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Pre-Hausdorff spaces

By JAY STINE (Dallas) and M. V. MIELKE (Coral Gables)

Abstract. This paper introduces three separation conditions for topological spaces, called $T_{0,1}$, $T_{0,2}$ ("pre-Hausdorff"), and $T_{1,2}$. These conditions generalize the classical T_1 and T_2 separation axioms, and they have advantages over them topologically which we discuss. We establish several different characterizations of pre-Hausdorff spaces, and a characterization of Hausdorff spaces in terms of pre-Hausdorff. We also discuss some classical Theorems of general topology which can or cannot be generalized by replacing the Hausdorff condition by pre-Hausdorff.

Introduction

The notion of separation is fundamental to topology. Even so, the classical separation axioms $(T_0, \dots, T_4)$ are sometimes overlooked in a first course or, alternatively, some consider the T_2 axiom (for instance) as being sufficiently weak that all spaces are assumed T_2 and no further consideration is given to separation. While this may be reasonable in some settings, it is certainly not in others. Analysis often takes place in the setting of metric spaces, which are T_2, whereas geometry often uses pseudometric (more generally, uniform) spaces which are not necessarily T_2. HERRLICH argues (in [7]) that "there are sufficient reasons for topologists to pay serious attention to non-Hausdorff spaces ... finite Hausdorff spaces are rare and not very interesting ... a 14-element set carries just a single Hausdorff topology but 98, 484, 324, 257, 128, 207, 032, 183 T_0 topologies". In this paper we define a generalized Hausdorff separation condition called pre-Hausdorff, which is satisfied by many important non-Hausdorff spaces. In [15]

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it is shown that a uniform space is T_0 if and only if it is T_2, and the proof of this reveals that all uniform spaces are pre-Hausdorff. Following Herrlich's argument above, the worthiness of studying pre-Hausdorff spaces can be justified by their abundance: a 14-element set carries 190, 899, 322 distinct pre-Hausdorff topologies (see Corollary 2.3).

The paper is organized as follows. Section 1 contains definitions of three separation axioms for topological spaces and examples to show how they are related. We prove that the categories formed by the spaces which satisfy these axioms are topological categories and, further, that these categories are reflective in the category of topological spaces. In Section 2, we give several characterizations of pre-Hausdorff spaces in terms of Hausdorff separation and some equivalence relations. Finally, in Section 3, we consider some classical Theorems of general topology which can or cannot be generalized by replacing the Hausdorff condition by pre-Hausdorff.

Throughout the paper, TOP will be used to denote the category of topological spaces and continuous functions. For $i = 0, 1, 2$, $T_i - TOP$ will denote the full subcategories of TOP consisting of the T_i spaces (see [4], page 138).

1. $T_{i,j}$-spaces

Definition 1.1. A topological space X is called a $T_{i,j}$-space (for $0 \leq i < j \leq 2$) if and only if each pair of points $a, b \in X$ which has a T_i-separation in X also has a T_j-separation in X.

Notation 1.2. The categories consisting of the $T_{i,j}$-spaces, along with continuous functions, will be denoted $T_{i,j} - TOP$.

Example 1.3. $T_{0,2}$ spaces have been referred to as pre-Hausdorff spaces in the literature (see [14]).

Example 1.4. $T_{0,1}$ spaces have been refered to as R_0-spaces in the literature (see [8]). An R_0-space is a topological space X which satisfies: $x \in \overline{\{y\}}$ (the topological closure of $\{y\}$) if and only if $y \in \overline{\{x\}}$, for all pairs of points $x, y \in X$. Evidently then, every neighborhood of x contains y if and only if every neighborhood of y contains x. Now if x and y have no T_0 separation, then this condition is satisfied. On the other hand, if x and y do have a T_0 separation, say x has a neighborhood not containing y, then y must have a neighborhood not containing x; i.e., x and y must have a T_1 separation. Thus R_0-spaces are exactly the $T_{0,1}$-spaces.

