Publ. Math. Debrecen
73/3-4 (2008), 341–360

Homogeneous geodesics of four-dimensional generalized symmetric pseudo-Riemannian spaces

By BARBARA DE LEO (Lecce) and ROSA ANNA MARINOSCI (Lecce)

Abstract. We investigate the set of homogeneous geodesics on generalized symmetric pseudo-Riemannian spaces of dimension $n = 4$. In particular, we prove that in any pseudo-Riemannian generalized symmetric space of dimension $n = 4$ there are just four linearly independent homogeneous geodesics through each point.

1. Introduction

A (connected) pseudo-Riemannian manifold (M, g) is *homogeneous* if there exists a group G of isometries acting transitively on it [O'N]. Such (M, g) can be then identified with $(G/H, g)$, where H is the isotropy group at a fixed point o of M. In general we can have more than one choice for G. For any fixed $M = G/H$, G acts effectively on G/H. The pseudo-Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is called a *pseudo-Riemannian homogeneous spaces*.

If the metric g is positive definite then Riemannian homogeneous space $(G/H, g)$ is always a *reductive homogeneous space*; this means that denoted by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively, there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $Ad(H)(\mathfrak{m}) \subset \mathfrak{m}$ (where $Ad : H \times \mathfrak{g} \to \mathfrak{g}$ is the adjoint representation of H on $\mathfrak{g}$). If the metric g is indefinite, the reductive decomposition of the Lie algebra $\mathfrak{g}$ may not exist (see an example in 4.4 [FMeP]). If the metric g is indefinite and the Lie algebra $\mathfrak{g}$ has a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$,

Mathematics Subject Classification: 53C20,53C22, 53C30.
Key words and phrases: pseudo-Riemannian homogeneous spaces, homogeneous geodesics.
Authors supported by funds of M.I.U.R., G.N.S.A.G.A and the University of Salento.

then the subspace $\mathfrak{m}$ may be identified with the tangent space $T_o(M)$ via the projection $\pi : G \to G/H(= M)$; by this identification the scalar product g_o on $T_o(M)$ defines a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ which is $Ad(H) - invariant$.

In the Riemannian case, a geodesic $\gamma(t)$ through the origin o is called *homogeneous* if it is an orbit of a one-parameter subgroup of G, that is there exists a vector $X \in \mathfrak{g}$ such that $\gamma(t) = \exp(tX)(o)$ (see [KV]). In the pseudo-Riemannian case this definition must be modified in the following way (see [DK], [FMeP]): a geodesic $\gamma(s)$ through the origin o is homogeneous if there exists a reparametrization $\gamma^*(t)$ of $\gamma(s)$ such that $\gamma^*(t) = \exp(tX)(o)$ for any $X \in \mathfrak{g}$. The vector X is called a *geodesic vector*.

A very useful characterization of geodesic vectors is the following: a vector $X \in \mathfrak{g}$ is a geodesic vector if and only if

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}}\rangle = k\langle X_{\mathfrak{m}}, Z\rangle \quad \text{for all } Z \in \mathfrak{m}, \tag{1.1}$$

where $k \in \mathbb{R}$ is some constant.

In the Riemannian case, the constant k of formula (1.1) must be equal to zero; the proof of the above characterization is due to O. Kowalski and L. Vanhecke [KV]. We refer to [AA], [CM1], [CKM], [KNVl], [KS], [KVl1], [M], [S] for some examples and further references concerning homogeneous geodesics in homogeneous Riemannian spaces.

In a physical context the above characterization appears (without proof) in [P] and in [FMeP] in the framework of Lorentzian geometry; in these papers the authors use the above characterization for studying the Penrose limits (or plane-waves limits) of a reductive homogeneous spacetime along light-like homogeneous geodesics.

The study of homogeneous geodesics of pseudo-Riemannian homogeneous spaces, in a mathematical context, is very recent. A rigorous proof of formula (1.1) appears for the first time in a recent paper of Z. Dusek and O. Kowalski [DK]. We refer to [AS], [C], [CM2], [CM3], [Me] for more results on this topic.

As concerns the existence of homogeneous geodesics, O. Kowalski and J. Szenthe proved in [KS] that every homogeneous Riemannian space admits at least one homogeneous geodesic and this result cannot be improved in general [KVl 1]. In the pseudo-Riemannian case there is a still open conjecture saying that in any homogeneous pseudo-Riemannian space, at least one homogeneous geodesic must exist [KVl 2]. More recently, homogeneous geodesics in affine homogeneous manifolds have been studied by O. Kowalski and Z. Vlášek [KVl 2], [KVl 3] and the problem concerning the existence of homogeneous geodesics in a locally homogeneous affine manifold has been definitively solved. In fact, they have found an infinite family of locally affinely homogeneous connections which do not admit any homogeneous geodesic.

In the present paper the authors focus their attention to the study of homogeneous geodesics in generalized symmetric pseudo-Riemannian spaces of dimension $n = 4$.

A *generalized symmetric pseudo-Riemannian space* is a connected pseudo-Riemannian manifold (M, g) admitting a *regular s-structure*, that is a family $\{s_x : x \in M\}$ of isometries of (M, g) (called symmetries), such that

$$s_x \circ s_y = s_z \circ s_x, \qquad z = s_x(y), \tag{1.2}$$

for all points x, y of M.

An s-structure $\{s_x : x \in M\}$ is said of *order* k $(k \geq 2)$ if $(s_x)^k = id_M$ and k is the least integer with this property. We say that an s-structure is of *infinity order* if such k does not exist. *Order* of a generalized symmetric pseudo-Riemannian space is the infimum of all integers $k \geq 2$ such that M admits a regular s-structure of order k (it may be that $k = \infty$).

In 1982 J. Černý and K. Kowalski classified all generalized symmetric pseudo-Riemannian spaces of dimension $n \leq 4$ [CK].

In dimension $n = 2$ all generalized symmetric pseudo-Riemannian spaces are symmetric, consequently all geodesics are homogeneous.

In dimension $n = 3$ a generalized symmetric pseudo-Riemannian space may be identified with $\mathbb{R}^3$ endowed with a special metric whose possible signatures are $(3, 0)$, $(0, 3)$, $(2, 1)$, $(1, 2)$. The set of all homogeneous geodesics in a three-dimensional Riemannian generalized symmetric space have been studied in the framework of [M] and in three-dimensional Lorentzian generalized symmetric spaces in the framework of [CM2] and [CM3].

