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Approximation ratio of the digits in Oppenheim series expansion

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Abstract. This paper is concerned with the Hausdorff dimensions of some sets determined by the approximation ratio of the digits in Oppenheim series expansion. We give a general characterization on the Hausdorff dimensions of such sets. As it's corollaries, we answer the questions posed by Galambos.

1. Introduction

Oppenheim series expansion is a generalized tool to the representation of real numbers by infinite series, including LÜROTH [12], ENGEL, SYLVESTER expansions [3] and Cantor product [14] as special cases, which is given by the following algorithm. For any $x \in (0,1]$, set

$$x = x_1,\ d_n = [1/x_n] + 1, \quad x_n = 1/d_n + \gamma_n \cdot x_{n+1}, \tag{1}$$

where $\gamma_n = \gamma_n(d_1, \dots, d_n)$ is some positive rational valued function and $[y]$ denotes the integer part of y. Then it leads to an infinite series expansion for each $x \in (0,1]$ with the form

$$x \sim \frac{1}{d_1} + \gamma_1 \frac{1}{d_2} + \dots + \gamma_1 \gamma_2 \dots \gamma_n \frac{1}{d_{n+1}} + \dots, \tag{2}$$

which is called the Oppenheim expansion of x, see [13]. The expansion (2) is termed as the restricted Oppenheim expansion of x if γ_n depends on the last denominator d_n only and the function

$$h_n(d) = \gamma_n(d) \cdot d(d-1) \tag{3}$$

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is integer-valued for all $d \geq 2$ and $n \geq 1$. In this restricted case, a sufficient and necessary condition for the series on the right hand side in (2) to be the expansion of its sum by the algorithm (1) is, see [13],

$$d_{n+1} \geq \gamma_n d_n(d_n - 1) + 1 \quad \text{for all } n \geq 1. \tag{4}$$

In the present paper, we deal with the restricted Oppenheim expansion only.

The representation (2) under (1) was first studied by Oppenheim [13], where he established the arithmetical properties, including the question of rationality of this expansion. The foundations of the metric theory were laid down by Galambos [6], [7], [8], [10], [11], see also the monographs of Galambos [9], Schweiger [15], Vervaat [16], Dajani and Kraaikamp [2]. From [9], Chapter 6, it can be seen that the integer approximations $T_n(x)$ to the ratios $d_n(x)/h_{n-1}(d_{n-1}(x))$ given by

$$T_n(x) < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq T_n(x) + 1, \quad n \geq 1, \tag{5}$$

where $h_0 \equiv 1$, plays an important role in the metric theory of Oppenheim expansion. They are stochastically independent and are distributed as the denominators in the Lüroth expansion. Galambos ([9], Chapter 6) showed that

$$\frac{1}{n \log n}(T_1(x) + \cdots + T_n(x)) \to 1, \quad \text{as } n \to \infty \tag{6}$$

in probability but it has Lebesgue measure zero where the above convergence actually occurs. Let

$$\mathbb{B}_m = \{x \in (0,1] : 1 \leq T_n(x) \leq m \text{ for all } n \geq 1\}, \qquad m \geq 2,$$
$$\mathbb{D} = \Big\{x \in (0,1] : \frac{1}{n \log n}(T_1(x) + \cdots + T_n(x)) \to 1, \text{ as } n \to \infty\Big\},$$

and for any $k > 0$, let

$$\mathbb{D}_k = \{x \in (0,1] : T_1(x) + \cdots + T_n(x) \leq kn \log n, \text{ for sufficiently large } n\}.$$

Galambos, see [9], page 132–133, posed the questions to calculate the Hausdorff dimension of the sets $\mathbb{B}_m$, $\mathbb{D}$ and $\mathbb{D}_k$ above.

In this paper, we give a general characterization on the Hausdorff dimension of the set determined by the approximation ratio $\frac{d_{n+1}(x)}{h_n(d_n(x))}$. More precisely, denote

$$C = \Big\{x \in (0,1] : L_n < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \leq M_n \text{ for all } n \geq 1\Big\},$$

where $\{L_n : n \geq 1\}$ and $\{M_n : n \geq 1\}$ are two given integer-valued sequences, then what is the Hausdorff dimension of C? We get the point under some natural restrictions on L_n and M_n. By (5), it is easy to check that

$$C = \{x \in (0,1] : L_n \leq T_n(x) \leq M_n - 1 \text{ for all } n \geq 1\}.$$

So, as corollaries, we answer the questions posed by Galambos.

Remark 1.1. Since $T_j(x) \geq 1$ for all $j \geq 1$, we have $T_1(x)+\cdots+T_n(x) \geq n$ for all $n \geq 1$. Thus we should modify GALAMBOS' question to calculate the Hausdorff dimension of E_k (see [9], page 133) to calculate the Hausdorff dimension of $\mathbb{D}_k$, where E_k is defined as the set of points with the inequality in $\mathbb{D}_k$ holding for all $n \geq 1$.