Example 1.5. Clearly T_j spaces are always $T_{i,j}$, but T_i spaces need not be $T_{i,j}$. A Sierpinski space (i.e., a two-point set , say $X = \{0,1\}$, with one proper open set, say $\{1\}$), for instance, is T_0, but neither $T_{0,1}$ nor $T_{0,2}$; while a T_1 space which is not T_2 will not be $T_{1,2}$. Furthermore, a $T_{i,j}$ space need not be either T_i or T_j. An indiscrete space with more than one element, for instance, is $T_{i,j}$ for each i, j, but is not T_0.

Example 1.6. Clearly $T_{0,2}$ spaces are both $T_{0,1}$ and $T_{1,2}$. However, a $T_{1,2}$ space need not be $T_{0,1}$ (and, consequently, not $T_{0,2}$ either), as in the case of a Sierpinski space, for example. Furthermore, a $T_{0,1}$ space may be neither $T_{0,2}$ nor $T_{1,2}$. This is the case if, for example, a space is T_1 but not T_2.

The following theorem shows that the categories $T_{i,j} - TOP$ have a desirable property that is not shared by the categories $T_i - TOP$.

Theorem 1.7. *The full subcategories $T_{i,j} - TOP$ are themselves topological over SET (the category of sets and functions). Moreover, their inclusions into TOP preserve initial lifts and, consequently, they preserve all limits.*

Proof. We prove the Theorem for $T_{0,2} - TOP$, the cases $T_{0,1} - TOP$ and $T_{1,2} - TOP$ being similar. Clearly the restriction of the forgetful functor $U : TOP \to SET$ is both concrete and has set-theoretic fibers. So we show that the structure induced on a set from an arbitrary family of $T_{0,2}$ spaces yields a $T_{0,2}$ space. This will also show that initial lifts in $T_{0,2} - TOP$ are computed as they are in TOP and, thus, the inclusion functor preserves them. Suppose that (X,τ) is the induced topological space on a set X from a family $\{(X_i,\tau_i)\}_{i\in I}$ of $T_{0,2}$ spaces via a family of functions $\{f_i : X \to X_i\}_{i\in I}$. Further suppose that $x, y \in X$ have a T_0-separation in τ by, say, $U_x \in \tau$, where $x \in U_x$ and $y \notin U_x$. We can assume that U_x is a basis element of τ so that $U_x = \bigcap_{j=1}^{n} f_{i_j}^{-1}(V_j)$, where each V_j is open in X_{i_j} for each $j = 1, 2, \ldots, n$. Then $\exists k$, $1 \le k \le n$, with $f_{i_k}(x) \in V_k$ and $f_{i_k}(y) \notin V_k$; i.e., $f_{i_k}(x)$ and $f_{i_k}(y)$ have a T_0-separation in X_{i_k}. Since X_{i_k} is $T_{0,2}$, $\exists$ neighborhoods U and W of $f_{i_k}(x)$ and $f_{i_k}(y)$ (resp.) such that $U \cap W = \emptyset$. Therefore (X,τ) is $T_{0,2}$. □

Corollary 1.8. *The inclusion functors* $\mathrm{inc}_{i,j} : T_{i,j} - TOP \to TOP$ *each have a left adjoint $L_{i,j}$.*

Proof. Note that any indiscrete space with two elements forms a small (one element) cogenerating set for any of the categories $T_{i,j} - TOP$. Since the functors $\mathrm{inc}_{i,j}$ are continuous by Theorem 1.7, the result follows immediately from the Corollary on page 126 of [12]. □

Note: An explicit description of the left adjoint to $\mathrm{inc}_{0,2}$ is given below, in the discussion following Theorem 2.18. Another description of $L_{0,2}$ using transfinite recursion can be found in [19], where this approach is then adapted to give an explicit description of the functor $L_{0,1} : TOP \to T_{0,1} - TOP$. It is also shown there that these left adjoints are retractions.

2. Pre-Hausdorff spaces

This section is concerned specifically with $T_{0,2} - TOP$, the category of pre-Hausdorff spaces. In [18] Steiner defines principal topologies in terms of ultratopologies, and proves that a topological space is principal if and only if arbitrary intersections of open sets are open (such spaces are also referred to as Alexandroff spaces in the literature, see [2]; however the term Alexandroff space also appears in a different context, see [3]). The following result will be used to gain insight into finite pre-Hausdorff spaces, and to count the number of distinct pre-Hausdorff topologies on a a finite set. We note that in this result, as in [11] page 113, we make a distinction between regular spaces and T_3 spaces; namely, that regular spaces need not have closed points.