In dimension $n = 4$ a generalized symmetric pseudo-Riemannian space may be identified with $\mathbb{R}^4$ endowed with a special metric of four types: A, B, C, D. The metric of Type A has possible signatures $(4, 0)$, $(0, 4)$, $(2, 2)$; the metric of Type B has always signature $(2, 2)$; in the case C the possible signatures are $(3, 1)$ and $(1, 3)$; the metric of Type D has always signature $(2, 2)$.

The aim of this paper is to investigate the set of homogeneous geodesics of generalized symmetric pseudo-Riemannian spaces of dimension $n = 4$. A basic property of these spaces is that all of them are reductive homogeneous. This property allows us to calculate the set of all homogeneous geodesics of these spaces, by using formula (1.1), and then to describe their behaviour.

In the following we shall use the terminology “space-like”, “time-like” and “light-like” for vectors also in the case of non-Lorentzian pseudo-Riemannian manifolds, with the obvious meaning.

As result of our study we obtain that in all generalized symmetric pseudo-Riemannian spaces of dimension $n = 4$ the set of homogeneous geodesics is non empty. In particular we prove the following result:

Theorem 1.1. *In any four-dimensional generalized symmetric Riemannian space of Type A, there exist for each point four linearly independent (never orthogonal) homogeneous geodesics* [KNVl].

a) *In any four-dimensional generalized symmetric pseudo-Riemannian space of Type A, with metric g of signature $(+,+,-,-)$, the set of all geodesic vectors consists of one plane of time-like geodesic vectors and one-parameter family of light-like geodesic vectors. In particular, there exists a four-parameter family of quadruplets of linearly independent light-like homogeneous geodesics through each point.*

b) *Each four-dimensional generalized symmetric pseudo-Riemannian space of Type B, admits always homogeneous geodesics and all of them are light-like. Besides there exist always four linearly independent light-like homogeneous geodesics through each point.*

c) *In any four-dimensional generalized symmetric Lorentzian space of Type C, the set of all geodesic vectors consists of one plane of space-like geodesic vectors and another plane containing two straight lines of light-like geodesic vectors.*

Through each point, there are just four linearly independent homogeneous geodesics.

d) *In any four-dimensional generalized symmetric pseudo-Riemannian space of Type D there is a two-parameter family of quadruplets of linearly independent light-like homogeneous geodesics through each point.*

2. Preliminaries on homogeneous geodesics and generalized symmetric pseudo-Riemannian spaces

Let (M, g) be a (connected) homogeneous pseudo-Riemannian manifold. Then, its full isometry group $I(M)$ acts transitively on it and M can be identified with $(G/H, g)$, where $G \subset I(M)$ is a subgroup of $I(M)$ acting transitively on M and H is the isotropy group at a fixed point $o \in M$. In general the group G is not unique.

Differently from the Riemannian case, the Lie algebra $\mathfrak{g}$ of G need not admit a reductive decomposition. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and

H respectively, and let $\mathfrak{m}$ be a complement of $\mathfrak{h}$ in $\mathfrak{g}$. If $\mathfrak{m}$ is stable under the action of $\mathfrak{h}$, then $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is called a *reductive split*, and $(\mathfrak{g}, \mathfrak{h})$ a *reductive pair*. It is important to stress that reductivity is not an intrinsic property of (M, g), but of the description of M as coset space G/H. In fact, the so-called *Kaigorodov space* is an example of a homogeneous Lorentzian manifold which has two different coset descriptions, but only one of them is reductive [FMeP]. Nevertheless, a homogeneous pseudo-Riemannian manifold (M, g) is called reductive if there exists a Lie group G acting transitively on M via isometries, with isotropy group H, such that $(\mathfrak{g}, \mathfrak{h})$ is reductive. Consider now a reductive homogeneous pseudo-Riemannian manifold $(M = G/H, g)$, where $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ is a reductive split. As already mentioned in the Introduction, a geodesic $\gamma(s)$ through the origin $o \in G/H$ is homogeneous if there exists a reparametrization $\gamma^*(t)$ of $\gamma(s)$ such that $\gamma^*(t) = \exp(tX)(o)$ for any $X \in \mathfrak{g}$. The vector X is called a geodesic vector of $(M = G/H, g)$.

In order to find all homogeneous geodesics of a reductive homogeneous pseudo-Riemannian space $(M = G/H, g)$, $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, we need to determine all geodesic vectors through a point. Since the canonical projection $p : G \to G/H (= M)$ induces an isomorphism between the subspace $\mathfrak{m}$ and the tangent space $T_o(M)$, the metric g_o on $T_o(M)$ induces a metric $\langle\ ,\ \rangle$ on $\mathfrak{m}$, which is $\mathrm{Ad}(H)$-invariant. The following characterization holds:

Proposition 2.1 ([P], [DK]). *A non-zero vector $X \in \mathfrak{g}$ is a geodesic vector if and only if*

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Y \rangle, \tag{2.1}$$

for all $Y \in \mathfrak{m}$ and some $k \in \mathbb{R}$ (the subscript $\mathfrak{m}$ denotes the projection into $\mathfrak{m}$).

When $X_{\mathfrak{m}}$ is either space-like ($\langle X_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle > 0$) or time-like ($\langle X_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle < 0$), applying (2.1) with $Y = X_{\mathfrak{m}}$ we get $k = 0$, while for a light-like vector $X_{\mathfrak{m}}$, k may be any real constant. Note also that if $\mathfrak{h} = 0$, then $\mathfrak{g} = \mathfrak{m}$ and (2.1) simplifies as follows:

$$\langle [X, Y], X \rangle = k \langle X, Y \rangle, \tag{2.2}$$

for all $Y \in \mathfrak{g}$.

A finite family $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of homogeneous geodesics through $o \in M$ is said to be *linearly independent* (respectively *orthogonal*) if the corresponding initial tangent vectors at o are linearly independent (respectively *orthogonal*).

The following result is obvious.