The Hausdorff dimension of $\mathbb{B}_m$ has been considered in [17]. Some other exceptional sets associated with the Oppenheim series expansion were discussed in [18], [20], [21].

We will use $|\cdot|$ to denote the diameter of a set, $\dim_{\mathrm{H}}$ to denote the Hausdorff dimension and 'cl' the closure of a subset of $(0,1]$ respectively.

2. Main results

In this section, we collect some elementary properties on Oppenheim series expansion and state our main results.

Definition 2.1. Let $d_1, d_2, \ldots, d_n$ be an admissible sequence, i.e., $d_1 \geq 2$ and $d_{j+1} \geq h_j(d_j) + 1$ for all $1 \leq j < n$. We call the set

$$I(d_1, \ldots, d_n) := \{x \in (0,1] : d_1(x) = d_1, \ldots, d_n(x) = d_n\}$$

an n-th order admissible interval.

Proposition 2.2 ([9]). *Let $d_1, d_2, \ldots, d_n$ be an admissible sequence. Then the n-th order admissible interval $I(d_1, \ldots, d_n)$ is the interval with two endpoints*

$$\frac{1}{d_1} + \gamma_1(d_1)\frac{1}{d_2} + \cdots + \gamma_1(d_1)\ldots\gamma_{n-1}(d_{n-1})\frac{1}{d_n},$$

and

$$\frac{1}{d_1} + \gamma_1(d_1)\frac{1}{d_2} + \cdots + \gamma_1(d_1)\ldots\gamma_{n-1}(d_{n-1})\frac{1}{d_n} + \gamma_1(d_1)\ldots\gamma_{n-1}(d_{n-1})\frac{1}{d_n(d_n-1)}.$$

Thus

$$|I(d_1,\dots,d_n)| = \gamma_1(d_1)\dots\gamma_{n-1}(d_{n-1})\frac{1}{d_n(d_n-1)} = \prod_{j=0}^{n-1}\frac{h_j(d_j)}{d_{j+1}(d_{j+1}-1)}.$$

Definition 2.3. We call $\{h_n, n\ge 1\}$ is of order t, if there exist two constants $0<c_1\le c_2$ such that

$$c_1 d^t \le h_n(d) \le c_2 d^t$$

for all $d\ge 2$ and $n\ge 1$.

Let $\{L_n, n\ge 1\}$ and $\{M_n, n\ge 1\}$ be two positive integer sequences which are non-decreasing and satisfy

$$L_n < M_n, \quad M_n \ge 3, \quad \text{and} \quad \sup_{n\ge 1}\frac{M_n}{L_n} := \alpha < \infty. \tag{7}$$

Recall that

$$\begin{aligned} C &= \Big\{x\in(0,1] : L_n < \frac{d_n(x)}{h_{n-1}(d_{n-1}(x))} \le M_n \text{ for all } n\ge 1\Big\} \\ &= \{x\in(0,1] : L_n \le T_n(x) \le M_n - 1 \text{ for all } n\ge 1\}. \end{aligned}$$

Theorem 2.4. *Assume $h_n(d)\ge d-1$ for all $d\ge 2$ and $n\ge 1$.*

(1) *When $\{M_n, n\ge 1\}$ is bounded, we have* $\dim_{\mathrm{H}} C = 1$.

(2) *When $M_n\to\infty$ as $n\to\infty$, we have*

  (i) *If* $\lim_{n\to\infty}\frac{\log M_{n+1}}{\log M_n} = 1$, *then* $\dim_{\mathrm{H}} C = 1$.

  (ii) *If* $\lim_{n\to\infty}\frac{\log M_{n+1}}{\log M_n} = b > 1$, *$\{h_n, n\ge 1\}$ is of order t and* $\lim_{n\to\infty}\frac{\log(M_n-L_n)}{\log M_n} = \beta$, *then* $\dim_{\mathrm{H}} C = \frac{\beta(b-t)+t}{(2b-\beta b+\beta)(b-t)+t}$ *if $b>t$ and* $\dim_{\mathrm{H}} C = 1$ *if $b\le t$.*

Remark 2.5. It would become clear from the details of the proof that the assumptions of monotonicity and $\sup_{n\ge 1}\frac{M_n}{L_n}<\infty$ in (7) is actually not necessary. But, from the dimensional number of C at the case $b>t$, the extra assumption $\lim_{n\to\infty}\frac{\log(M_n-L_n)}{\log M_n} = \beta$ is not superfluous.

