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A new characterization of q-convexifiable Banach spaces

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Abstract. In this paper an atomic decomposition theorem for Banach-space-valued weak Hardy regular martingale space $wH_p(X)$ is given. As an application, we show that a Banach space X is q-convexifiable if and only if $\|S^{(q)}(f)\|_{wL_p} \leq C\|f^*\|_{wL_p}$ $(0 < p < \infty,$ $2 \leq q < \infty)$ for each X-valued regular martingale $f = (f_n)_{n\geq 0}$.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $(\mathcal{F}_n)_{n\geq 0}$ a non-decreasing sequence of sub-σ-algebras of $\mathcal{F}$ satisfying $\mathcal{F} = \sigma(\bigcup_{n\geq 0} \mathcal{F}_n)$. The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ are denoted by $\mathbb{E}$ and $\mathbb{E}_n$, respectively. We say that $(\mathcal{F}_n)_{n\geq 0}$ satisfies the regular condition, if $\chi(F) \leq d\mathbb{E}_{n-1}[\chi(F)]$, $\forall F \in \mathcal{F}_n$, $n = 1, 2, 3, \dots$, where $\chi(A)$ denotes the characteristic function of the set A and $d \geq 1$. In the following we always assume that $(\mathcal{F}_n)_{n\geq 0}$ satisfies the regular condition and d denotes the constant in the definition above.

Let X be a Banach space with norm $\|\cdot\|$ and $f = (f_n)_{n\geq 0}$ an X-valued regular martingale relative to $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_n)_{n\geq 0})$. For convenience, we assume

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that $f_0 = 0$, denote $df_i = f_i - f_{i-1} (i \geq 1)$, and

$$f_n^* = \sup_{0 \leq i \leq n} \|f_i\|, \quad f^* = \sup_n f_n^* = \sup_{n \geq 0} \|f_n\|,$$

$$S_n^{(p)}(f) = \left(\sum_{i=1}^{n} \|df_i\|^p \right)^{\frac{1}{p}}, \; S^{(p)}(f) = \sup_n S_n^{(p)}(f) = \left(\sum_{i=1}^{\infty} \|df_i\|^p \right)^{\frac{1}{p}}, \quad (0 < p < \infty).$$

Let $0 < p < \infty$, the space of all X-valued measurable functions f satisfying

$$\|f\|_{wL_p(X)} = \sup_{y>0} y \mathbb{P}(\|f\| > y)^{\frac{1}{p}} < \infty$$

is called a weak $L_p(X)$-space, denoted by $wL_p(X)$. We write $wL_p(X)$ as wL_p if X is the scalar field. It is well-known that $\|.\|_{wL_p(X)}$ is a quasi-norm on $wL_p(X)$. Notice that $L_p(X) \subset wL_p(X)$ since $\|f\|_{wL_p(X)} \leq \|f\|_{L_p(X)}$. If $0 < p < \infty$, we define X-valued weak regular martingale Hardy space $wH_p(X)$ as follows:

$$wH_p(X) = \{f = (f_n)_{n \geq 0} : \|f\|_{wH_p(X)} = \|f^*\|_{wL_p} < \infty\}.$$

It is clear that if X is the scalar field and $\| \cdot \|_{wL_p}$ is replaced by $\| \cdot \|_{L_p}$ we get the familiar martingale Hardy spaces H_p in the definition above (see F. Weisz [2]).

Definition 1.1 ([5]). Let $0 < \alpha$, $r \leq \infty$. An X-valued measurable function a is called a $(3, \alpha, r)$-atom if there exists a stopping time ν (ν is called the stopping time associated with a) such that

(i) $a_n = \mathbb{E}_n[a] = 0$ if $\nu \geq n$,

(ii) $\|a^*\|_r \leq \mathbb{P}(\nu \neq \infty)^{\frac{1}{r} - \frac{1}{\alpha}}$.

Throughout this paper, we denote the set of integers and the set of non-negative integers by $\mathbb{Z}$ and $\mathbb{N}$, respectively. We use C_α to denote constants which depend only on α and may denote different constants at different occurrences.