Proposition 2.2. *A finite family $\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$ of homogeneous geodesics through $p_o \in M$ is linearly independent (respectively orthogonal) if the $\mathfrak{m}$-components of the corresponding geodesic vectors are linearly independent (respectively orthogonal).*

Now we recall some basic facts about generalized symmetric pseudo-Riemannian spaces (we refer to [CK], [K1] and [K2] for more details).
In the following we are going to consider *proper* generalized symmetric pseudo-Riemannian spaces, i.e. those generalized symmetric pseudo-Riemannian spaces which are not locally symmetric and not direct product of generalized symmetric pseudo-Riemannian spaces.

For our study a basic property of such spaces is in the next proposition.

Proposition 2.3. *Any generalized symmetric pseudo-Riemannian space admits at least one structure of a reductive homogeneous space with an invariant metric.*

In low dimensions, the full classification of generalized symmetric pseudo-Riemannian spaces is due to J. Černý and O. Kowalski; the authors proved in their paper [CK] the following theorems.

Theorem 2.4. *Any proper, simply connected generalized symmetric pseudo-Riemannian space (M, g) of dimension $n = 3$ is of order 4. It is indecomposable, and described (up to an isometry) as follows:*

The underlying homogeneous space G/H is the matrix group

$$\begin{pmatrix} e^{-t} & 0 & x \\ 0 & e^{t} & y \\ 0 & 0 & 1 \end{pmatrix},$$

(M, g) is the space $\mathbb{R}^3(x, y, t)$ with the pseudo-Riemannian metric

$$g = \pm(e^{2t}dx^2 + e^{-2t}dy^2) + \lambda dt^2, \tag{2.3}$$

where $\lambda \neq 0$ is a real constant. The possible signatures of g are $(3,0)$, $(0,3)$, $(2,1)$, $(1,2)$. The typical symmetry of order 4 at the initial point $(0,0,0)$ is the transformation

$$x' = -y, \quad y' = x, \quad t' = -t \tag{2.4}$$

Theorem 2.5. *All proper, simply connected generalized symmetric pseudo-Riemannian spaces (M, g) of dimension $n = 4$ are of order 3, or 4, or infinity. All these spaces are indecomposable, and belong (up to an isometry) to the following four types:*

Type A.
The underlying homogeneous space is G/H *where*

$$G = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with ad-bc=1.
(M,g) *is the space* $\mathbb{R}^4(x,y,u,v)$ *with the pseudo-Riemannian metric*

$$\begin{aligned} g = &\pm[(-x + \sqrt{1+x^2+y^2}\,)du^2 + (x + \sqrt{1+x^2+y^2}\,)dv^2 - 2y^2 du\, dv] \\ &+ \lambda[(1+y^2)dx^2 + (1+x^2)dy^2 - 2xy\, dx\, dy]/(1+x^2+y^2), \end{aligned} \tag{2.5}$$

where $\lambda \neq 0$ *is a real constant. The order is* $k = 3$ *and possible signatures are* $(4,0)$, $(0,4)$, $(2,2)$. *The typical symmetry of order* 3 *at the initial point* $(0,0,0,0)$ *is the transformation*

$$\begin{aligned} u' &= -(1/2)u - (\sqrt{3}/2)v, & v' &= (\sqrt{3}/2)u - (1/2)v, \\ x' &= -(1/2)x + (\sqrt{3}/2)y, & y' &= -(\sqrt{3}/2)x - (1/2)y. \end{aligned} \tag{2.6}$$

Type B.
The underlying homogeneous space is G/H *where*

$$G = \begin{pmatrix} e^{-(x+y)} & 0 & 0 & a \\ 0 & e^x & 0 & b \\ 0 & 0 & e^y & c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 & -w \\ 0 & 1 & 0 & -2w \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(M,g) *is the space* $\mathbb{R}^4(x,y,u,v)$ *with the pseudo-Riemannian metric*

$$g = \lambda(dx^2 + dy^2 + dx\, dy) + e^{-y}(2dx + dy)dv + e^{-x}(dx + 2dy)du, \tag{2.7}$$

where λ *is a real constant. The order is* $k = 3$ *and the signature is always* $(2,2)$. *The typical symmetry of order* 3 *at the initial point* $(0,0,0,0)$ *is the transformation*

$$u' = -ue^{(y-x)} - v, \quad v' = ue^{-(y+2x)}x' = y, \quad y' = -(x+y). \tag{2.8}$$

Type C.
The underlying homogeneous space G/H is the matrix group

$$\begin{pmatrix} e^{-t} & 0 & 0 & x \\ 0 & e^{t} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(M,g) *is the space* $\mathbb{R}^4(x,y,z,t)$ *with the pseudo-Riemannian metric*

$$g = \pm(e^{2t}dx^2 + e^{-2t}dy^2) + dz\,dt. \tag{2.9}$$

The order is $k=4$ *and possible signatures are* $(1,3)$, $(3,1)$. *The typical symmetry of order 4 at the initial point* $(0,0,0,0)$ *is the transformation*

$$x' = -y, \qquad y' = x, \qquad z' = -z, \qquad t' = -t. \tag{2.10}$$

Type D.
The underlying homogeneous space is G/H where

$$G = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix} \qquad H = \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where ad-bc=1.
(M,g) *is the space* $\mathbb{R}^4(x,y,u,v)$ *with the pseudo-Riemannian metric*

$$\begin{aligned} g = {} & (\sinh(2u) - \cosh(2u)\sin(2v))dx^2 + (\sinh(2u) + \cosh(2u)\sin(2v))dy^2 \\ & - 2\cosh(2u)\cos(2v)dx\,dy + \lambda(du^2 - \cosh^2(2u)dv^2), \end{aligned} \tag{2.11}$$

where $\lambda \neq 0$ *is a real constant. The order is infinite and the signature is* $(2,2)$. *The typical symmetry at the initial point* $(0,0,0,0)$ *is induced by the automorphism of* G *of the form:*

$$a' = a, \quad b' = (1/\alpha^2)b, \quad c' = \alpha^2 c, \quad d' = d, \quad x' = (1/\alpha)x, \quad y' = \alpha y, \tag{2.12}$$

where $\alpha \neq 0, \pm 1$.

In the next section at first we shall calculate the set of all homogeneous geodesics for all the spaces classified in Theorem 2.5 and after we shall describe their behaviour.

