Publ. Math. Debrecen
44 / 1–2 (1994), 79–103

Measurable solutions of a (2,2)–type nonlinear functional equation of sum form with several unknown functions

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To the memory of Professor András Rapcsák

Abstract. We determine the measurable solutions of the functional equation

$$\text{(E)}\quad f_{11}(xy)+f_{12}\left(x(1-y)\right)+f_{21}\left((1-x)y\right)+f_{22}\left((1-x)(1-y)\right)= \\ =g(x)h(y)\quad (x,y\in\,]0,1[\,)$$

where $f_{ij},g,h:\,]0,1[\,\to\mathbf{C}$ $(i,j=1,2)$ are (unknown) functions. In [9] we proved that the solution of (E) is equivalent to the solution of a system of equations consisting of 4 equations which are of the above type but their left hand sides contain only one function. The measurable solution of these individual equations can be found in [8], [9], [10]. The solution of (E) is obtained by finding the solution of the above mentioned system. There are 19 solution classes.

1. Introduction

In [9] we proved that the functions $f_{ij},g,h:\,]0,1[\,\to\mathbf{C}$ $(i,j{=}1,2)$ satisfy the functional equation

$$\text{(E)}\quad f_{11}(xy)+f_{12}\left(x(1-y)\right)+f_{21}\left((1-x)y\right)+f_{22}\left((1-x)(1-y)\right)= \\ =g(x)h(y)\quad (x,y\in\,]0,1[\,)$$

if and only if the functions $F_{ij},\ G_i,\ H_j$ $(i,j=1,2)$ defined by

$$(1)\quad F_{ij}(x):=\tfrac{1}{4}\left[f_{11}(x)+(-1)^{j+1}f_{12}(x)+\right.$$

1980 *Mathematics Subject Classification* (1985 *Revision*): 39 B 22, 39 B 99.
Keywords: Functional equation, functional equation of sum form, measurable solution.
Research supported by the Hungarian National Research Science Foundation, Operating Grant Number OTKA 1652.

$$+(-1)^{i+1}f_{21}(x)+(-1)^{i+j+2}f_{22}(x)\big] \quad (x\in\,]0,1[\,)$$

(2) $$G_i(x):=\tfrac{1}{2}\left[g(x)+(-1)^{i+1}g(1-x)\right] \quad (x\in\,]0,1[\,)$$

(3) $$H_j(x):=\tfrac{1}{2}\left[h(x)+(-1)^{j+1}h(1-x)\right] \quad (x\in\,]0,1[\,)$$

satisfy for $x,y\in\,]0,1[$ the following system of equations:

(S) $$\begin{cases} F_{11}(xy)+F_{11}\left(x(1-y)\right)+F_{11}\left((1-x)y\right)+ \\ \qquad +F_{11}\left((1-x)(1-y)\right)=G_1(x)H_1(y) \\ F_{12}(xy)-F_{12}\left(x(1-y)\right)+F_{12}\left((1-x)y\right)- \\ \qquad -F_{12}\left((1-x)(1-y)\right)=G_1(x)H_2(y) \\ F_{21}(xy)+F_{21}\left(x(1-y)\right)-F_{21}\left((1-x)y\right)- \\ \qquad -F_{21}\left((1-x)(1-y)\right)=G_2(x)H_1(y) \\ F_{22}(xy)-F_{22}\left(x(1-y)\right)-F_{22}\left((1-x)y\right)+ \\ \qquad +F_{22}\left((1-x)(1-y)\right)=G_2(x)H_2(y) \end{cases}$$

In possession of the solutions of (S) the solutions of equation (E) can be obtained by

(4) $$f_{ij}(x)=F_{11}(x)+(-1)^{j+1}F_{12}(x)+(-1)^{i+1}F_{21}(x)+ \\ +(-1)^{i+j+2}F_{22}(x)$$

(5) $$g(x)=G_1(x)+G_2(x)$$

(6) $$h(x)=H_1(x)+H_2(x) \quad (i,j=1,2;\ x\in\,]0,1[\,).$$

Equation (E) arises as generalization of functional equations of sum form characterizing information measures having the sum property (see e.g. [1], [8]; concerning related equations see [2], [3], [4], [5], [6], [7]). Equation (E) is also of interest from the functional equationist's point of view because of its complexity.