Corollary 2.6 ([17])**.** *If $h_n(d)\ge d-1$ for all $d\ge 2$ and $n\ge 1$, then for any $m\ge 2$,*

$$\dim_{\mathrm{H}}\{x\in(0,1] : 1\le T_n(x)\le m \text{ for all } n\ge 1\} = 1.$$

Proof. This is a direct consequence of Theorem 2.4. □

Corollary 2.7. *Suppose $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. Then we have*

$$\dim_{\mathrm{H}}\{x \in (0,1] : \lim_{n\to\infty} \frac{T_1(x)+\cdots+T_n(x)}{n\log n} = 1\} = 1.$$

Proof. Choose

$$L_n = [\log(n+6)], \quad M_n = \left[\log(n+6) + (\log(n+6))^{1-\frac{1}{\sqrt{\log\log(n+6)}}}\right] + 2$$

for all $n \geq 1$. Applying Theorem 2.4, we get the desired result immediately. □

Corollary 2.8. *Suppose $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. Then for any $k > 0$,*

$$\dim_{\mathrm{H}}\{x \in (0,1] : T_1(x)+\cdots+T_n(x) \leq kn\log n, \text{ for sufficiently large } n\} = 1.$$

Proof. Choose

$$L_n = \left[\frac{k}{2}\log n\right], \quad M_n = \left[\frac{k}{2}\left(\log n + (\log n)^{1-\frac{1}{\sqrt{\log\log n}}}\right)\right]$$

when n is large enough. Then the desired result is an easy consequence of Theorem 2.4. □

At the end of this section, we state the Billingsley theorem (see [1], [4], [5], [19]), which will be used to obtain the lower bound of the Hausdorff dimension of a fractal set.

Lemma 2.9. *Let $E \subset (0,1]$ be a Borel set and μ be a measure with $\mu(E) > 0$. If for any $x \in E$,*

$$\liminf_{r\to 0} \frac{\log\mu(B(x,r))}{\log r} \geq s,$$

where $B(x,r)$ denotes the open ball with center at x and radius r, then $\dim_{\mathrm{H}} E \geq s$.

3. Proof of Theorem 2.4

We fix some notation at first.

For any admissible sequence $d_1, d_2, \ldots, d_n$, let

$$J(d_1,\ldots,d_n) = \bigcup_{L_{n+1}h_n(d_n) < d_{n+1} \leq M_{n+1}h_n(d_n)} \mathrm{cl}\, I(d_1,\ldots,d_n,d_{n+1}),$$

and call $J(d_1, \ldots, d_n)$ an n-th order basic interval. By Proposition 2.2, we have

$$|J(d_1, \ldots, d_n)| = \left(\frac{1}{L_{n+1}} - \frac{1}{M_{n+1}}\right) |I(d_1, \ldots, d_n)|. \tag{8}$$

Fix $k \geq 1$ and $\bar{d}_1, \ldots, \bar{d}_k$ an admissible sequence. For any $n \geq k$, let

$$\begin{aligned} D_n(\bar{d}_1, \ldots, \bar{d}_k) = \Big\{ & (d_1, \ldots, d_n) \in \mathbb{N}^n : d_j = \bar{d}_j, 1 \leq j \leq k, \\ & L_{j+1} < \frac{d_{j+1}}{h_j(d_j)} \leq M_{j+1}, k \leq j < n \Big\}. \end{aligned}$$

In the following, we shall give a bound estimation of the gap $G^l(d_1, \ldots, d_n)$ which is the gap between $J(d_1, \ldots, d_n)$ and the closest n-th order basic interval which lies on the left hand side of $J(d_1, \ldots, d_n)$, and give a bound estimation of the gap $G^r(d_1, \ldots, d_n)$ which is the gap between $J(d_1, \ldots, d_n)$ and the closest n-th order basic interval, (if it exsits, otherwise we set $G^r(d_1, \ldots, d_n) = \infty$), which lies on the right hand side of $J(d_1, \ldots, d_n)$. For $G^l(d_1, \ldots, d_n)$, it is clear that $G^l(d_1, \ldots, d_n)$ is not less than the distance between the left endpoint of $J(d_1, \ldots, d_n)$ and the left endpoint of $I(d_1, \ldots, d_n)$. Thus by Proposition 2.2,

$$G^l(d_1, \ldots, d_n) \geq \frac{1}{M_{n+1}} |I(d_1, \ldots, d_n)|.$$

For $G^r(d_1, \ldots, d_n)$, let $J(\tilde{d}_1, \ldots, \tilde{d}_n)$ be the n-th order basic interval which lies on the right hand side of $J(d_1, \ldots, d_n)$ and closest to it. Let $j_0 = \min\{j : d_j \neq \tilde{d}_j\}$. Then $d_j = \tilde{d}_j$ for $1 \leq j < j_0$ and $d_{j_0} > \tilde{d}_{j_0}$. Moreover, it is clear that $G^r(d_1, \ldots, d_n)$ is not less than the distance between the left endpoint of $J(\tilde{d}_1, \ldots, \tilde{d}_{j_0})$ and the left endpoint of $I(\tilde{d}_1, \ldots, \tilde{d}_{j_0})$. Thus, by Proposition 2.2, we have

$$\begin{aligned} G^r(d_1, \ldots, d_n) &\geq \frac{1}{M_{j_0+1}} I(\tilde{d}_1, \ldots, \tilde{d}_{j_0}) \geq \frac{1}{M_{j_0+1}} I(d_1, \ldots, d_{j_0}) \\ &\geq \frac{1}{M_{j_0+1}} I(d_1, \ldots, d_n) \geq \frac{1}{M_{n+1}} I(d_1, \ldots, d_n). \end{aligned} \tag{9}$$

Write

$$G(d_1, \ldots, d_n) = \frac{1}{M_{n+1}} I(d_1, \ldots, d_n). \tag{10}$$

Now we are in the position to show Theorem 2.4. We divide the proof into three propositions.