3. Homogeneous geodesics of four-dimensional generalized symmetric pseudo-Riemannian spaces of Type A

According to Theorem 2.5, as homogeneous space, a four-dimensional generalized symmetric pseudo-Riemannian space of Type A is $M = G/H$ where

$$G = \begin{pmatrix} a & b & u \\ c & d & v \\ 0 & 0 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with *ad-bc=1.* The metric g has *essentially* (up to a renumeration of a basis) two possible signatures: $(+,+,+,+)$, $(+,+,-,-)$.

Case 1. The metric g has signature $(+,+,+,+)$.

The Lie algebra $\mathfrak{g}$ of the Lie group G has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\{X_1, Y_1, X_2, Y_2, B\}$ is a basis of $\mathfrak{g}$ with $\{X_1, Y_1, X_2, Y_2\}$ orthonormal basis of $\mathfrak{m}$ and $\{B\}$ basis of $\mathfrak{h}$; the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ is given by the following table:

$[\ ,\ ]$	X_1	Y_1	X_2	Y_2	B
X_1	0	0	$-X_1$	Y_1	Y_1
Y_1	0	0	Y_1	X_1	$-X_1$
X_2	X_1	$-Y_1$	0	$-2B$	$-2Y_2$
Y_2	$-Y_1$	$-X_1$	$2B$	0	$2X_2$
B	$-Y_1$	X_1	$2Y_2$	$-2X_2$	0

Table I

The set of all homogeneous geodesics of this space was studied in details in [KNVl]. In that paper the authors proved that there exists a continuum of quadruplets of linearly independent homogeneous geodesics through the origin but never an orthogonal quadruplet.

Case 2. The metric g has signature $(+,+,-,-)$.

In this case the Lie algebra $\mathfrak{g}$ of the Lie group G has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\{X_1, Y_1, X_2, Y_2, B\}$ is a basis of $\mathfrak{g}$ with $\{X_1, Y_1, X_2, Y_2\}$ basis of $\mathfrak{m}$ and $\{B\}$ basis of $\mathfrak{h}$ such that the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ and the scalar product on

$\mathfrak{m}$ are given, respectively, by the following tables:

$[\,,\,]$	X_1	Y_1	X_2	Y_2	B
X_1	0	0	$-\delta X_1$	δY_1	Y_1
Y_1	0	0	δY_1	δX_1	$-X_1$
X_2	δX_1	$-\delta Y_1$	0	$-2\delta^2 B$	$-2Y_2$
Y_2	$-\delta Y_1$	$-\delta X_1$	$2\delta^2 B$	0	$2X_2$
B	$-Y_1$	X_1	$2Y_2$	$-2X_2$	0

Table II

where $\delta > 0$ is a real constant,

$\langle\,,\,\rangle$	X_1	Y_1	X_2	Y_2
X_1	1	0	0	0
Y_1	0	1	0	0
X_2	0	0	-2	0
Y_2	0	0	0	-2

Table III

(see [CK], [K1] and [K2] for more details).
Using Table II and Table III to compute (1.1), we get that $X \in \mathfrak{g}$ is a geodesic vector if its components (a, b, c, d, α) with respect to the basis $\{X_1, Y_1, X_2, Y_2, B\}$ satisfy the system:

$$\begin{cases} (ac - bd)\delta - b\alpha = ka & (a^2 - b^2)\delta + 4\alpha d = 2kc \\ (bc + ad)\delta - a\alpha = -kb & ab\delta + 2\alpha c = -kd. \end{cases} \tag{3.1}$$

If $k = 0$ then we consider the two possibilities: $\alpha = 0$ and $\alpha \neq 0$.

For $\alpha = 0$, we get the solution $(0, 0, c, d, 0)$, $(c, d) \in \mathbb{R}^2$, but $(c, d) \neq (0, 0)$.

Suppose $\alpha \neq 0$, then from the third and fourth equation of (3.1) we get

$$\begin{cases} d = [(b^2 - a^2)\delta]/(4\alpha) \\ c = (-ab\delta)/(2\alpha); \end{cases} \tag{3.2}$$

and substituting c and d of (3.2) in the first and second equation of (3.1), we have

$$\begin{cases} b[(a^2 + b^2)\delta^2 + 4\alpha^2] = 0 \\ a[(a^2 + b^2)\delta^2 + 4\alpha^2] = 0 \end{cases} \tag{3.3}$$

which implies $a = b = 0$ (because $\alpha \neq 0$) and consequently from (3.2) $c = d = 0$; so we get the solution $(0, 0, 0, 0, \alpha)$.

Summarizing, the only solutions of (3.1) for $k = 0$ are: $(0, 0, c, d, 0)$ and $(0, 0, 0, 0, \alpha)$, where $(c, d) \in \mathbb{R}^2$, but $(c, d) \neq (0, 0)$ and $\alpha \in \mathbb{R} - \{0\}$.
Suppose $k \neq 0$. Then only light-like solutions can occur, so we must add to (3.1) the condition

$$a^2 + b^2 = 2(c^2 + d^2). \tag{3.4}$$

From the third and fifth equation of the new system

$$\begin{cases} (ac - bd)\delta - b\alpha = ka & ab\delta + 2\alpha c = -kd \\ (bc + ad)\delta - a\alpha = -kb & a^2 + b^2 = 2(c^2 + d^2), \\ (a^2 - b^2)\delta + 4\alpha d = 2kc & \end{cases} \tag{3.5}$$

we obtain

$$\begin{cases} \delta a^2 = (kc - 2\alpha d) + \delta(c^2 + d^2) \\ \delta b^2 = -(kc - 2\alpha d) + \delta(c^2 + d^2); \end{cases} \tag{3.6}$$

and from the fourth equation of (3.5)

$$\delta^2 a^2 b^2 = (kd + 2\alpha c)^2, \tag{3.7}$$

consequently from (3.6) and (3.7) we get

$$(c^2 + d^2)[\delta^2(c^2 + d^2) - (k^2 + 4\alpha^2)] = 0. \tag{3.8}$$

If $(c^2 + d^2) = 0$ then we have the solution $(0, 0, 0, 0, \alpha)$.
If $\delta^2(c^2 + d^2) - (k^2 + 4\alpha^2) = 0$ then

$$\delta^2(c^2 + d^2) - 4\alpha^2 = k^2. \tag{3.9}$$

From the first two equations of (3.5), rewritten in the form

$$b(d\delta + \alpha) = a(c\delta - k) \qquad a(d\delta - \alpha) = -b(c\delta + k) \tag{3.10}$$

we obtain

$$ab(d^2\delta^2 - \alpha^2) = -ab(c^2\delta^2 - k^2) \tag{3.11}$$

or equivalently, taking into account (3.9),

$$3ab\alpha^2 = 0. \tag{3.12}$$

If in (3.12) $\alpha = 0$, the system (3.5) becomes:

$$\begin{cases} (ac - bd)\delta = ka & (a^2 - b^2)\delta = 2kc \\ (bc + ad)\delta = -kb & a^2 + b^2 = 2(c^2 + d^2). \\ ab\delta = -kd & \end{cases} \tag{3.13}$$

From the third and fourth equation of (3.13), we get $c = (a^2 - b^2)\delta/(2k)$, $d = -\delta ab/k$ and from (3.4) and (3.9) $a^2 + b^2 = 2k^2/\delta^2$.