It is well-known that the inequalities of Banach-space-valued martingales are closely connected with the geometrical properties of Banach spaces. For the related definitions see refs [6], [7]. In particular, we need the following lemmas:

Lemma 1.2 ([10]). *If $\{\mathcal{F}\}_{n \geq 0}$ satisfies the regular condition, then there exists stopping times $\nu_\lambda = \inf \left\{ n \in \mathbb{N} : \mathbb{E}_n[\chi(\gamma_{n+1} > \lambda)] \geq \frac{1}{d} \right\}$ such that*

$$\mathbb{P}(\nu_\lambda \neq \infty) \leq d \mathbb{P}(\sup_n \gamma_n > \lambda) \tag{1.1}$$

for all nonnegative, adapted sequences $(\gamma)_{n \geq 0}$ and $\lambda \geq \|\gamma_0\|_\infty$, and $\nu_{\lambda_1} \leq \nu_{\lambda_2}$ when $\lambda_1 \leq \lambda_2$.

Lemma 1.3 ([8]). *Let X be a Banach space and $2 \le q < \infty$. Then the following statements are equivalent:*

(i) *X is q-convexifiable;*

(ii) *There exists a constant $C > 0$ such that for all X-valued regular martingales:*

$$\|S^{(q)}(f)\|_r \le C\|f^*\|_r, \quad 0 < r < \infty;$$

(iii) *There exists a constant $C > 0$ such that for all X-valued regular martingales:*

$$\lambda^q \mathbb{P}(S^{(q)}(f) > \lambda) \le C\|f^*\|_q^q, \quad \lambda > 0.$$

2. Atomic decomposition

In the scalar-valued case, atomic decomposition for regular martingale Hardy space H_p was given by Long [10]. In this section we give a weak type vector-valued parallelism.

Theorem 2.1. *Let $0 < p < \infty$ and X a Banach space with Radon–Nikodym property. Then $f = (f_n)_{n\ge 0} \in wH_p(X)$ if and only if there exist a sequence $(a^k)_{k\in\mathbb{Z}}$ of $(3, p, \infty)$-atoms and the corresponding stopping times $(\nu_k)_{k\in\mathbb{Z}}$ such that*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k \mathbb{E}_n[a^k], \quad \forall n \in \mathbb{N}, \tag{2.1}$$

where $0 \le \mu_k \le A \cdot 2^k \mathbb{P}(\nu_k \ne \infty)^{\frac{1}{p}}$ for some constant $A > 0$ and $\sup_{k\in\mathbb{Z}} 2^k \mathbb{P}(\nu_k \ne \infty)^{\frac{1}{p}} < \infty$.

Moreover, the following equivalence of norms holds:

$$\|f\|_{wH_p(X)} \sim \inf \sup_{k\in\mathbb{Z}} 2^k \mathbb{P}(\nu_k \ne \infty)^{\frac{1}{p}}, \tag{2.2}$$

where the infimum is taken over all the preceding decompositions of f.

Proof. Assume that $f = (f_n)_{n\ge 0} \in wH_p(X)$. We define the following stopping times for all $k \in \mathbb{Z}$:

$$\nu_k = \inf\left\{ n \in \mathbf{N} : \mathbb{E}_n[\chi(F_k)] \ge \frac{1}{d} \right\} \quad (\inf \emptyset = \infty).$$

where $F_k = \{\|f_{n+1}\| > 2^k\}$. Obviously by Lemma 1.2, $\nu_k \uparrow \infty$ $(k \to \infty)$. Let $f^{\nu_k} = (f_{n\wedge\nu_k})_{n\geq 0}$ be the stopping martingale. Then

$$\sum_{k\in\mathbb{Z}}(f_n^{\nu_{k+1}} - f_n^{\nu_k}) = \sum_{k\in\mathbb{Z}}\left(\sum_{m=0}^{n}\chi(m \leq \nu_{k+1})df_m - \sum_{m=0}^{n}\chi(m \leq \nu_k)df_m\right)$$
$$= \sum_{m=0}^{n}\left(\sum_{k\in\mathbb{Z}}\chi(\nu_k < m \leq \nu_{k+1})df_m\right) = f_n \tag{2.3}$$

Now let

$$\mu_k = 2^k \cdot 3\mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}},\ a_n^k = \mu_k^{-1}(f_n^{\nu_{k+1}} - f_n^{\nu_k}), \quad k \in \mathbb{Z}, n \in \mathbb{N} \tag{2.4}$$

($a_n^k = 0$ if $\mu_k = 0$). It is clear that for any fixed $k \in \mathbb{Z}$, $a^k = (a_n^k)_{n\geq 0}$ is an X-valued martingale. Since $\chi(F_k) \leq d\mathbb{E}_n[\chi(F_k)] < 1$ on the set $\{\nu_k > n\}$, we have $\|f_{n+1}\| \leq 2^k$. i.e., $\|f_{\nu_k}\| \leq 2^k$. So by (2.4) we have

$$\|a_n^k\| = \mu_k^{-1}\|f_n^{\nu_{k+1}} - f_n^{\nu_k}\| \leq \mu_k^{-1}(\|f_{\nu_{k+1}\wedge n}\| + \|f_{\nu_k\wedge n}\|)$$
$$\leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}. \tag{2.5}$$