Proposition 3.1. *Suppose $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. If $\{M_n, n \geq 1\}$ is bounded, we have* $\dim_{\mathrm{H}} C = 1$.

Proposition 3.2. *Suppose $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. If $M_n \to \infty$, as $n \to \infty$, and $\lim_{n\to\infty} \frac{\log M_{n+1}}{\log M_n} = 1$, we have* $\dim_{\mathrm{H}} C = 1$.

Proposition 3.3. *Assume $h_n(d) \geq d-1$ for all $d \geq 2$ and $n \geq 1$. Suppose $M_n \to \infty$, as $n \to \infty$, $\lim_{n\to\infty} \frac{\log M_{n+1}}{\log M_n} = b > 1$, $\{h_n, n \geq 1\}$ is of order t and $\lim_{n\to\infty} \frac{\log(M_n - L_n)}{\log M_n} = \beta$, then $\dim_{\mathrm{H}} C = \frac{\beta(b-t)+t}{(2b-\beta b+\beta)(b-t)+t}$ if $b > t$ and $\dim_{\mathrm{H}} C = 1$ if $b \leq t$.*

The proof of Proposition 3.1 is the same as that in Proposition 3.2, except some minor modifications. Also it can be done with the ideas given in [17]. So, we show Proposition 3.2 and 3.3 in details only.

In the sequel, the following Stolz's formula is used several times, so we state it as a lemma here.

Lemma 3.4. *Let $\{a_n, b_n, n \geq 1\}$ be two real sequences. If a_n tends to infinity increasingly as $n \to \infty$ and there exists $-\infty \leq \alpha \leq \infty$ such that*

$$\lim_{n\to\infty} \frac{b_n - b_{n-1}}{a_n - a_{n-1}} = \alpha,$$

Then $\lim_{n\to\infty} \frac{b_n}{a_n} = \alpha$.

Proof of Proposition 3.2. By (7), the assumptions on M_n and Stolz's formula, we have

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n-1} \log\left[\left(\frac{1}{2\alpha}\right)^j \prod_{i=1}^{j} M_i\right]}{\sum_{j=1}^{n+1} 2\log M_j + \log M_{n+1}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n} \log M_i - n\log 2\alpha}{3\log M_{n+2} - \log M_{n+1}} = \infty. \tag{11}$$

For any $0 < \epsilon < 1$, let $\epsilon' = \frac{\epsilon}{1-\epsilon}$. By (11), there exists $k_1 \in \mathbb{N}$ such that for any $n \geq k_1$, we have

$$\prod_{j=1}^{n-1} \left(\frac{1}{(2\alpha)^j} \prod_{i=1}^{j} M_i\right)^{\epsilon'} \geq \prod_{j=1}^{n+1} M_j^2 \cdot M_{n+1}. \tag{12}$$

Set

$$E(M_1) = \left\{x \in (0,1] : d_1(x) = M_1,\ L_{n+1} < \frac{d_{n+1}(x)}{h_n(d_n(x))} \leq M_{n+1} \text{ for all } n \geq 1\right\}.$$

It is easy to see $E(M_1) \subset C$. From the definition of $D_n(\bar{d_1}, \dots, \bar{d_k})$ and $J(d_1, \dots, d_n)$, we have

$$E(M_1) = \bigcap_{n=1}^{\infty} \bigcup_{(d_1,\dots,d_n)\in D_n(M_1)} J(d_1, \dots, d_n).$$

Set $\mu(J(M_1)) = 1$. For any $n \geq 2$ and $J(d_1, \dots, d_n) \in D_n(M_1)$, set

$$\mu(J(d_1, \dots, d_n)) = \prod_{j=1}^{n-1} \left(\frac{1}{M_{j+1} - L_{j+1}} \cdot \frac{1}{h_j(d_j)} \right). \tag{13}$$

Then μ is a probability mass distribution supported on $E(M_1)$, because

$$\begin{aligned} &\sum_{L_{n+1}h_n(d_n)<d_{n+1}\leq M_{n+1}h_n(d_n)} \mu(J(d_1, \dots, d_n, d_{n+1})) \\ &= \mu(J(d_1, \dots, d_n)) \times \sum_{L_{n+1}h_n(d_n)<d_{n+1}\leq M_{n+1}h_n(d_n)} \frac{1}{M_{n+1} - L_{n+1}} \cdot \frac{1}{h_n(d_n)} \\ &= \mu(J(d_1, \dots, d_n)). \end{aligned}$$

In order to apply Lemma 2.9 to give a lower bound estimation on $\dim_H E(M_1)$, we will estimate the measure of arbitrary balls.