Theorem 2.1. *Suppose (X, τ) is a principal space. The following are equivalent:*

(1) *(X, τ) is pre-Hausdorff.*

(2) *(X, τ) is regular.*

(3) *(X, τ) has dimension 0; i.e., has a basis consisting of clopen sets (see [9], page 10, B).*

(4) *The topos of sheaves on X is Boolean; i.e., the negation operator $\neg : \tau \to \tau$ satisfies $\neg\neg = id$ (see [13], page 270).*

Proof. (1) $\Leftrightarrow$ (2) Suppose $A \subset X$ is closed and $p \in A^C$. Then p has a T_0-separation from each point $a \in A$. If X is pre-Hausdorff, then $(\forall a \in A)$ $(\exists N_a, N_{p_a} \in \tau)$ such that $a \in N_a$, $p \in N_{p_a}$, and $N_a \cap N_{p_a} = \varnothing$. Then $p \in U = \bigcap_{a \in A} N_{p_a}$, $A \subset V = \bigcup_{a \in A} N_a$, and $U \cap V = \varnothing$. Since X is principal we have that U is open and, consequently, X is regular.

Conversely, suppose that $x, y \in X$ have a T_0-separation by, say, $U_x \in \tau$, where $x \in U_x$ and $y \notin U_x$. Then U_x^C is closed so, if X is regular, there are disjoint open sets U and V such that $x \in U$ and $U_x^C \subset V$. Thus, X is pre-Hausdorff.
(2) $\Leftrightarrow$ (3) See [2], Theorem 2.9.

(3) $\Leftrightarrow$ (4) In the topos of sheaves on X, the negation operator $\neg : \tau \to \tau$ is defined by $\neg U = \text{interior}(U^C)$ (see [13], Chapter 2). Then $\neg\neg U = \text{interior}(\overline{U})$, and so $\neg\neg U = U$ iff $U = \text{interior}(\overline{U})$; i.e., iff U is a regular open set (see [4], page 92). It is easily shown that an open set is regular iff it is clopen. $\square$

Remarks 2.2.

(1) The proof of Theorem 2.1 shows that a regular space is pre-Hausdorff even if it is not principal. Clearly the converse is false; for if X is a Hausdorff space which is not T_3, then X is pre-Hausdorff but not regular.

(2) Also, a 0-dimensional space is pre-Hausdorff even if it is not principal. However this is not true conversely; for the set of real numbers $\mathbb{R}$ with the usual open interval topology is a (pre-)Hausdorff space which is not 0-dimensional. In fact, $\dim(\mathbb{R}) = 1$ (see [9], page 25, Example III).

In [5] it is shown that a principal topological space (X, τ) is regular if and only if the minimal basis for τ forms a partition of X. Consequently, we have the immediate

Corollary 2.3. *If X is a finite set, then the distinct pre-Hausdorff topologies on X are in one-to-one correspondence with the distinct partitions on X.*

In [5] there is an algorithm using matrices, and a computer program, to compute the number of regular (hence, pre-Hausdorff) topologies on a finite set. Alternatively, several methods for counting the number of partitions on a set with n-elements, the so-called "n-th Bell Number" $B(n)$, are well-known (see [17], page 33). The 14th Bell number, for instance, is $B(14) = 190,899,322$, which is accordingly the number of distinct pre-Hausdorff topologies on a 14-element set as mentioned in the introduction.

Finite (and other) pre-Hausdorff spaces can also be described using the notion of a Borel field.

Definition 2.4. A Borel field F (on a fixed set B) is a non-empty family of subsets of B such that F is closed with respect to complements and countable unions; i.e., F satisfies:

(1) If $A \in F$, then $A^C \in F$.

(2) If $\{A_i\}_{i=1}^{\infty} \subset F$, then $\bigcup_{i=1}^{\infty} A_i \in F$.