2. Solution of the equations of the system (S)

Slightly changing the notations let us write the individual equations of the sytem (S) in the form

(A) $$F_{11}(xy)+F_{11}(x(1-y))+F_{11}((1-x)y)+ \\ +F_{11}((1-x)(1-y))=G_1(x)H_1(y),$$

(B) $\quad F_{12}(xy) - F_{12}(x(1-y)) + F_{12}((1-x)y) -$
$$- F_{12}((1-x)(1-y)) = K_1(x)L_2(y),$$

(C) $\quad F_{21}(xy) + F_{21}(x(1-y)) - F_{21}((1-x)y) -$
$$- F_{21}((1-x)(1-y)) = K_2(x)L_1(y),$$

(D) $\quad F_{22}(xy) - F_{22}(x(1-y)) - F_{22}((1-x)y) +$
$$+ F_{22}((1-x)(1-y)) = G_2(x)H_2(y),$$

where $x, y \in \,]0,1[$.

Supposing the measurability of the functions F_{ij} equations (A), (B), (C), (D) have been solved in [8], [10], [10], [9] respectively.

Below we give the measurable solutions in the form of two tables.

A1, ... ,D6 refer to the solutions of equations (A), ... ,(D) respectively (first column of the tables). The next columns contain the functions G_1 or K_1, H_1 or L_1, G_2 or K_2, H_2 or L_2. The solutions F_{ij} can be written as

(7) $$F_{ij}(x) = F^*_{ij}(x) + F^{**}_{ij}(x) \quad (i,j = 1,2;\ x \in \,]0,1[\,)$$

where F^*_{ij} is the solution of the corresponding homogeneous equation. Hence f_{ij} can also be decomposed as

(8) $$f_{ij}(x) = f^*_{ij}(x) + f^{**}_{ij}(x) \quad (i,j = 1,2;\ x \in \,]0,1[\,)$$

where f^*_{ij} is the solution of the homogeneous equation

$$f_{11}(xy) + f_{12}\left(x(1-y)\right) + f_{21}\left((1-x)y\right) + f_{22}\left((1-x)(1-y)\right) = 0$$

corresponding to (E). By a result of Kannappan and Ng [5]

(9) $$f^*_{ij}(x) = a(x - \frac{1}{4}) + (-1)^{j+1}f + (-1)^{i+1}e + \\ + (-1)^{i+j}\left[b + c\log x + d(x^2 - x)\right] \quad (x \in \,]0,1[\,)$$

with arbitrary constants $a, b, c, d, e, f \in \mathbf{C}$. Further, corresponding to (7), we have

(10) $$f^{**}_{ij}(x) = F^{**}_{11}(x) + (-1)^{j+1}F^{**}_{12}(x) + \\ + (-1)^{i+1}F^{**}_{21}(x) + (-1)^{i+j}F^{**}_{22}(x) \quad (i,j = 1,2; x \in \,]0,1[\,).$$

In the last column of our second table the functions F^{**}_{ij} are given. We use the notations

$$A_k(x) := x^k + (1-x)^k \qquad B_k(x) := x^k - (1-x)^k \quad O(x) := 0$$
$$L_1(x) := \log x - \log(1-x) \quad P_k(x) := x^k \qquad (k \in \mathbf{C};\ x \in \,]0,1[\,).$$

	G_1	H_1	
A1	O	arb.	
A2	arb.$\neq O$	O	
A3	$a_1 A_\alpha$	$p_1 A_\alpha$	
A4	$a_2 A_2 + a_3 A_3$	$p_2 [a_2 A_2 + a_3 A_3]$	
A5	$a_4 A_4 + a_5 A_5$	$p_3 [a_4 A_4 + a_5 A_5]$	
	K_1		
B1	arb.		
B2	O		
B3	$s_1 A_1$		
B4	$s_2 A_\delta$		
B5	$k_1 A_1 + k_2 A_2$		
B6	$s_3 A_3$		
B7	$k_3 A_3 + k_4 A_4$		
		L_1	K_2
C1		O	arb.
C2		arb.$\neq O$	O
C3		$r_1 A_1$	$c_0 B_1 + 2c_1 L_1$
C4		$r_2 A_\gamma$	$c_2 B_\gamma$
C5		$e_1 A_1 + e_2 A_2$	$d_1 B_1$
C6		$r_3 A_3$	$d_2 B_2 + d_3 B_3$
C7		$e_3 A_3 + e_4 A_4$	$d_4 B_4$
			G_2
D1			O
D2			arb.$\neq O$
D3			$b_1 L_1$
D4			$b_2 B_\beta$
D5			$b_3 B_3 + b_4 B_4$
D6			$b_5 B_5 + b_6 B_6$