We claim first that, for any $(d_1, \dots, d_n) \in D_n(M_1)$,

$$|J(d_1, \dots, d_n)| \geq M_{n+1} \big(\mu(J(d_1, \dots, d_n))\big)^{1+\epsilon'}, \tag{14}$$

which, in fact, is the essential point in getting the desired result. Note that for any $(d_1, \dots, d_n) \in D_n(M_1)$,

$$h_n(d_n) \geq \frac{1}{2} d_n \geq \frac{1}{2} L_n h_{n-1}(d_{n-1}) \geq \cdots \geq \frac{1}{2^n} \prod_{j=1}^{n} L_j \geq \frac{1}{(2\alpha)^n} \prod_{j=1}^{n} M_j. \tag{15}$$

Combine (12) and (15), we have,

$$\Big| J(d_1, \dots, d_n) \Big| \geq \frac{1}{M_{n+1}^2} \prod_{j=1}^{n} \frac{1}{M_j^2} \prod_{j=1}^{n-1} \frac{1}{h_j(d_j)} \geq M_{n+1} \left(\prod_{j=1}^{n-1} \frac{1}{h_j(d_j)} \right)^{1+\epsilon'}. \tag{16}$$

So, we get the claims.

Let $r_0 = \min\{G(d_1, \dots, d_{k_1}), (d_1, \dots, d_{k_1}) \in D_{k_1}(M_1)\}$. Now we estimate $\mu(B(x, r))$ for any $x \in E(M_1)$ and $0 < r < r_0$. For any $x \in E(M_1)$, there exists a sequence $d_1, d_2, \ldots$ such that $(d_1, \ldots, d_n) \in D_n(M_1)$ and $x \in J(d_1, \ldots, d_n)$ for all $n \geq 1$. Choose $n \geq k_1$ such that

$$G(d_1, \ldots, d_{n+1}) \leq r < G(d_1, \ldots, d_n).$$

By the definition of $G(d_1, \ldots, d_n)$, we know that $B(x, r)$ can intersect only one n-th order basic interval which is $J(d_1, \ldots, d_n)$. For the number of $(n+1)$-th basic intervals that $B(x, r)$ can intersect, we distinguish two cases.

Case I. $G(d_1, \ldots, d_{n+1}) \leq r < |I(d_1, \ldots, d_{n+1})|$.
In this case, $B(x, r)$ can intersect at most six $(n+1)$-th order admissible intervals $I(d_1, \ldots, d_n, d_{n+1} + i)$, $-1 \leq i \leq 4$. This is because

$$r \leq \min \Big\{ \big|I(d_1, \ldots, d_{n+1} - 1)\big|,\ \sum_{i=1}^{4} \big|I(d_1, \ldots, d_{n+1} + i)\big| \Big\}.$$

for $d_{n+1} \geq h_n(d_n) + 1 \geq d_n \geq M_1 \geq 3$. Thus, by (14), we get

$$\begin{aligned} \mu(B(x, r)) &\leq 6\mu(J(d_1, \ldots, d_{n+1})) \leq 6 \left(\frac{1}{M_{n+2}} |J(d_1, \ldots, d_{n+1})| \right)^{1-\epsilon} \\ &\leq 6 \left(\frac{1}{M_{n+2}} |I(d_1, \ldots, d_{n+1})| \right)^{1-\epsilon} = 6|G(d_1, \ldots, d_{n+1})|^{1-\epsilon} \leq 6r^{1-\epsilon}. \quad (17) \end{aligned}$$

Case II. $|I(d_1, \ldots, d_{n+1})| \leq r < G(d_1, \ldots, d_n)$.
By Proposition 2.2, we have for any $(d_1, \ldots, d_n, d'_{n+1}) \in D_{n+1}(M_1)$,

$$\begin{aligned} |I(d_1, \ldots, d_n, d'_{n+1})| &= |I(d_1, \ldots, d_n)| \cdot \frac{h_n(d_n)}{d'_{n+1}(d'_{n+1} - 1)} \\ &\geq |I(d_1, \ldots, d_n)| \cdot \frac{1}{M_{n+1}^2 h_n(d_n)}, \end{aligned}$$

thus $B(x, r)$ can intersect at most

$$\ell := 4rM_{n+1}^2 h_n(d_n) \cdot |I(d_1, \ldots, d_n)|^{-1}$$

$(n+1)$-th order basic intervals. Therefore

$$\mu(B(x, r)) \leq \min \Big\{ \mu(J(d_1, \ldots, d_n)),\ \sum_i \mu(J(d_1, \ldots, d_n, i)) \Big\},$$

where the sum is over all i such that $\max\{d_{n+1} - \ell,\ h_n(d_n) + 1\} \leq i \leq d_{n+1} + \ell$. By (13) and (14), we have

$$\mu(B(x, r)) \leq \mu(J(d_1, \ldots, d_n))$$

$$\cdot \min\left\{1,\ 8rM_{n+1}^2 h_n(d_n)|I(d_1,\ldots,d_n)|^{-1}\frac{1}{M_{n+1}-L_{n+1}}\frac{1}{h_n(d_n)}\right\}$$
$$\le 8\left(\frac{1}{M_{n+1}}|J(d_1,\ldots,d_n)|\right)^{1-\epsilon}$$
$$\cdot 1^{\epsilon}(rM_{n+1}^2)^{1-\epsilon}|I(d_1,\ldots,d_n)|^{-(1-\epsilon)}\left(\frac{1}{M_{n+1}-L_{n+1}}\right)^{1-\epsilon}$$
$$= 8\left(\frac{1}{L_{n+1}}\right)^{1-\epsilon} r^{1-\epsilon} \le 8r^{1-\epsilon}. \tag{18}$$