Remarks 2.5.

(1) A Borel field is also known as a σ-algebra (see [16], page 17, for instance).

(2) If F is a Borel field on B, then clearly $B \in F$ and $\varnothing \in F$.

(3) It follows immediately from DeMorgan's laws that a Borel field is also closed with respect to countable intersections; i.e., if $\{A_i\}_{i=1}^{\infty} \subset F$, then $\bigcap_{i=1}^{\infty} A_i \in F$.

(4) If F is a Borel field on B and F is countable, then (B, F) is a topological space which has the following properties:

   (a) arbitrary intersections of open sets are open, and

   (b) every open set is also closed; i.e., every open set is clopen (both open and closed).

Corollary 2.6. *Suppose X is a finite set and τ is a family of subsets of X. τ is a Borel field if and only if (X, τ) is a pre-Hausdorff space.*

PROOF. Follows immediately from Remark 2.5 (4) and Theorem 2.1. □

Clearly this result is also true for any set X if τ is countable.

In [20], SZEKERES and BINET prove that the set of all Borel fields on a finite set is in one-to-one correspondence with the number of equivalence relations on that set. It is well known that the number of equivalence relations on a finite set are in one-to-one correspondence with the number of partitions on that set. Consequently, Corollary 2.6 is equivalent to Corollary 2.3.

Of the 190, 899, 322 distinct pre-Hausdorff topologies on a set with 14 elements there are, of course, many which are homeomorphic. To characterize homeomorphic pairs of finite pre-Hausdorff spaces, we look at the basis consisting of the "minimal" open sets which, by the proof of Corollary 2.3, forms a partition. For a finite pre-Hausdorff space X, we shall denote this partition which generates X by P_X. The following shows that for finite pre-Hausdorff spaces to be homeomorphic, their generating partitions must "look" the same.

Proposition 2.7. *Finite pre-Hausdorff spaces (X, τ) and (Y, σ) are homeomorphic if and only if there exists a bijective correspondence between P_X and P_Y that preserves the cardinality of the corresponding blocks.*

PROOF. If X and Y are homeomorphic, then the condition on their generating partitions follows immediately since a homeomorphism is a bijective open mapping. Conversely, suppose there exists a bijective correspondence between P_X and P_Y which preserves the cardinality of the corresponding blocks, and that $P_X = \{B_i\}_{i=1}^{k}$ and $P_Y = \{C_i\}_{i=1}^{k}$ are labeled so that B_i and C_i each have the same cardinality for all $i = 1, 2, \ldots, k$. Then, for each i, we can choose a bijection $f_i : B_i \to C_i$. The function $f : X \to Y$ defined by $f(x) = f_i(x)$, for $x \in B_i$, is clearly a homeomorphism. □

It follows from Proposition 2.7 that the number of non-homeomorphic pre-Hausdorff spaces on a set with n-elements is $p(n)$ = the number of partitions of n according to the following.

Definition 2.8. A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$.

The problem of computing $p(n)$ in general is complex and has received much attention from mathematicians, especially after the landmark paper by G. H. Hardy and S. Ramanujan in 1918 ([6]). A comprehensive summary of results can be found in [1], where there is also a table of values for $p(n)$ up to $p(100)$. Thus, the number of non-homeomorphic pre-Hausdorff topologies on a set with 14 elements is $p(14) = 135$ (see [1], page 238).

Suppose X is a topological space, and $B \subset X$. Recall that a point $b \in X$ is called a generic point of B provided $\overline{\{b\}} = B$, and that X is called sober provided every closed irreducible (i.e., cannot be decomposed into a union of two or more smaller closed subsets) subset of X has a unique generic point. See [10], page 230 for an interesting explanation of the term sober.

Theorem 2.9. *A topological space (X, τ) is Hausdorff if and only if X is both pre-Hausdorff and sober.*

Proof. The implication to the right is immediate because Hausdorff spaces are naturally pre-Hausdorff and, furthermore, they are always sober (see [13], page 475).