		F_{11}^{**}
A1		O
A2		O
A3		$p_1 a_1 P_\alpha$
A4		$p_2\left[\frac{3a_3+2a_2}{6}(2a_3P_3+3a_2P_2)-\frac{a_2a_3}{24}P_0\right]$
A5		$p_3\left[\frac{5a_5+2a_4}{10}(2a_5P_5+5a_4P_4)-\right.$ $\left.-\frac{a_4a_5}{30}(40P_3-15P_2+\frac{1}{2}P_0)\right]$
	L_2	F_{12}^{**}
B1	O	O
B2	arb.$\neq O$	O
B3	$i_0B_1+2i_1L_1$	$s_1\left[i_0P_1+i_1\log\right]$
B4	i_2B_δ	$s_2 i_2 P_\delta$
B5	j_1B_1	$j_1\left[k_1P_1+k_2P_2\right]$
B6	$j_2B_2+j_3B_3$	$s_3\left[j_3P_3+\frac{j_2}{2}(3P_2-P_1)\right]$
B7	j_4B_4	$j_4\left[k_4P_4+\frac{k_3}{2}(4P_3-3P_2+P_1)\right]$
		F_{21}^{**}
C1		O
C2		O
C3		$r_1\left[c_0P_1+c_1\log\right]$
C4		$r_2c_2P_\gamma$
C5		$d_1\left[e_1P_1+e_2P_2\right]$
C6		$r_3\left[d_3P_3+\frac{d_2}{2}(3P_2-P_1)\right]$
C7		$d_4\left[e_4P_4+\frac{e_3}{2}(4P_3-3P_2+P_1)\right]$
	H_2	F_{22}^{**}
D1	arb.	O
D2	O	O
D3	q_1L_1	$\frac{1}{2}q_1b_1\log^2$
D4	q_2B_β	$q_2b_2P_\beta$
D5	$q_3\left[b_3B_3+b_4B_4\right]$	$q_3\left[\frac{2b_4+b_3}{2}(b_4P_4+2b_3P_3)-\frac{b_3b_4}{2}P_1\right]$
D6	$q_4\left[b_5B_5+b_6B_6\right]$	$q_4\left[\frac{3b_6+b_5}{3}(b_6P_6+3b_5P_5)-\right.$ $\left.-\frac{b_5b_6}{6}(15P_4-10P_3+3P_1)\right]$

For $i \geq 1$ all constants $a_i, b_i, c_i, d_i, e_i, i_i, j_i, h_i, p_i, q_i, r_i, s_i \in \mathbf{C}$ are arbitrary $\neq 0$ while c_0, i_0 can be zero as well. Further $3a_3 + 2a_2 \neq 0,$, $5a_5 + 2a_4 \neq 0, 2b_4 + b_3 \neq 0$, $3b_6 + b_5 \neq 0$, α, β, γ, δ are arbitrary $\neq 0$ constants, moreover $\beta \neq 1$ holds too. Due to these restrictions the degree of G_1, H_1 in solution A4 is 2, in solution A5 is 4; the degree of G_2, H_2 in D5 is 3, in D6 is 5; the degree of K_1 and L_2 (or L_1 and H_2) in solution B5 (or in C5) is 2 and 1, in B6 (or in C6) is 2 and 3, finally in B7 (or in C7) is 4 and 3.

3. Solution of equation (E)

Each solution of the system (S) can be obtained from a quadruple (Ai,Bj,Ck,Dl) of solutions of the equations (A), (B), (C), (D) respectively such that

$$K_1 = G_1 \quad L_1 = H_1 \quad K_2 = G_2 \quad L_2 = H_2 \tag{11}$$

hold. Let us call a quadruple (Ai,Bj,Ck,Dl) *admissible* if (11) holds. Thus, to find the solutions of the system (S),*we have to determine the set of all admissible quadruples*

$$\{(\text{Ai,Bj,Ck,Dl}) \mid i = 1, \dots, 5; j, k = 1, \dots, 7; l = 1, \dots, 6\}$$

and then the solutions of (E) can be obtained by (5), (6), (8), (9), (10). There are alltogether $5 \cdot 7^2 \cdot 6 = 1470$ quadruples. We divide them into 16 groups according to the symmetry properties of the functions g, h.