By (17), (18) and Lemma 2.9, we have

$$\dim_{\mathrm{H}} E(M_1) \ge 1-\epsilon.$$

Since $\epsilon > 0$ is arbitrary and $E(M_1) \subset C$, we have

$$\dim_{\mathrm{H}} C = 1. \qquad \square$$

Proof of Proposition 3.3. For any $n \ge 1$ and $\bar{d_1} \ge 2$, let

$$H_n(\bar{d_1}) = \left[\left(\frac{c_1}{\alpha}\right)^{1+t+\cdots+t^{n-1}} M_n^t M_{n-1}^{t^2}\ldots M_2^{t^{n-1}}\bar{d_1}^{\,t^n}\right],$$
$$G_n(\bar{d_1}) = \left[c_2^{1+t\cdots+t^{n-1}} M_n^t M_{n-1}^{t^2}\ldots M_2^{t^{n-1}}\bar{d_1}^{\,t^n}\right]+1.$$

We divide the proof into two parts.

Part I. $b > t$. Write $s_0 = \frac{\beta(b-t)+t}{(2b-\beta b+\beta)(b-t)+t}$. For this case, we know

$$\lim_{n\to\infty}\frac{\sum_{j=2}^n \log(M_j-L_j)+\sum_{j=2}^n \log G_{j-1}(\bar{d_1})}{2\sum_{j=2}^n \log M_{j+1}+\sum_{j=2}^n \log G_{j-1}(\bar{d_1})-\log(M_{n+1}-L_{n+1})} = s_0, \tag{19}$$

which gives, for any $s > s_0$ and n large enough,

$$\prod_{j=2}^n \Big((M_j-L_j)G_{j-1}(\bar{d_1})\Big)^{1-s}\alpha^{2ns}\prod_{j=2}^n\Big(\frac{M_{j+1}-L_{j+1}}{M_{j+1}^2}\Big)^s \le 1. \tag{20}$$

For any $\bar{d_1} \ge 2$, let

$$E(\bar{d_1}) = \Big\{x\in(0,1] : d_1(x) = \bar{d_1},\ L_{n+1} < \frac{d_{n+1}(x)}{h_n(d_n(x))} \le M_{n+1},\ \text{for all } n \ge 1\Big\}.$$

Then

$$C = \bigcup_{\bar{d_1}=2}^{\infty} E(\bar{d_1}).$$

For any $\bar{d_1} \geq 2$, $x \in E(\bar{d_1})$ and $n \geq 1$, since $\{h_n, n \geq 1\}$ is of order t, then

$$h_n(d_n(x)) \leq c_2 d_n^t(x) \leq c_2 M_n^t h_{n-1}^t(d_{n-1}(x)).$$

By iteration, we have for any $x \in E(\bar{d_1})$ and $n \geq 1$,

$$H_n(\bar{d_1}) < h_n(d_n(x)) \leq G_n(\bar{d_1}).$$

Note that

$$E(\bar{d_1}) = \bigcap_{n=1}^{\infty} \bigcup_{(d_1,\ldots,d_n)\in D_n(\bar{d_1})} J(d_1, \ldots, d_n),$$

which follows that

$$\begin{aligned}
\mathbf{H}^s(E(\bar{d_1})) &\leq \liminf_{n\to\infty} \sum_{(d_1,\ldots,d_{n-1},d_n)\in D_n(\bar{d_1})} \left|J(d_1, \ldots, d_{n-1}, d_n)\right|^s \\
&= \liminf_{n\to\infty} \sum_{(d_1,\ldots,d_{n-1})\in D_{n-1}(\bar{d_1})} \left|J(d_1, \ldots, d_{n-1})\right|^s \\
&\quad \cdot \sum_{L_n < \frac{d_n}{h_{n-1}(d_{n-1})} \leq M_n} \left(\frac{M_{n+1} - L_{n+1}}{M_{n+1}L_{n+1}} \frac{M_n L_n}{M_n - L_n} \frac{h_{n-1}(d_{n-1})}{d_n(d_n-1)} \right)^s \\
&\leq \liminf_{n\to\infty} \sum_{(d_1,\ldots,d_{n-1})\in D_{n-1}(\bar{d_1})} \left|J(d_1, \ldots, d_{n-1})\right|^s \\
&\quad \cdot \big((M_n - L_n)h_{n-1}(d_{n-1})\big)^{1-s} \alpha^{2s} \left(\frac{M_{n+1} - L_{n+1}}{M_{n+1}^2} \right)^s \leq \cdots \\
&\leq \liminf_{n\to\infty} \prod_{j=2}^{n} \big((M_j - L_j)h_{j-1}(d_{j-1})\big)^{1-s} \alpha^{2ns} \prod_{j=2}^{n} \left(\frac{M_{j+1} - L_{j+1}}{M_{j+1}^2} \right)^s \\
&\leq \liminf_{n\to\infty} \prod_{j=2}^{n} \big((M_j - L_j)G_{j-1}(\bar{d_1})\big)^{1-s} \alpha^{2ns} \prod_{j=2}^{n} \left(\frac{M_{j+1} - L_{j+1}}{M_{j+1}^2} \right)^s \\
&\leq 1. \qquad\qquad \text{(by (20))}
\end{aligned}$$