Now suppose that X is both pre-Hausdorff and sober, but not Hausdorff. Then $\exists x, y \in X$ such that $x \neq y$, and x and y have no T_2 separation in τ. Then x and y have no T_0 separation in τ either, which implies $\overline{\{x\}} = \overline{\{y\}}$. But then $\overline{\{x\}}$ is a closed irreducible subset of X with more than one generic point. $\square$

It is well-known that a topological space X is Hausdorff if and only if the diagonal $\Delta_X (= \{(x, x) : x \in X\})$ is closed in the product space X^2. Analogous results for a pre-Hausdorff space (X, τ) are given in terms of the following relation R_0 on X.

$$R_0 = \{(x, y) : x \text{ and } y \text{ have no } T_0 \text{ separation in } \tau\} \subset X^2.$$

Clearly R_0 is an equivalence relation.

Theorem 2.10. *The following are equivalent:*

(1) *X is pre-Hausdorff.*

(2) *R_0 is closed in X^2.*

(3) *$R_0 = \overline{\Delta_X}$.*

(4) *The quotient space $\frac{X}{R_0}$ is Hausdorff.*

PROOF. We show that each of (2), (3), and (4) is equivalent to (1).

(2) If R_0 is closed and $x, y \in X$ have a T_0 separation in τ, then $(x, y) \notin R_0 = \overline{R_0}$. So $\exists U, V \in \tau$ such that $(x, y) \in U \times V$ and $(U \times V) \cap R_0 = \varnothing$. But this implies that U and V are disjoint, for if $p \in U \cap V$, then $(p, p) \in (U \times V) \cap R_0$. Therefore x and y have a T_2 separation, and X is pre-Hausdorff.

Conversely, if X is pre-Hausdorff and $(x, y) \in R_0^C$, then x and y have a T_2 separation in τ by, say, N_x and N_y. Then $(x, y) \in N_x \times N_y \subset R_0^C$, so R_0 is closed.

(3) That (3) implies (1) follows immediately from (2). For the reverse implication, we have $\Delta_X \subset R_0 = \overline{R_0}$ (since X is pre-Hausdorff) so that $\overline{\Delta_X} \subset R_0$. For the reverse inclusion, suppose a point $(x, y) \in R_0 = \overline{R_0}$ has neighbourhood $U \times V$ in X^2. Then $(x, x) \in U \times V \cap \Delta_X$, so that $(x, y) \in \overline{\Delta_X}$.

(4) Suppose $\frac{X}{R_0}$ is Hausdorff, and that distinct points $x, y \in X$ have a T_0 separation by a neighbourhood of x. Then $[x](= \{z : z\ R_0\ x\}) \neq [y]$, so that $[x]$ and $[y]$ have a T_2 separation in $\frac{X}{R_0}$. Since the canonical map $q : X \to \frac{X}{R_0}$ is continuous, x and y have a T_2 separation in X.

Conversely if X is pre-Hausdorff, then R_0 is closed by (2). Furthermore, $q : X \to \frac{X}{R_0}$ is easily seen to be an open map. Consequently $\frac{X}{R_0}$ is Hausdorff (see [4], 1.6, page 140). □

We now show that any space can be universally retracted onto a Hausdorff space in the sense of adjunction as follows.

Lemma 2.11. *If (X, τ) is any topological space, (Y, σ) is a T_0 space, and $f : X \to Y$ is continuous, then f factors uniquely through the quotient map $q : X \to \frac{X}{R_0}$; i.e., $\exists!$ continuous $\overline{f} : \frac{X}{R_0} \to Y$ such that $f = \overline{f} \circ q$.*

PROOF. Define $\overline{f}([x]) = f(x)$. Then $\overline{f}$ is well-defined since Y is T_0, and $\overline{f}$ is continuous since $\frac{X}{R_0}$ is equipped with the coinduced topology; i.e., the quotient topology in TOP. □

Theorem 2.12. *The inclusion functor* $\mathrm{inc}_{2,2} : T_2 - Top \hookrightarrow T_{0,2} - Top$ *has a left adjoint $L_{2,2}$ which is a retract.*