Furthermore, $\|a^{k*}\|_\infty \leq \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}$. By assumption X has the Radon–Nikodym property, so there exists an X-valued integrable function a^k such that $\mathbb{E}_n[a^k] = a_n^k$ $(n \geq 0)$. It is clear that $a_n^k = 0$ if $n \leq \nu_k$. And by (2.5) we have $\|a^{k*}\|_\infty < \mathbb{P}(\nu_k \neq \infty)^{-\frac{1}{p}}$. So a^k is a $(3, p, \infty)$-atom. It follows from the definition of μ_k that $0 \leq \mu_k \leq A \cdot 2^k\mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}$ with $A = 3$. By (2.3) we get

$$f_n = \sum_{k\in\mathbb{Z}}\mu_k\big(\mu_k^{-1}(f_n^{\nu_{k+1}} - f_n^{\nu_k})\big) = \sum_{k\in\mathbb{Z}}\mu_k a_n^k = \sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}_n[a^k].$$

Hence (2.1) holds. Since $\{\|f_n\|\}$ is a nonnegative and adapted sequence, by (1.1) we have $\mathbb{P}(\nu_k \neq \infty) \leq d\mathbb{P}(f^* > 2^k)$. It follows that for any $k \in \mathbb{Z}$ we have

$$2^k\mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}} \leq d2^k\mathbb{P}(f^* > 2^k)^{\frac{1}{p}} \leq d\|f^*\|_{wL_p}$$
$$= d\|f\|_{wH_p(X)} < \infty, \tag{2.6}$$

which implies $\sup_{k\in\mathbb{Z}} 2^k\mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}} < \infty$.

Conversely, assume that $f = (f_n)_{n\geq 0}$ has a decomposition of the form (2.1). Let $M = \sup_{k\in\mathbb{Z}} 2^k\mathbb{P}(\nu_k \neq \infty)^{\frac{1}{p}}$. For any fixed $y > 0$ choose $j \in \mathbb{Z}$ such that $2^j \leq y < 2^{j+1}$. Let

$$f_n = \sum_{k\in\mathbb{Z}}\mu_k\mathbb{E}_n[a^k] = \sum_{k=-\infty}^{j-1}\mu_k a_n^k + \sum_{k=j}^{\infty}\mu_k a_n^k =: g_n + h_n, \quad n \in \mathbb{N}.$$

Thus $f^* \le g^* + h^*$, and it follows that

$$\mathbb{P}(f^* > 2Ay) \le \mathbb{P}(g^* > Ay) + \mathbb{P}(h^* > Ay).$$

Since $(a^k)_{k\in\mathbb{Z}}$ are $(3, p, \infty)$-atoms, $\|a^{k*}\|_\infty \le \mathbb{P}(\nu_k \ne \infty)^{-\frac{1}{p}}$ and we have

$$g^* \le \sum_{k=-\infty}^{j-1} \mu_k a^{k*} \le \sum_{k=-\infty}^{j-1} \mu_k \|a^{k*}\|_\infty \le \sum_{k=-\infty}^{j-1} A \cdot 2^k \le A2^j.$$

Follows which we have

$$\mathbb{P}(g^* > Ay) \le \mathbb{P}(g^* > A2^j) = 0.$$

On the other hand, since $a_n^k = \mathbb{E}_n[a^k] = 0$ if $n \le \nu_k$, we know $a^{k*} = 0$ on the set $\{\nu_k = \infty\}$. Moreover, $h^* \le \sum_{k=j}^\infty \mu_k a^{k*}$. So we have $\{h^* > 0\} \subset \bigcup_{k=j}^\infty \{\nu_k \ne \infty\}$. Consequently,

$$\begin{aligned}\mathbb{P}(f^* > 2Ay) \le \mathbb{P}(h^* > Ay) \le \mathbb{P}(h^* > 0) &\le \sum_{k=j}^\infty \mathbb{P}(\nu_k \ne \infty) \le \sum_{k=j}^\infty M^p 2^{-kp} \\ &\le C_p M^p 2^{-(j+1)p} \le C_p M^p y^{-p}.\end{aligned}$$