If in (3.12) $ab = 0$, then we get the solutions

$$(0, 0, 0, 0, \alpha), (0, \pm\sqrt{2}(k/\delta), -k/\delta, 0, 0), (\pm\sqrt{2}(k/\delta), 0, k/\delta, 0, 0). \tag{3.14}$$

We conclude that in case $k \neq 0$, the only solutions of the system (3.5) are

$$(0, 0, 0, 0, \alpha), (a, b, (a^2 - b^2)\delta/(2k), (-ab)\delta/k, 0) \tag{3.15}$$

where $a^2 + b^2 = 2k^2/\delta^2$.

As consequence we get that X is a geodesic vector of a four-dimensional generalized symmetric pseudo-Riemannian space of Type A, with metric g of signature $(+, +, -, -)$, if and only if its m-component $(=X_m)$ admits one of the following forms:

$$X_m = cX_2 + dY_2, \tag{3.16}$$

$(c, d) \in \mathbb{R}^2, (c, d) \neq (0, 0)$,

$$X_m = aX_1 + bY_1 + [(a^2 - b^2)\delta/(2k)]X_2 - (ab\delta/k)Y_2, \tag{3.17}$$

$(a, b) \in \mathbb{R}^2$ but $a^2 + b^2 = (2k^2/\delta^2)$ and $k \neq 0$.

Now substituting $a = \lambda \cos t$, $b = \lambda \sin t$, where $\lambda = \sqrt{2}k/\delta$, we can rewrite (3.17) in the new form

$$X_m = \lambda \left[\cos t X_1 + \sin t Y_1 + \left(\frac{1}{\sqrt{2}} \cos 2t \right) X_2 - \left(\frac{1}{\sqrt{2}} \sin 2t \right) Y_2 \right], \tag{3.18}$$

with $\lambda \neq 0$.

Then from (3.18) we obtain a *one-parameter family of light-like geodesic vectors depending on the free parameter 't'.* In particular, there exists a four-parameter family of quadruplets of linearly independent light-like homogeneous geodesics through each point.[1]

We note also that all geodesic vectors of the form (3.16) form a plane of time-like geodesic vectors .

From the above results we get property a) of Theorem 1.1.

[1]The authors are obliged to Zdenek Vlášek from the Charles University in Prague for his calculations in MAPLE justifying the above result.

4. Homogeneous geodesics of four-dimensional generalized symmetric pseudo-Riemannian spaces of Type B

According to Theorem 2.5, as homogeneous space, a four-dimensional generalized symmetric pseudo-Riemannian space of Type B is $M = G/H$, where G is the matrix group

$$\begin{pmatrix} e^{-(x+y)} & 0 & 0 & a \\ 0 & e^{x} & 0 & b \\ 0 & 0 & e^{y} & c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and H is the matrix group

$$\begin{pmatrix} 1 & 0 & 0 & -w \\ 0 & 1 & 0 & -2w \\ 0 & 0 & 1 & 2w \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra $\mathfrak{g}$ of the Lie group G has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ and $\{X_1, Y_1, X_2, Y_2, A\}$ is a basis of $\mathfrak{g}$ with $\{X_1, Y_1, X_2, Y_2\}$ basis of $\mathfrak{m}$ and $\{A\}$ basis of $\mathfrak{h}$ such that the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ and the scalar product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ are given by the following tables:

$[\ ,\ ]$	X_1	Y_1	X_2	Y_2	A
X_1	0	0	$-X_1$	$\pm A + Y_1$	0
Y_1	0	0	$\mp A + Y_1$	X_1	0
X_2	X_1	$\pm A - Y_1$	0	0	$2Y_1$
Y_2	$\mp A - Y_1$	$-X_1$	0	0	$-2X_1$
A	0	0	$-2Y_1$	$2X_1$	0

Table IV

and

$\langle\ ,\ \rangle$	X_1	Y_1	X_2	Y_2
X_1	0	0	-1	0
Y_1	0	0	0	-1
X_2	-1	0	2λ	0
Y_2	0	-1	0	2λ .

Table V

The signature of the metric g is $(+,+,-,-)$ (see $[CK]$ for more details).
Using Table IV and Table V to compute (1.1), we obtain that X is geodesic if and only if its components $(x_1, y_1, x_2, y_2, \alpha)$ with respect to the basis $\{X_1, Y_1, X_2, Y_2, A\}$ satisfy the system:

$$\begin{cases} {x_2}^2 - {y_2}^2 = kx_2 & x_1x_2 - y_1y_2 + 2\alpha y_2 = k(-x_1 + 2\lambda x_2) \\ 2x_2y_2 = -ky_2 & x_1y_2 + x_2y_1 + 2\alpha x_2 = k(y_1 - 2\lambda y_2). \end{cases} \tag{4.1}$$

It is easy to prove that when $k = 0$, the solutions of (4.1) are $(x_1, y_1, 0, 0, \alpha)$.
If $k \neq 0$, only light-like solutions can occur, so we must add to (4.1) the condition

$$x_1x_2 + y_1y_2 - \lambda(x_2^2 + y_2^2) = 0. \tag{4.2}$$

In order to solve the new system

$$\begin{cases} {x_2}^2 - {y_2}^2 = kx_2 & x_1x_2 - y_1y_2 + 2\alpha y_2 = k(-x_1 + 2\lambda x_2) \\ (2x_2 + k)y_2 = 0 & x_1y_2 + x_2y_1 + 2\alpha x_2 = k(y_1 - 2\lambda y_2) \\ & x_1x_2 + y_1y_2 - \lambda(x_2^2 + y_2^2) = 0 \end{cases} \tag{4.3}$$

we can see easily that $y_2 = 0$ and $2x_2 + k = 0$ cannot occur together; so we shall consider the two cases separately.