In the last column the indices $i; j; k$ and l assume the values of $3, 4, 5; 3, 4, 5, 6, 7; 3, 4, 5, 6, 7$ and $3, 4, 5, 6$ respectively. We remark that the number of quadruples in the above table is only 394. The number of quadruples decreased since we partly took into consideration condition (11). In groups 1–11 all quadruples are admissible. Let us call a pair (Ai,Bj) of solutions a *good pair* if $K_1 = G_1$ holds for it (i.e. if the parameters of the solutions are chosen such that $K_1 = G_1$ holds). In the opposite case (when the parameters cannot be chosen such that $K_1 = G_1$ holds) (Ai,Bj) will be called a *bad pair*. Similarly the pairs (Ai,Ck), (Bj,Dl) and (Ck,Dl) will be called good pairs if $L_1 = H_1, L_2 = H_2$, and $K_2 = G_2$ hold respectively. In order that the quadruples in groups 12–15 be admissible it is necessary and sufficient that the pairs (Ck,Dl) (in group 12), (Bj,Dl) (in group 13), (Ai,Bj) (in group 14), (Ai,Ck) (in group 15) be good.

Group	G_1 and K_1	H_1 and L_1	G_2 and K_2	H_2 and L_2	Solution
1	O	O	O	O	$(A1, B1, C1, D1)$
2	$\neq O$	O	O	O	$(A2, B1, C1, D1)$
3	O	$\neq O$	O	O	$(A1, B1, C2, D1)$
4	O	O	$\neq O$	O	$(A1, B1, C1, D2)$
5	O	O	O	$\neq O$	$(A1, B2, C1, D1)$
6	O	O	$\neq O$	$\neq O$	$(A1, B2, C1, Dl)$
7	O	$\neq O$	O	$\neq O$	$(A1, B2, C2, D1)$
8	O	$\neq O$	$\neq O$	O	$(A1, B1, Ck, D2)$
9	$\neq O$	O	O	$\neq O$	$(A2, Bj, C1, D1)$
10	$\neq O$	O	$\neq O$	O	$(A2, B1, C1, D2)$
11	$\neq O$	$\neq O$	O	O	$(Ai, B1, C2, D1)$
12	O	$\neq O$	$\neq O$	$\neq O$	$(A1, B2, Ck, Dl)$
13	$\neq O$	O	$\neq O$	$\neq O$	$(A2, Bj, C1, Dl)$
14	$\neq O$	$\neq O$	O	$\neq O$	$(Ai, Bj, C2, D1)$
15	$\neq O$	$\neq O$	$\neq O$	O	$(Ai, B1, Ck, D2)$
16	$\neq O$	$\neq O$	$\neq O$	$\neq O$	(Ai, Bj, Ck, Dl)

Bellow we shall determine the solutions of (E) for each of the 16 groups.

Groups 1-5., 7., and 10. For the quadruples belonging to these groups we have $G_1+G_2=O$ or $H_1+H_2=O$ hence either $g=O$ or $h=O$ and thus

$$\text{(E1)} \begin{cases} f_{ij}^{**}(x) & = O(x) \\ g(x) & = O(x) \\ h(x) & = \text{arbitrary}, \end{cases} \quad \text{or} \quad \text{(E2)} \begin{cases} f_{ij}^{**}(x) & = O(x) \\ g(x) & = \text{arbitrary}, \ g \neq O \\ h(x) & = O(x). \end{cases}$$

Group 6. This group has the quadruples (A1,B2,C1,Dl) (l=3, ..., 6) hence

$$\text{(E3)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+j}\frac{1}{2}q_1 b_1 \log^2 x \\ g(x) = b_1\left(\log x - \log(1-x)\right) \\ h(x) = q_1\left(\log x - \log(1-x)\right), \end{cases}$$

$$\text{(E4)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+j} q_2 b_2 x^{\beta} \\ g(x) = b_2\left(x^{\beta} - (1-x)^{\beta}\right) \\ h(x) = q_2\left(x^{\beta} - (1-x)^{\beta}\right), \end{cases}$$

$$\text{(E5)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+j} q_3\left[\frac{1}{2}(2b_4 + b_3)(b_4x^4 + 2b_3x^3) - \frac{1}{2}b_3b_4x\right] \\ g(x) = b_3\left(x^3 - (1-x)^3\right) + b_4\left(x^4 - (1-x)^4\right) \\ h(x) = q_3\left[b_3\left(x^3 - (1-x)^3\right) + b_4\left(x^4 - (1-x)^4\right)\right], \end{cases}$$

$$\text{(E6)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+j} q_4\left[\frac{1}{3}(3b_6 + b_5)(b_6x^6 + 3b_5x^5) - \right. \\ \qquad\qquad \left. - \frac{1}{6}b_5b_6(15x^4 - 10x^3 + 3x)\right] \\ g(x) = b_5\left(x^5 - (1-x)^5\right) + b_6\left(x^6 - (1-x)^6\right) \\ h(x) = q_4\left[b_5\left(x^5 - (1-x)^5\right) + b_6\left(x^6 - (1-x)^6\right)\right]. \end{cases}$$