PROOF. Define $L_{2,2}(X) = \frac{X}{R_0}$. By Lemma 2.11, the quotient map $q : X \to \frac{X}{R_0}$ provides a universal arrow from any pre-Hausdorff space X to the Hausdorff space $\frac{X}{R_0}$. The object $\frac{X}{R_0}$ and the universal arrow q completely determine the left adjoint to $\mathrm{inc}_{2,2}$ (see [12], Theorem 2 (ii), page 81). $L_{2,2}$ is a retract by Theorem 2.10 (3). □

Corollary 2.13. *The inclusion functor* $\mathrm{inc}_2 : T_2 - TOP \hookrightarrow TOP$ *has a left adjoint* L_2 *which is a retract.*

PROOF. Combining Corollary 1.8 with Theorem 2.12, we define $L_2 = L_{2,2} \circ L_{0,2}$. □

The functor L_2 can be described without the use of $L_{0,2}$ and $L_{2,2}$. To this end, we now construct L_2 directly by way of forming quotients by an equivalence relation. We take a general approach which also shows $T_0 - TOP$ and $T_1 - TOP$ to be reflective, and gives an explicit description of the left adjoints to their inclusions into TOP.

Definition 2.14. Let (X, τ) be a topological space. For each $i = 0, 1, 2$, define a relation R_i on X by:

$$(x, y) \in R_i \text{ iff } \forall Y \in T_i - TOP, \forall \text{ continuous } f : X \to Y,\ f(x) = f(y).$$

Remark 2.15. R_0 as defined in 2.14 equals R_0 as defined above.

Lemma 2.16. *For each* $i = 0, 1, 2$ *we have the following:*

(1) R_i *is an equivalence relation.*

(2) *If* $Y \in T_i - TOP$ *and* $f : X \to Y$ *is continuous, then* f *factors through the quotient map* $q : X \to \frac{X}{R_i}$.

(3) $\frac{X}{R_i} \in T_i - TOP$.

PROOF.

(1) Straightforward.

(2) Given a continuous function $f : X \to Y$ with $Y \in T_i - TOP$, define $\overline{f} : \frac{X}{R_i} \to Y$ by $\overline{f}([x]) = f(x)$. Then $\overline{f}$ is well-defined by definition of R_i, and $\overline{f}$ is continuous since $\frac{X}{R_i}$ has the quotient topology in TOP.

(3) Suppose that $[x] \neq [y]$ in $\frac{X}{R_i}$. Then $\exists Y \in T_i - TOP$ and $\exists$ continuous $f : X \to Y$ with $f(x) \neq f(y)$, which implies that $f(x)$ and $f(y)$ have a T_i-separation in Y.

Then $[x]$ and $[y]$ have a T_i-separation in $\frac{X}{R_i}$ via the inverse image of $\overline{f} : \frac{X}{R_i} \to Y$.

□

Theorem 2.17. *For each* $i = 0, 1, 2$, *the inclusion functor* $\mathrm{inc}_i : T_i - TOP \hookrightarrow TOP$ *has a left adjoint* $L_i : TOP \to T_i - TOP$. *Moreover, each* L_i *is a retract.*

Proof. Define $L_i((X,\tau)) = \frac{X}{R_i}$. Then, by Lemma 2.16, $L_i((X,\tau)) \in T_i - TOP$, and the quotient map $q : X \to \frac{X}{R_i}$ is universal among all arrows from X into a T_i-space. If $X \in T_i - TOP$ then, clearly, $L_i(X) = X$. $\square$

The functor $L_{0,2}$ of Corollary 1.8 can also be explicitly described using the equivalence relation R_2. Indeed, if (X,τ) is a topological space, then $\frac{X}{R_2}$ is Hausdorff by 2.16 (3). So (X,τ_2), the topological space induced from $\frac{X}{R_2}$ via $q : X \to \frac{X}{R_2}$ will be pre-Hausdorff. It is readily shown that the assignment $(X,\tau) \longmapsto (X,\tau_2)$ is left adjoint to the inclusion $T_{0,2} - TOP \hookrightarrow TOP$.

3. Replacing Hausdorff with pre-Hausdorff

There are many known results in topology which concern Hausdorff spaces. Given such a result, a natural question is whether or not the result remains true when Hausdorff is replaced with pre-Hausdorff. In this section we point out some standard Theorems which can be generalized to the pre-Hausdorff setting, and some which cannot.