If $y_2 = 0$, we obtain two solutions $(0, 0, 0, 0, \alpha)$, $(\lambda k, y_1, k, 0, 0)$.

If $2x_2 + k = 0$, we get the solutions $(-2\lambda k \mp \sqrt{3}y_1, y_1, -k/2, (\mp\sqrt{3}/2)k, 0)$. Therefore, X is a geodesic vector of a generalized symmetric pseudo-Riemannian space of Type B if and only if its m-component $(=X_m)$ is one of the following forms:

$$X_m = x_1X_1 + y_1Y_1 \tag{4.4}$$

$$X_m = \lambda kX_1 + y_1Y_1 + kX_2 \tag{4.5}$$

$$X_m = (-2\lambda k \mp \sqrt{3}y_1)X_1 + y_1Y_1 - (k/2)X_2 \mp (\sqrt{3}/2)kY_2 \tag{4.6}$$

but $k \neq 0$.

We note that each geodesic vector of (4.4), (4.5), (4.6) is a light-like vector; in particular the four geodesic vectors
$\{X_1, Y_1, k(\lambda X_1 + X_2), -k(2\lambda X_1 + (1/2)X_2 \pm (\sqrt{3}/2)Y_2)\}$ are linearly independent.

The above results prove completely property *b)* of Theorem 1.1.

5. Homogeneous geodesics of four-dimensional generalized symmetric pseudo-Riemannian spaces of Type C

As stated in Theorem 2.5, as homogeneous space, a four-dimensional generalized symmetric pseudo-Riemannian space of Type C is the matrix group G

$$\begin{pmatrix} e^{-t} & 0 & 0 & x \\ 0 & e^{t} & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

According to [CK], the Lie bracket on the Lie algebra $\mathfrak{g}$ of G is given by:

$$[X, V_2] = -X, \quad [Y, V_2] = Y, \quad [X, Y] = [X, V_1] = [Y, V_1] = [V_1, V_2] = 0 \tag{5.1}$$

where $\{X, Y, V_1, V_2\}$ is a basis of $\mathfrak{g}$, and any scalar product $\langle \ , \ \rangle$ on $\mathfrak{g}$ is of the form

$$\begin{aligned} \langle X, X\rangle = \langle Y, Y\rangle = \pm 1, \langle V_1, V_2\rangle = \pm 1, \langle X, Y\rangle = \langle X, V_1\rangle \\ = \langle X, V_2\rangle = \langle Y, V_1\rangle = \langle Y, V_2\rangle = \langle V_1, V_1\rangle = \langle V_2, V_2\rangle = 0. \end{aligned} \tag{5.2}$$

Case I: $\langle X, X\rangle = \langle Y, Y\rangle = 1, \langle V_1, V_2\rangle = \pm 1, \langle X, Y\rangle = \langle X, V_1\rangle = \langle X, V_2\rangle = \langle Y, V_1\rangle = \langle Y, V_2\rangle = \langle V_1, V_1\rangle = \langle V_2, V_2\rangle = 0$.

In this case the metric has signature $(+,+,+,-)$. Using (5.1) and (5.2) to compute (1.1), we obtain that $W \in \mathfrak{g}$ is a geodesic vector if and only if its components (a, b, c, d) with respect to the basis $\{X, Y, V_1, V_2\}$ satisfy

$$\begin{cases} a(d-k) = 0 & dk = 0 \\ b(d+k) = 0 & b^2 - a^2 = \pm kc. \end{cases} \tag{5.3}$$

We note that if $k \neq 0$, then the above system admits only the null solution. If $k = 0$, then the solutions of the system are either of the form $(a, \pm a, c, 0)$ or of the form $(0, 0, c, d)$. Consequently, W must be one of the following forms:

$$W = a(X \pm Y) + cV_1, \qquad \forall (a, c) \in \mathbb{R}^2, \ a \neq 0, \tag{5.4}$$

$$W = cV_1 + dV_2, \qquad \forall (c, d) \in \mathbb{R}^2 - \{(0, 0)\}. \tag{5.5}$$

We note that each geodesic vector of the form (5.4) is a space-like vector. Each geodesic vector of the form (5.5) is a light-like vector if either $W = cV_1$ or $W = dV_2$. Otherwise it is space-like or time-like.

Note that the four geodesic vectors $\{X+Y, X-Y, V_1, V_2\}$ are linearly independent.

As consequence we get property c) of Theorem 1.1.

Case II: $\langle X,X\rangle = \langle Y,Y\rangle = -1, \langle V_1,V_2\rangle = \pm 1, \langle X,Y\rangle = \langle X,V_1\rangle = \langle X,V_2\rangle = \langle Y,V_1\rangle = \langle Y,V_2\rangle = \langle V_1,V_1\rangle = \langle V_2,V_2\rangle = 0$.

In this case the metric has signature $(-,-,-,+)$ thus the study of this case may be reduced to Case I, by reversing the metric.

6. Homogeneous geodesics of four-dimensional generalized symmetric pseudo-Riemannian spaces of Type D

According to Theorem 2.5, as homogeneous space, a four-dimensional generalized symmetric pseudo-Riemannian space of Type D is $M = G/H$, where

$$G = \begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $ad - bc = 1$. The Lie algebra $\mathfrak{g}$ of the Lie group G has a reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$; $\{U_1, U_2, U_3, U_4, A\}$ is a basis of $\mathfrak{g}$ with $\{U_1, U_2, U_3, U_4\}$ basis of $\mathfrak{m}$ and $\{A\}$ basis of $\mathfrak{h}$ such that the Lie bracket $[\ ,\ ]$ on $\mathfrak{g}$ and the scalar product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ are given by the following tables:

$[\ ,\ ]$	U_1	U_2	U_3	U_4	A
U_1	0	0	0	$-U_2$	U_1
U_2	0	0	$-U_1$	0	$-U_2$
U_3	0	U_1	0	$-A$	$2U_3$
U_4	U_2	0	A	0	$-2U_4$
A	$-U_1$	U_2	$-2U_3$	$2U_4$	0