Group 8. From the quadruples (A1,B1,Ck,D2) ($k = 3, \dots, 7$) we get

$$\text{(E7)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1} r_1\left[c_0x + c_1\log x\right] \\ g(x) = c_0\left(x - (1-x)\right) + 2c_1\left(\log x - \log(1-x)\right) \\ h(x) = r_1\left(x + (1-x)\right), \end{cases}$$

$$\text{(E8)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1} r_2 c_2 x^{\gamma} \\ g(x) = c_2\left(x^{\gamma} - (1-x)^{\gamma}\right) \\ h(x) = r_2\left(x^{\gamma} + (1-x)^{\gamma}\right), \end{cases}$$

$$\text{(E9)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1} d_1(e_1x + e_2x^2) \\ g(x) = d_1\left(x - (1-x)\right) \\ h(x) = e_1\left(x + (1-x)\right) + e_2\left(x^2 + (1-x)^2\right), \end{cases}$$

$$\text{(E10)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1} r_3\left[d_3x^3 + \frac{1}{2}d_2(3x^2 - x)\right] \\ g(x) = d_2\left(x^2 - (1-x)^2\right) + d_3\left(x^3 - (1-x)^3\right) \\ h(x) = r_3\left(x^3 + (1-x)^3\right), \end{cases}$$

$$\text{(E11)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1} d_4 \left[e_4 x^4 + \frac{1}{2} e_3 (4x^3 - 3x^2 + x)\right] \\ \quad g(x) = d_4 \left(x^4 - (1-x)^4\right) \\ \quad h(x) = e_3 \left(x^3 + (1-x)^3\right) + e_4 \left(x^4 + (1-x)^4\right). \end{cases}$$

Group 9. The quadruples (A2,Bj,C1,D1) ($j = 3, \dots, 7$) give that

$$\text{(E12)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{j+1} s_1 \left[i_0 x + i_1 \log x\right] \\ \quad g(x) = s_1 \left(x + (1-x)\right) \\ \quad h(x) = i_0 \left(x - (1-x)\right) + 2i_1 \left(\log x - \log(1-x)\right), \end{cases}$$

$$\text{(E13)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{j+1} s_2 i_2 x^\delta \\ \quad g(x) = s_2 \left(x^\delta + (1-x)^\delta\right) \\ \quad h(x) = i_2 \left(x^\delta - (1-x)^\delta\right), \end{cases}$$

$$\text{(E14)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{j+1} j_1 \left[k_1 x + k_2 x^2\right] \\ \quad g(x) = k_1 \left(x + (1-x)\right) + k_2 \left(x^2 + (1-x)^2\right) \\ \quad h(x) = j_1 \left(x - (1-x)\right), \end{cases}$$

$$\text{(E15)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{j+1} s_3 \left[j_3 x^3 + \frac{1}{2} j_2 (3x^2 - x)\right] \\ \quad g(x) = s_3 \left(x^3 + (1-x)^3\right) \\ \quad h(x) = j_2 \left(x^2 - (1-x)^2\right) + j_3 \left(x^3 - (1-x)^3\right), \end{cases}$$

$$\text{(E16)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{j+1} j_4 \left[k_4 x^4 + \frac{1}{2} k_3 (4x^3 - 3x^2 + x)\right] \\ \quad g(x) = k_3 \left(x^3 + (1-x)^3\right) + k_4 \left(x^4 + (1-x)^4\right) \\ \quad h(x) = j_4 \left(x^4 - (1-x)^4\right). \end{cases}$$

Group 11. The elements of this group are (Ai,B1,C2,D1) ($i = 3, 4, 5$) from which

$$\text{(E17)}\quad \begin{cases} f_{ij}^{**}(x) = p_1 a_1 x^\alpha \\ \quad g(x) = a_1 \left(x^\alpha + (1-x)^\alpha\right) \\ \quad h(x) = p_1 \left(x^\alpha + (1-x)^\alpha\right), \end{cases}$$

$$\text{(E18)}\quad \begin{cases} f_{ij}^{**}(x) = p_2 \left[\frac{1}{6}(3a_3 + 2a_2)(2a_3 x^3 + 3a_2 x^2) - \frac{1}{24} a_2 a_3\right] \\ \quad g(x) = a_2 \left(x^2 + (1-x)^2\right) + a_3 \left(x^3 + (1-x)^3\right) \\ \quad h(x) = p_2 \left[a_2 \left(x^2 + (1-x)^2\right) + a_3 \left(x^3 + (1-x)^3\right)\right], \end{cases}$$

