together with (3.2) and (3.6) we have the result that $b'b=a'a+u$.

Now, multiplying the second equality of (ii) with $1-aa'$ from the left side, we get

$$(1-aa')(au+vb)=(1-aa')(bb'b+ab'b),$$

which is the same as

$$a(1-a'a)(u-b'b)=(1-aa')(bb'-v)b.$$

Multiplying the last equality with b' from the right side we get

$$(1-aa')(1-v)bb'=-a(1-a'a)(1-u)b'. \tag{3.7}$$

Because of (3.1), it follows that $(1-aa')(1-v)bb'=0$, or

$$q(1-v)(1-n)=0. \tag{3.8}$$

Finally, using again Theorem 2.3 together with (3.4) and (3.8) it follows that $bb'=aa'+v$.

The proof of (i)$\Leftrightarrow$ (iii) is just the same as (i)$\Leftrightarrow$ (ii), replacing the role of a and b and taking $-u$ and $-v$ instead of u and v, respectively. Or, in other words we prove the result taking (i) to be $a'a=b'b-u$ and $aa'=bb'-v$. □

Theorem 3.2 gives a characterization of the elements in a ring which have related idempotents differing by a suitable choice of u and v. If a is generalized Drazin invertible element in R and if a^π is the spectral idempotent of a then $a^D = a^{(2)}_{p,1-p}$ for $p = 1 - a^\pi$. See [6] and [7] for the definition of quasinilpotent elements and the generalized Drazin inverse in rings.

Now, as a corollary we obtain one partial result from the main Theorem 3.2 from [3].

Corollary 3.3. *Let a and b are generalized Drazin invertible elements in $\mathcal{R}$ and $s \in \mathcal{R}$ such that $1 - s^2 \in R^{-1}$. If $a^\pi + s \in R^\bullet$ then the following conditions are equivalent:*

(i) $b^\pi = a^\pi + s$;

(ii) $(1+s)b^D - a^D(1-s) = a^D(a-b)b^D$.

(iii) $b^D(1+s) - (1-s)a^D = b^D(a-b)a^D$

Proof. Let $a' = a^D$ and $b' = b^D$. Using Theorem 3.2 with $u = v = -s$ the result follows. □

Also, as a corollary we obtain the first result from Theorem 4.2 in [4].

Corollary 3.4. *Let $a, b \in R$ and let $p, q \in R^\bullet$ be such that $a^{(2)}_{p,q}$ and $b^{(2)}_{p,q}$ exist. Then the following hold*

$$a^{(2)}_{p,q} - b^{(2)}_{p,q} = b^{(2)}_{p,q}(b-a)a^{(2)}_{p,q} = a^{(2)}_{p,q}(b-a)b^{(2)}_{p,q}.$$

Proof. With $a' = a^{(2)}_{p,q}$ and $b' = b^{(2)}_{p,q}$ and $u = v = 0$ from (ii) and (iii) in Theorem 3.2 the result follows. □

4. Perturbation of outer generalized invertible elements in Banach algebras

In this section we assume that $\mathcal{R}$ is a complex Banach algebra with the unit 1. Results from Theorem 3.2 are also valid in complex Banach algebras. Now we state the following upper bound for $\|b' - a'\|/\|a'\|$.

Theorem 4.1. *Let $a, b, u, v \in \mathcal{R}$, $p, q, m, n \in \mathcal{R}^\bullet$, $a' = a^{(2)}_{p,q}$ and $b' = b^{(2)}_{m,n}$. Let $b'b = a'a + u$ and $bb' = aa' + v$.*

If $\|u\| + \|a'(a-b)\| < 1$ and $\|v\| < 1$, then

$$\frac{\|b' - a'\|}{\|a'\|} \le \frac{\|a'(a-b)\| + \|u\| + \|v\|}{1 - \|u\| - \|a'(a-b)\|}.$$

PROOF. From $\|u\|, \|v\| < 1$ it follows that $1-u^2, 1-v^2 \in R^{-1}$. Then using the first equation from (ii) in Theorem 3.2 we have $b'-a' = ub'+a'v+a'(a-b)b' = (a'(a-b)+u)(b'-a') + a'(a-b)a' + ua' + a'v$. Applying the norm here we get

$$\|b'-a'\| \leq (\|a'(a-b)\| + \|u\|)\|b'-a'\| + (\|a'(a-b)\| + \|u\| + \|v\|)\|a'\|$$

and the result follows. □

As a corollary we obtain Theorem 5.3 in [3].

Corollary 4.2. *Let a and b are generalized Drazin invertible elements in R. If $\|b^\pi - a^\pi\| + \|a^D(b-a)\| < 1$, then*

$$\|\frac{\|b^D - a^D\|}{\|a^D\|} \leq \frac{\|a^D(b-a)\| + 2\|b^\pi - a^\pi\|}{1 - \|b^\pi - a^\pi\| - \|a^D(b-a)\|}.$$

Again, as a corollary we obtain the second result in Theorem 4.2 in [4].

Corollary 4.3. *Let a and b are elements in a Banach algebra R, and $p, q \in R^\bullet$ be such that $a^{(2)}_{p,q}$ and $b^{(2)}_{p,q}$ exist. Then if $\|a^{(2)}_{p,q}\| \, \|b-a\| < 1$ then*

$$\frac{\|b^{(2)}_{p,q} - a^{(2)}_{p,q}\|}{\|a^{(2)}_{p,q}\|} \leq \frac{\|a^{(2)}_{p,q}(b-a)\|}{1 - \|a^{(2)}_{p,q}(b-a)\|}.$$

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