It follows that

$$\|f\|_{wH_p(X)}^p \le C_p M^p = C_p \sup_{k\in\mathbb{Z}} 2^{kp} \mathbb{P}(\nu_k \ne \infty) < \infty, \tag{2.7}$$

which shows that $f \in wH_p(X)$. Combine (2.6) with (2.7) we obtain that

$$C_p \sup_{k\in\mathbb{Z}} 2^k \mathbb{P}(\nu_k \ne \infty)^{\frac{1}{p}} \le \|f\|_{wH_p(X)} \le C_p \sup_{k\in\mathbb{Z}} 2^k \mathbb{P}(\nu_k \ne \infty)^{\frac{1}{p}}.$$

Thus (2.2) holds. The proof of the theorem is completed. □

3. A characterization of q-convexifiable spaces

In this section we give an application of Theorem 2.1. A characterization for q-convexifiable Banach spaces is given.

Theorem 3.1. *Let $2 \le q < \infty$ and X a Banach space. Then the following statements are equivalent:*

(i) *X is q-convexifiable;*

(ii) *There exists a constant $C>0$ such that for all X-valued regular martingales:*

$$\|S^{(q)}(f)\|_{wL_p} \le C\|f^*\|_{wL_p},\ 0<p<\infty. \tag{3.1}$$

Proof. (i)⇒(ii). Assume that X is a q-convexifiable Banach space and $f\in wH_p(X)$. Since q-convexity of X implies that X has Radon–Nikodym property, then f can be decomposed into the sum of a sequence of $(3,p,\infty)$-atoms by Theorem 2.1. For any fixed $y>0$ choose $j\in\mathbb{Z}$ such that $2^j\le y<2^{j+1}$ and let

$$f=\sum_{k\in\mathbb{Z}}\mu_k a^k=\sum_{k=-\infty}^{j-1}\mu_k a^k+\sum_{k=j}^{\infty}\mu_k a^k=:g+h.$$

Thus $\mathbb{P}(S^{(q)}(f)>2y)\le\mathbb{P}(S^{(q)}(g)>y)+\mathbb{P}(S^{(q)}(h)>y)$. For $0<p<\infty$, choose $\max\{p,2\}<q<\infty$, by Lemma 1.3, Theorem 2.1 and notice that $a^{k^*}=0$ on the set $\{\nu_k=\infty\}$, we have

$$\begin{aligned}\|S^{(q)}(g)\|_q&\le\sum_{k=-\infty}^{j-1}\mu_k\|S^{(q)}(a^k)\|_q\le C_q\sum_{k=-\infty}^{j-1}\mu_k\|a^{k*}\|_q\\&\le C_q\sum_{k=-\infty}^{j-1}A\cdot 2^{k(1-\frac{p}{q})}2^{\frac{kp}{q}}\mathbb{P}(\nu_k\ne\infty)^{\frac{1}{q}}\\&\le C_q\sum_{k=-\infty}^{j-1}A\cdot 2^{k(1-\frac{p}{q})}\|f\|_{wH_p(X)}^{\frac{p}{q}}\le C_{p,q}y^{1-\frac{p}{q}}\|f\|_{wH_p(X)}^{\frac{p}{q}}.\end{aligned}$$

It follows that

$$\mathbb{P}(S^{(q)}(g)>y)\le y^{-q}\mathbb{E}[S^{(q)}(g)^q]\le C_{p,q}y^{-p}\|f\|_{wH_p(X)}^p. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}\mathbb{P}(S^{(q)}(h)>y)&\le\mathbb{P}(S^{(q)}(h)>0)\le\sum_{k=j}^{\infty}\mathbb{P}(S^{(q)}(a^k)>0)\\&\le\sum_{k=j}^{\infty}\mathbb{P}(\nu_k\ne\infty)\le\sum_{k=j}^{\infty}2^{-kp}\cdot 2^{kp}\mathbb{P}(\nu_k\ne\infty)\\&\le C_p y^{-p}\|f\|_{wH_p(X)}^p.\end{aligned} \tag{3.3}$$

Combine (3.2) with (3.3) we get $\mathbb{P}(S^{(q)}(f)>2y)\le C_{p,q}y^{-p}\|f\|_{wH_p(X)}^p$. Thus (3.1) is obtained.

(ii)⇒(i). Assume that (3.1) holds. Since $\|f^*\|_{wL_q}\le\|f^*\|_q$, it is easy to obtain that

$$\lambda^q\mathbb{P}(S^{(q)}(f)>\lambda)\le C\|f^*\|_q^q,\quad \lambda>0.$$

By Lemma 1.3 we know that X is q-convexifiable. The proof is completed. □

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