$$\text{(E19)}\quad \begin{cases} f_{ij}^{**}(x) = p_3\left[\frac{1}{10}(5a_5+2a_4)(2a_5x^5+5a_4x^4) - \right.\\ \qquad\qquad \left. -\ \frac{1}{30}a_4a_5(40x^3-15x^2+\frac{1}{2})\right]\\ g(x) = a_4\left(x^4+(1-x)^4\right)+a_5\left(x^5+(1-x)^5\right)\\ h(x) = p_3\left[a_4\left(x^4+(1-x)^4\right)+a_5\left(x^5+(1-x)^5\right)\right]. \end{cases}$$

Group 12. We have to find those quadruples (A1,B2,Ck,Dl) ($k=3,\dots,7;\ l=3,\dots,6$) for which (Ck,Dl) are good pairs.

(C3,D3) is a good pair if and only if $c_0=0, 2c_1=b_1$, hold hence

$$\text{(E20)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1}\frac{1}{2}b_1\left[r_1\log x+(-1)^{j+1}q_1\log^2 x\right]\\ g(x) = b_1\left(\log x-\log(1-x)\right)\\ h(x) = q_1\left(\log x-\log(1-x)\right)+r_1\left(x+(1-x)\right). \end{cases}$$

The pairs (C3,D4), (C3,D5), (C3,D6) and (C4,D3), (C5,D3), (C6,D3), (C7,D3) are bad since the logarithmic term L_1 appears in C3 (with coefficient $2c_1\neq 0$), and it does not in D4, D5, and D6 and similarly, D3 has logarithmic term L_1 while C4, C5, C6, and C7 have not.

(C4,D4) is a good pair since $c_2B_\gamma=b_2B_\beta$ holds if $\gamma=\beta$ and $c_2=b_2$, or if $\gamma=1$, $\beta=2$ and $c_2=b_2$. These give the next two solutions:

$$\text{(E21)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1}b_2x^\beta\left[r_2+(-1)^{j+1}q_2\right]\\ g(x) = b_2\left(x^\beta-(1-x)^\beta\right)\\ h(x) = q_2\left(x^\beta-(1-x)^\beta\right)+r_2\left(x^\beta+(1-x)^\beta\right) \end{cases}$$

$$\text{(E22)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1}b_2x\left[r_2+(-1)^{j+1}q_2\right]\\ g(x) = b_2\left(x^2-(1-x)^2\right)\\ h(x) = q_2\left(x^2-(1-x)^2\right)+r_2\left(x+(1-x)\right). \end{cases}$$

To check whether some pairs are good or not we shall use the following propositions.

Proposition 1. *The equation*

$$\sum_{k=0}^{5}\beta_nA_n(x)=0\quad (x\in\ ]0,1[\,)$$

holds with some constants $\beta_k\in\mathbf{C}$ $(k=0,\dots,5)$ *if and only if*

$$\begin{aligned} \beta_3 &= -4\beta_2-10(\beta_1+2\beta_0)\\ \beta_4 &= \ \ \ 5\beta_2+15(\beta_1+2\beta_0)\\ \beta_5 &= -2\beta_2-\ \ 6(\beta_1+2\beta_0) \end{aligned}$$

is satisfied (for example $\beta_1+2\beta_0$, β_2 are arbitrary and $\beta_3, \beta_4, \beta_5$ are given by above equations).

Proposition 2. *The equation*

$$\sum_{k=0}^{3} \gamma_k A_k(x) = 0 \quad (x \in \ ]0,1[\,)$$

holds with some constants $\gamma_k \in \mathbf{C} \quad (k = 0, \ldots, 3)$ if and only if

$$\gamma_2 = -3(\gamma_1 + 2\gamma_0), \quad \gamma_3 = \quad 2(\gamma_1 + 2\gamma_0)\,.$$

is satisfied.

Proposition 3. *The equation*

$$\sum_{k=1}^{6} \gamma_k B_k(x) = 0 \qquad (x \in \ ]0,1[\,)$$

holds with some constants $\gamma_k \in \mathbf{C} \quad k = 1, \ldots, 6$ if and only if

$$\gamma_4 = -3\gamma_3 - 5(\gamma_2 + \gamma_1), \quad \gamma_5 = 3\gamma_3 + 6(\gamma_2 + \gamma_1)$$
$$\gamma_6 = -\gamma_3 - 2(\gamma_2 + \gamma_1)$$

is valid.