Pre-Hausdorff topologies share some invariance properties with Hausdorff topologies:

Proposition 3.1.

(1) *Each subspace of a pre-Hausdorff space is also pre-Hausdorff.*

(2) *The Cartesian product of pre-Hausdorff spaces is also pre-Hausdorff.*

Proof. The proof of (1) is straightforward, (2) follows immediately from Theorem 1.7. $\square$

In the following result, as in [11] page 112, we make a distinction between normal spaces and T_4 spaces; namely, normal spaces need not have closed points. Recall that every compact Hausdorff space is normal (see [11], page 141).

Theorem 3.2. *Every compact pre-Hausdorff space is normal.*

Proof. Suppose (X,τ) is a compact pre-Hausdorff space. Since the theorem is trivially true when τ is the indiscrete topology, we assume that it is not and choose a closed set $A \subset X$ and a point $x \notin A$. For each $y \in A$, A^C provides a T_0 separation of x and y. Since X is pre-Hausdorff, x and y have a T_2 separation by, say $N_y \ni y$ and $N_{x,y} \ni x$. Then $\{N_y\}_{y\in A}$ is an open cover of A. Since A is compact, $\exists y_1,\ldots,y_n \in A$ such that $U = \bigcap_{i=1}^n N_{x,y_i}$ and $V = \bigcup_{i=1}^n N_{y_i}$ are open, and they provide a disjoint separation of x and A.

Now suppose that A and B are disjoint, closed sets in X. By the above we have, $\forall a \in A$, $\exists U_a$, $U_{a,B} \in \tau$ such that $a \in U_a$, $B \subset U_{a,B}$, and $U_a \cap U_{a,B} = \varnothing$. Since A is compact, $\exists a_1, \dots, a_n$ such that $U_A = \bigcup_{i=1}^{n} U_{a_i}$ and $V_B = \bigcap_{i=1}^{n} U_{a_i,B}$ are both open, and they provide a disjoint separation of A and B. $\square$

A special property of Hausdorff topologies is the following:

Each finite subset of a Hausdorff space is closed.

This is clearly not shared with pre-Hausdorff topologies; for if X is a finite indiscrete space with more than one element, for instance, and A is any non-void proper subset of X, then A is not closed.

Many important results involve mating compactness with the Hausdorff property. An intriguing feature of compact Hausdorff spaces is that they are essentially algebraic. Indeed, a well-known result is that the category of compact Hausdorff spaces is algebraic (or "monadic") over the category of sets (see [12], chapter 6). A fact which is crucial in proving this is the following:

If X is compact and Y is Hausdorff, then any continuous function $f : X \to Y$ is a closed map.

This result is clearly false for pre-Hausdorff spaces; for example, if we map a compact space (X, τ) which is not indiscrete into X with the indiscrete topology by the identity function, then we have a continuous bijection which is not a closed map. Consequently, this identity map is not a homeomorphism.

It is easily shown that if a category A is algebraic over a category B (i.e., A is isomorphic to a category of T-algebras, where T is a monad in X determined by an adjunction), and $L \dashv R : A \to B$ is the adjoint pair of functors which determines the isomorphism, then A satisfies: if $f : a \to b$ is any morphism in A, and $R(f) : R(a) \to R(b)$ is an isomorphism in B, then f is an isomorphism in A. In the case of compact Hausdorff spaces over the category of sets, this reveals the well-known fact that a continuous bijection of compact Hausdorff spaces is a homeomorphism. Since the example in the preceding paragraph shows a continuous bijection from a compact space to a pre-Hausdorff space which is not a homeomorphism, we conclude that the category of compact pre-Hausdorff spaces is not algebraic over the category of sets.

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JAY STINE
DEPARTMENT OF MATHEMATICS
MISERICORDIA UNIVERSITY
DALLAS, PA 18612 USA

E-mail: jstine@misericordia.edu

M. V. MIELKE
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MIAMI
CORAL GABLES, FL. 33124
USA

E-mail: mielke@math.miami.edu

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