Therefore, $\dim_H E(\bar{d_1}) \leq s$. By the arbitrariness of $s > s_0$, we have

$$\dim_{\mathrm{H}} C \leq \sup_{\bar{d_1}\geq 2} \dim_{\mathrm{H}} E(\bar{d_1}) \leq \frac{\beta(b-t)+t}{(2b-\beta b+\beta)(b-t)+t}.$$

Now we prove the inverse inequality. Fix $\bar{d_1} \geq 2$. Let $\{t_n, n \geq 1\}$ be an integer sequence with $3 \leq t_n \leq (M_{n+1} - L_{n+1})G_n(\bar{d_1})$ for all $n \geq 1$. It should be noticed first that

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} \log(M_{j+1} - L_{j+1}) + \sum_{j=1}^{n} \log H_j(\bar{d_1})}{2\sum_{j=1}^{n+1} \log M_j + \log M_{n+2} + \sum_{j=1}^{n} \log G_j(\bar{d_1})} = \frac{\beta(b-t)+t}{b(b-t)+b} \geq s_0.$$

Thus, for any $s' < s_0$, there exists $k_3 \in \mathbb{N}$ such that for any $n \geq k_3$ and $(d_1, \dots, d_n) \in D_n(\bar{d_1})$, we have

$$\prod_{j=1}^{n} \frac{1}{M_{j+1} - L_{j+1}} \frac{1}{h_j(d_j)} \leq \left(\frac{1}{M_{n+2}} \prod_{j=0}^{n} \frac{h_j(d_j)}{d_{j+1}(d_{j+1}-1)} \right)^{s'}. \tag{21}$$

For any $n \geq 1$, denote

$$F_{\bar{d_1}}(t_n) = \frac{\sum_{j=1}^{n} \log(M_{j+1} - L_{j+1}) + \sum_{j=1}^{n} \log H_j(\bar{d_1}) - \log t_n}{2\sum_{j=1}^{n+1} \log M_j + \sum_{j=1}^{n} \log G_j(\bar{d_1}) - \log t_n + \log 12}.$$

Since $F_{\bar{d_1}}(\cdot)$ is monotonic decreasing with respect to t_n, we have

$$\liminf_{n\to\infty} F_{\bar{d_1}}(t_n) \geq \liminf_{n\to\infty} F_{\bar{d_1}}((M_{n+1} - L_{n+1})G_n(\bar{d_1})) = s_0.$$

Thus, for $s' < s_0$, there exists $k_4 \in \mathbb{N}$ such that for any $n \geq k_4$, $3 \leq t_n \leq (M_{n+1} - L_{n+1})G_n(\bar{d_1})$ and $(d_1, \dots, d_n) \in D_n(\bar{d_1})$, we have

$$t_n \prod_{j=1}^{n} \frac{1}{M_{j+1} - L_{j+1}} \frac{1}{h_j(d_j)} \leq \left(\frac{t_n}{12} \prod_{j=1}^{n} \frac{1}{M_{j+1}^2 h_j(d_j)} \right)^{s'}. \tag{22}$$

We can do as the same way as in the proof of Proposition 3.2 to define a probability measure μ supported on $E(\bar{d_1})$, i.e., $\mu(J(\bar{d_1})) = 1$, and

$$\mu(J(d_1, \dots, d_n)) = \prod_{j=1}^{n-1} \left(\frac{1}{M_{j+1} - L_{j+1}} \cdot \frac{1}{h_j(d_j)} \right),$$

for any $n \geq 2$ and $(d_1, \dots, d_n) \in D_n(\bar{d_1})$.

In fact, by (19), s_0 is just the Billingsley dimension of $E(\bar{d_1})$ with respect to the measure μ defined above. In the following, what we will done is just to check that $\dim_H E(\bar{d_1})$ coincides with its Billingsley dimension.(see also [19] for cases when Hausdorff dimension coincides with its Billingsley dimension.)