Proposition 1 can be proved easily by writing the sum $\sum \beta_n A_n$ as a polynomial and comparing the coefficients of the two sides (a proof can also be found in [8]). Proposition 2 is a special case of proposition 1. Finally the last proposition follows from proposition 1 if we take into consideration the relations $B_k' = kA_k,\ B_k(\frac{1}{2}) = 0,\ B_k(x) = \int_{1/2}^{x} kA_k(t)dt.$

□

(C4,D5) is a bad pair since if $c_2 B_\gamma = b_3 B_3 + b_4 B_4$ were true then the right hand side would be a polynomial of degree three hence $\gamma = 3$ or $\gamma = 4$. By proposition 3 we obtain that in the first case $b_4 = 0$ (and $c_2 = b_3$), in the second $b_3 = 0$ (and $c_2 = b_4$), which contradicts to our assumptions on the constants b_3, b_4.

(C4,D6) is a bad pair again since the equation $c_2 B_\gamma = b_5 B_5 + b_6 B_6$ leads to $b_6 = 0$ or $b_5 = 0$ which is a contradiction.

The pair (C5,D4) is a good one since $d_1 B_1 = b_2 B_\beta$ holds exactly if $\beta = 2$, $d_1 = b_2$. This gives the solution

$$\text{(E23)} \qquad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1} b_2 \left[e_1 x + \left(e_2 + (-1)^{j+1} q_2 \right) x^2 \right] \\ g(x) = b_2 \left(x^2 - (1-x)^2 \right) \\ h(x) = q_2 \left(x^2 - (1-x)^2 \right) + e_1 \left(x + (1-x) \right) + \\ \qquad + e_2 \left(x^2 + (1-x)^2 \right). \end{cases}$$

The pairs (C5,D5) and (C5,D6) are bad ones since from the equations $d_1B_1 = b_3B_3 + b_4B_4$ and $d_1B_1 = b_5B_5 + b_6B_6$ it follows by proposition 3 that $b_3 + 2b_4 = 0$ and $b_5 + 3b_6 = 0$ which was excluded.

(C6,D4) is a good pair since from $d_2B_2 + d_3B_3 = b_2B_\beta$ by comparing the degrees of the two sides $\beta = 3$ or $\beta = 4$. In the first case proposition 3 gives that $d_2 = 0$, $d_3 = b_2$ which is impossible. If $\beta = 4$ then we get $d_2 = -b_2$, $d_3 = 2b_2$ which supplies the solution

$$\text{(E24)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1}b_2\left[\frac{1}{2}r_3(4x^3-3x^2+x) + (-1)^{j+1}q_2x^4\right] \\ \quad g(x) = b_2\left(x^4-(1-x)^4\right) \\ \quad h(x) = q_2\left(x^4-(1-x)^4\right) + r_3\left(x^3+(1-x)^3\right). \end{cases}$$

(C6,D5) is a good pair again since the equation $d_2B_2+d_3B_3 = b_3B_3+b_4B_4$ holds if and only if $d_2 = -b_4$, $d_3 = b_3 + 2b_4$ which gives the solution

$$\text{(E25)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1}r_3\left[(2b_4+b_3)x^3 - \frac{1}{2}b_4(3x^2-x)\right] + \\ \qquad\qquad + (-1)^{i+j}q_3\left[\frac{1}{2}(2b_4+b_3)(b_4x^4+2b_3x^3) - \frac{1}{2}b_3b_4x\right] \\ \quad g(x) = b_3\left(x^3-(1-x)^3\right) + b_4\left(x^4-(1-x)^4\right) \\ \quad h(x) = q_3\left[b_3\left(x^3-(1-x)^3\right) + b_4\left(x^4-(1-x)^4\right)\right] + \\ \qquad\qquad + r_3\left(x^3+(1-x)^3\right). \end{cases}$$

(C6,D6) is a bad pair since from the equation $d_2B_2 + d_3B_3 = b_5B_5 + b_6B_6$ by proposition 3 it follows that $3b_6 + b_5 = 0$ which has been excluded. (C7,D4) is however a good pair since $d_4B_4 = b_2B_\beta$ holds with $\beta = 4$, $b_2 = d_4$ giving the solution

$$\text{(E26)}\quad \begin{cases} f_{ij}^{**}(x) = (-1)^{i+1}b_2\left[\frac{1}{2}e_3(4x^3-3x^2+x) + \left(e_4 + (-1)^{j+1}q_2\right)x^4\right] \\ \quad g(x) = b_2\left(x^4-(1-x)^4\right) \\ \quad h(x) = q_2\left(x^4-(1-x)^4\right) + e_3\left(x^3+(1-x)^3\right) + \\ \qquad\qquad + e_4\left(x^4+(1-x)^4\right). \end{cases}$$

(C7,D5) and (C7,D6) are bad pairs since the equation $G_2 = K_2$ leads to $b_3 = 0, d_4 = b_4$ and $b_6 = b_5 = d_4 = 0$ which contradicts to our assumptions on the constants.