Table VI

and

$\langle\ ,\ \rangle$	U_1	U_2	U_3	U_4
U_1	0	1	0	0
U_2	1	0	0	0
U_3	0	0	0	λ
U_4	0	0	λ	0

Table VII

with λ real constant, $\lambda \neq 0$ (see [CK] for more details). The metric corresponding to the above scalar product $\langle \, , \rangle$ has signature $(+,+,-,-)$.
Using Table VI and Table VII to compute (1.1), we obtain that $X \in \mathfrak{g}$, is a geodesic vector if and only if its components (a,b,c,d,α) with respect to the basis $\{U_1,U_2,U_3,U_4,A\}$ satisfy the system

$$\begin{cases} ad - b\alpha = kb & -b^2 - 2\lambda\alpha d = k\lambda d \\ bc + a\alpha = ka & -a^2 + 2\lambda\alpha c = k\lambda c. \end{cases} \tag{6.1}$$

If $k = 0$, the system (6.1) becomes:

$$\begin{cases} ad - b\alpha = 0 & -b^2 - 2\lambda\alpha d = 0 \\ bc + a\alpha = 0 & -a^2 + 2\lambda\alpha c = 0. \end{cases} \tag{6.2}$$

We study the two cases: $\alpha = 0$ and $\alpha \neq 0$.

If $\alpha = 0$, the only solutions of (6.1) are $(0,0,c,d,0)$.

If $\alpha \neq 0$, from the third and fourth equation of (6.2) we get:

$$d = -b^2/(2\lambda\alpha) \qquad c = a^2/(2\lambda\alpha) \tag{6.3}$$

now, substituting (6.3) in the first two equations of (6.2), we obtain:

$$b(ab + 2\lambda\alpha^2) = 0 \qquad a(ab + 2\lambda\alpha^2) = 0. \tag{6.4}$$

If $ab + 2\lambda\alpha^2 = 0$, the solutions of (6.2) are $(a,b,a^2/2\lambda\alpha,-b^2/2\lambda\alpha,\alpha)$.

If $ab + 2\lambda\alpha^2 \neq 0$, the only solution is $(0,0,0,0,\alpha)$.

Suppose $k \neq 0$; only light-like solutions can occur, so we must add to (6.2) the condition

$$ab + \lambda cd = 0. \tag{6.5}$$

Consider the new system

$$\begin{cases} ad - b\alpha = kb & -a^2 + 2\lambda\alpha c = k\lambda c \\ bc + a\alpha = ka & ab + \lambda cd = 0; \\ -b^2 - 2\lambda\alpha d = k\lambda d \end{cases} \tag{6.6}$$

from the first two equations of (6.6), rewritten respectively in the form $ad = b(k+\alpha)$ and $bc = a(k-\alpha)$ we obtain:

$$abcd = ab(k^2 - \alpha^2) \tag{6.7}$$

and from the third and fourth equation of (6.6), rewritten in the form $-b^2 = \lambda d(k+2\alpha)$ and $-a^2 = \lambda c(k-2\alpha)$ respectively, we get:

$$a^2b^2 = \lambda^2 cd(k^2 - 4\alpha^2). \tag{6.8}$$

We rewrite (6.7) and (6.8) taking account of (6.5) and we obtain:

$$ab[ab + \lambda(k^2 - \alpha^2)] = 0 \qquad ab[ab + \lambda(k^2 - 4\alpha^2)] = 0. \tag{6.9}$$

Suppose in (6.9) $ab = 0$, then it is either $a = 0$ or $b = 0$.

If $a = 0$, the solutions are: $(0, 0, 0, 0, \alpha)$, $(0, 0, 0, d, \alpha = -k/2)$, $(0, b, 0, b^2/\lambda k, \alpha = -k)$, $(0, 0, c, 0, \alpha = k/2)$.

If $b = 0$, we get the solutions: $(0, 0, 0, 0, \alpha)$, $(0, 0, 0, d, \alpha = -k/2)$, $(0, 0, c, 0, \alpha = k/2)$, $(a, 0, a^2/\lambda k, 0, \alpha = k)$.

Suppose in (6.9) $ab \neq 0$ then we must have $ab + \lambda(k^2 - \alpha^2) = 0$, and $ab + \lambda(k^2 - 4\alpha^2) = 0$, from which we get:

$$3\alpha^2\lambda = 0 \tag{6.10}$$

or equivalently $\alpha = 0$ (because $\lambda \neq 0$). So the system (6.6) becomes

$$\begin{cases} ad = bk & -a^2 = \lambda kc \\ bc = ak & ab + \lambda cd = 0. \\ -b^2 = \lambda kd \end{cases} \tag{6.11}$$

Calculating c and d from the third and fourth equation of (6.11), and replacing them in the first two, we have

$$\begin{cases} b(\lambda k^2 + ab) = 0 & c = -a^2/\lambda k \\ a(\lambda k^2 + ab) = 0 & ab + \lambda cd = 0 \\ d = -b^2/\lambda k \end{cases} \tag{6.12}$$

from which we get the solutions $(a, b, -a^2/\lambda k, -b^2/\lambda k, 0)$, but $\lambda k^2 + ab = 0$. Hence, X is a geodesic vector of a four-dimensional generalized symmetric pseudo-Riemannian space of Type D if and only if its m-component ($=X_m$) has one of the following forms:

$$X_m = cU_3 + dU_4, \qquad (c, d) \neq 0, \tag{6.13}$$

$$X_m = aU_1 + bU_2 + \left(\frac{a^2}{2\lambda\alpha}\right)U_3 - \left(\frac{b^2}{2\lambda\alpha}\right)U_4, \quad 2\lambda\alpha^2 + ab = 0, \tag{6.14}$$

$$X_m = bU_2 + \left(\frac{b^2}{k\lambda}\right)U_4, \tag{6.15}$$

$$X_m = aU_1 + \left(\frac{a^2}{k\lambda}\right) U_3, \tag{6.16}$$

$$X_m = aU_1 + bU_2 - (a^2/k\lambda)U_3 - (b^2/k\lambda)U_4, \qquad ab + k^2\lambda = 0. \tag{6.17}$$

but $k \neq 0$.

We note that each geodesic vector of the form (6.15), (6.16), (6.17) is a light-like geodesic vector.