Let $k_2 = \max\{k_3, k_4\}$. Then for any $n \geq k_2$ and $(d_1, \dots, d_{n+1}) \in D_{n+1}(\bar{d_1})$, by (21), we have

$$\mu(J(d_1, \dots, d_{n+1})) \leq \left(\frac{1}{M_{n+2}} \prod_{j=0}^{n} \frac{h_j(d_j)}{d_{j+1}(d_{j+1}-1)} \right)^{s'}. \tag{23}$$

Now we estimate $\mu(B(x,r))$ for any $x \in E(\bar{d_1})$ and $r > 0$ small enough. For any $x \in E(\bar{d_1})$, there exists a sequence $d_1, d_2, \dots$ such that $(d_1, \dots, d_n) \in D_n(\bar{d_1})$ and $x \in J(d_1, d_2, \dots, d_n)$ for all $n \geq 1$. For any $0 < r < \min\{G(d_1, \dots, d_{k_2}),$ $(d_1, \dots, d_{k_2}) \in D_{k_2}(\bar{d_1})\}$, choose $n \geq k_2$ such that

$$G(d_1, \dots, d_{n+1}) \leq r < G(d_1, \dots, d_n).$$

Case I. $G(d_1, \dots, d_{n+1}) \leq r < |I(d_1, \dots, d_{n+1})|$.
In this case, by (23), we have

$$\mu(B(x,r)) \leq 6\mu(J(d_1, \dots, d_{n+1})) \leq 6(G(d_1, \dots, d_{n+1}))^{s'} \leq 6r^{s'}. \tag{24}$$

Case II. $|I(d_1, \dots, d_{n+1})| \leq r < G(d_1, \dots, d_n)$.
Denote by $t_n(r)$ the number of $(n+1)$-th order admissible intervals that the ball $B(x,r)$ can intersect. Then evidently that $1 \leq t_n(r) \leq (M_{n+1} - L_{n+1})h_n(d_n)$. If $t_n(r) \leq 5$, then by (24),

$$\mu(B(x,r)) \leq 5\mu(J(d_1, \dots, d_{n+1})) \leq 5r^{s'}. \tag{25}$$

If $t_n(r) \geq 6$, then $B(x,r)$ contains at least $[\frac{t_n(r)}{3}]$ many $(n+1)$-th order admissible intervals, thus

$$r \geq \frac{t_n(r)}{12} \prod_{j=0}^{n-1} \frac{h_j(d_j)}{d_{j+1}(d_{j+1}-1)} \frac{1}{M_{n+1}^2 h_n(d_n)} \geq \frac{t_n(r)}{12} \prod_{j=1}^{n} \frac{1}{M_{j+1}^2 h_j(d_j)}.$$

By (22), we have

$$\mu(B(x,r)) \leq t_n(r) \prod_{j=1}^{n} \frac{1}{M_{j+1} - L_{j+1}} \frac{1}{h_j(d_j)} \leq r^{s'}. \tag{26}$$

Combine (24), (25) (26) and Lemma 2.9, we have $\dim_{\mathrm{H}} E(\bar{d_1}) \geq s'$. Since $E(\bar{d_1}) \subset C$ and $s' < s_0$ is arbitrary, we have $\dim_{\mathrm{H}} C \geq s_0$.

Part II. $b \le t$. Choose $\bar{d_1} \ge 2$ such that $\log \bar{d_1}^{t-1} > \log \frac{\alpha}{c_1}$, and let

$$E(\bar{d_1}) = \Big\{ x \in (0,1] : d_1(x) = \bar{d_1}, L_{n+1} < \frac{d_{n+1}(x)}{h_n(d_n(x))} \le M_{n+1} \text{ for all } n \ge 1 \Big\}.$$

By the choice of $\bar{d_1}$, we have

$$\liminf_{n\to\infty} \frac{\sum_{j=1}^{n-1} \log H_j(\bar{d_1})}{\log M_{n+1} + 2\sum_{j=1}^{n+1} \log M_j} \ge \liminf_{n\to\infty} \frac{\sum_{j=1}^{n-1} \log(M_j^t \dots M_2^{t^{j-1}})}{\log M_{n+1} + 2\sum_{j=1}^{n+1} \log M_j} = \infty. \tag{27}$$

For any $\epsilon > 0$, let $\epsilon' = \frac{\epsilon}{1-\epsilon}$. By (27), there exists $k_5 \in \mathbb{N}$ such that for any $n \ge k_5$,

$$M_{n+1} \prod_{j=1}^{n+1} M_j^2 \le \Big(\prod_{j=1}^{n-1} H_j(\bar{d_1}) \Big)^{\epsilon'}. \tag{28}$$

As a consequence, for any $n \ge k_5$ and $(d_1, \dots, d_n) \in D_n(\bar{d_1})$, we have

$$|J(d_1, \dots, d_n)| \ge \prod_{j=1}^{n+1} \frac{1}{M_j^2} \prod_{j=1}^{n-1} \frac{1}{h_j(d_j)} \ge M_{n+1} \Big(\prod_{j=1}^{n-1} \frac{1}{h_j(d_j)} \Big)^{1+\epsilon'}. \tag{29}$$

Combine this and formula (16) in Proposition 3.2, we can get $\dim_{\mathrm{H}} C = 1$ by following the proof in Proposition 3.2 step by step. The proof of Proposition 3.3 is finished. □

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