Each geodesic vector of the form (6.13) is:

– a light-like vector if and only if either $c = 0$ or $d = 0$,

– a space-like (time-like) vector if and only if $\lambda cd > 0$ ($\lambda cd < 0$).

Each geodesic vector of the form (6.14) is either space-like or time-like vector; more precisely it is space-time (time-like) vector if and only if $\lambda < 0 \quad (\lambda > 0)$.

We note also that the four light-like geodesic vectors

$$\left\{ cU_3, dU_4, aU_1 + \left(\frac{a^2}{k\lambda}\right) U_3,\ bU_2 + \left(\frac{b^2}{k\lambda}\right) U_4 \right\}$$

where $a \neq 0$, $b \neq 0$, $c \neq 0$, $d \neq 0$, are always linearly independent. The corresponding family of four straight lines depends obviously on two arbitrary parameters a, b. So property d) of Theorem 1.1 is completely proved.

References

[AA] D. Alekseevsky and A. Arvanitoyeorgos, Metrics with homogeneous geodesics on flag manifolds, *Comm. Math. Univ. Carolinae* **43**, no. 2 (2002), 189–199.

[AS] D. Alekseevsky and J. Szenthe, Orbits of stable type in Lorentzian G-manifolds, in preparation.

[C] G. Calvaruso, Homogeneous structures on three-dimensional Lorentzian manifolds, *Journal Geom. Phys.* **57** (2007), 1279–1291.

[CK] J. Černý and K. Kowalski, Classification of generalized symmetric pseudo-Riemannian spaces of dimension $n \leq 4$, *Tensor, N. S.* **38**, no. 8 (1982), 256–267.

[CM1] G. Calvaruso and R. Marinosci, Homogeneous geodesics in five-dimensional generalized symmetric spaces, *Balkan J. Geom. Appl.* **8**, no. 1 (2002), 1–19.

[CM2] G. Calvaruso and R. Marinosci, Homogeneous geodesics of three-dimensional unimodular Lorentzian Lie groups, *Mediterr. J. Math.* **3** (2006), 467–481.

[CM3] G. Calvaruso and R. Marinosci, Homogeneous geodesics of non-unimodular Lorentzian Lie groups and naturally reductive Lorentzian spaces in dimension three, *Adv. Geom.* **8** (2008), 473–489.

[CKM] G. Calvaruso, O. Kowalski and R. Marinosci, Homogeneous geodesics in solvable Lie groups, *Acta Math. Hungarica* **101**, no. 4 (2003), 313–322.

[DK] Z. Dušek and O. Kowalski, On light-like homogeneous geodesics, *Publ. Math. Debrecen* **71**, no. 1–2 (2007), 245–252.

[FMeP] J. Figueroa-O'Farrill, P. Meessen and S. Philip, Homogeneity and plane wave limits, 2005, 1–32, arXiv: hep-th/0504069.

[KoNo] S. Kobayashi and K. Nomizu, Foundations of Differential Geometry, Vol. I, II, *Interscience Publ.*, *New York*, 1963, 1969.

[K1] O. Kowalski, Classification of generalized symmetric Riemannian spaces of dimension $n \leq 5$, *Rozpravy ČSAV, Řada MPV 85* **8** (1975).

[K2] O. Kowalski, Generalized Symmetric Spaces, Vol. 805, Lectures Notes in Math., *Springer-Verlag*, *Berlin, Heidelberg, New York*, 1980.

[KNVl] O. Kowalski, S. Nikčević and Z. Vlášek, Homogeneous geodesics in homogeneous Riemannian manifolds (examples), Geometry and Topology of Submanifolds, (Beijing/Berlin, 1999), *World Sci. Publishing co.*, *River Edge, NJ*, 2000, 104–112.

[KS] O. Kowalski and J. Szenthe, On the existence of homogeneous geodesics in homogeneous Riemannian manifolds, *Geom. Dedicata* **81** (2000), 209–214, Erratum: *Geom. Dedicata* **84**, (2001), 331–332..

[KV] O. Kowalski and L. Vanhecke, Riemannian manifolds with homogeneous geodesics, *Boll. Un. Mat. Ital.* **5** (1991), 189–246.

[KVl 1] O. Kowalski and Z. Vlášek, Homogeneous Riemannian manifolds with only one homogeneous geodesic, *Publ. Math. Debrecen* **62**, no. 3–4 (2003), 437–446.

[KVl 2] O. Kowalski and Z. Vlášek, On homogeneous geodesics in affine homogeneous manifolds, 2007, (*preprint*).

[KVl 3] O. Kowalski and Z. Vlášek, Homogeneous geodesics in (locally) homogeneous manifolds with affine connections, 2007, Preprint.

[M] R. A. Marinosci, Homogeneous geodesics in a three-dimensional Lie group, *Comm. Math. Univ. Carolinae* **43**, no. 2 (2002), 261–270.

[Me] P. Meessen, Homogeneous Lorentzian spaces whose null-geodesics are canonically homogeneous, *Letters Math. Phys.* **75** (2006), 209–212.

[O'N] B. O'Neill, Semi-Riemannian Geometry, *Academic Press*, *New York*, 1983.

[P] S. Philip, Penrose limits of homogeneous spaces, *J. Geom. Phys.* **56** (2006), 1516–1533.

[S] J. Szenthe, On the set of homogeneous geodesics of a left-invariant metric, *Universitatis Iagellonicae Acta Mathematica*, Fasciculus XL (2002), 171–181.

[TV] F. Tricerri and L. Vanhecke, Homogeneous Structures on Riemannian Manifolds, Vol. 83, London Math. Soc. Lecture Notes Series, *Cambridge Univ. Press*, 1983.

BARBARA DE LEO
UNIVERSITÁ DEL SALENTO
DIPARTIMENTO DI MATEMATICA "E. DE GIORGI"
VIA PROVINCIALE LECCE-ARNESANO
73100 LECCE
ITALY

E-mail: barbara.deleo@unile.it

ROSA ANNA MARINOSCI
UNIVERSITÁ DEL SALENTO
DIPARTIMENTO DI MATEMATICA "E. DE GIORGI"
VIA PROVINCIALE LECCE-ARNESANO
73100 LECCE
ITALY

E-mail: rosanna.marinosci@unile.it

(Received February 2, 2008; revised June 25, 2008)