Group 13. Those quadruples (A2,Bj,C1,Dl) belong to this group for which (Bj,Dl) $(j = 3,\dots,7; l = 3,\dots,6)$ are good pairs. The situation is similar to the group 12 (only the notation of the constants is different) hence the good pairs and the corresponding parameter values are the following:

(B3,D3)	$i_0 = 0,$	$2i_1 = q_1,$	
(B4,D4)	$\delta = \beta,$	$i_2 = q_2,$	
(B4,D4)	$\delta = 1,$	$\beta = 2,$	$i_2 = q_2,$
(B5,D4)	$\beta = 2,$	$j_1 = q_2,$	
(B6,D4)	$\beta = 4,$	$j_2 = -b_2,$	$j_3 = 2b_2,$
(B6,D5)	$j_2 = -q_3 b_4,$	$j_3 = q_3(b_3 + 2b_4),$	
(B7,D4)	$\beta = 4,$	$j_4 = q_2.$	

From these we obtain the following solutions:

$$\text{(E27)}\begin{cases} f_{ij}^{**}(x) = (-1)^{j+1}\frac{1}{2}q_1\left[s_1\log x + (-1)^{i+1}b_1\log^2 x\right] \\ \quad g(x) = b_1\left(\log x - \log(1-x)\right) + s_1\left(x + (1-x)\right) \\ \quad h(x) = q_1\left(\log x - \log(1-x)\right), \end{cases}$$

$$\text{(E28)}\begin{cases} f_{ij}^{**}(x) = (-1)^{j+1}q_2 x^\beta\left[s_2 + (-1)^{i+1}b_2\right] \\ \quad g(x) = b_2\left(x^\beta - (1-x)^\beta\right) + s_2\left(x^\beta + (1-x)^\beta\right) \\ \quad h(x) = q_2\left(x^\beta - (1-x)^\beta\right), \end{cases}$$

$$\text{(E29)}\begin{cases} f_{ij}^{**}(x) = (-1)^{j+1}q_2 x\left[s_2 + (-1)^{i+1}b_2\right] \\ \quad g(x) = b_2\left(x^2 - (1-x)^2\right) + s_2\left(x + (1-x)\right) \\ \quad h(x) = q_2\left(x^2 - (1-x)^2\right), \end{cases}$$

$$\text{(E30)}\begin{cases} f_{ij}^{**}(x) = (-1)^{j+1}q_2\left[k_1 x + \left(k_2 + (-1)^{i+1}b_2\right)x^2\right] \\ \quad g(x) = b_2\left(x^2 - (1-x)^2\right) + k_1\left(x + (1-x)\right) + \\ \qquad\qquad + k_2\left(x^2 + (1-x)^2\right) \\ \quad h(x) = q_2\left(x^2 - (1-x)^2\right), \end{cases}$$

$$\text{(E31)}\begin{cases} f_{ij}^{**}(x) = (-1)^{j+1}q_2\left[\frac{1}{2}s_3(4x^3 - 3x^2 + x) + (-1)^{i+1}b_2 x^4\right] \\ \quad g(x) = b_2\left(x^4 - (1-x)^4\right) + s_3\left(x^3 + (1-x)^3\right) \\ \quad h(x) = q_2\left(x^4 - (1-x)^4\right), \end{cases}$$

$$\text{(E32)} \begin{cases} f_{ij}^{**}(x) = (-1)^{j+1}s_3q_3\left[(2b_4+b_3)x^3 - \frac{1}{2}b_4(3x^2-x)\right] + \\ \qquad + (-1)^{i+j}q_3\left[\frac{1}{2}(2b_4+b_3)(b_4x^4+2b_3x^3) - \frac{1}{2}b_3b_4x\right] \\ g(x) = b_3\left(x^3-(1-x)^3\right) + b_4\left(x^4-(1-x)^4\right) + \\ \qquad + s_3\left(x^3+(1-x)^3\right) \\ h(x) = q_3\left[b_3\left(x^3-(1-x)^3\right) + b_4\left(x^4-(1-x)^4\right)\right], \end{cases}$$

$$\text{(E33)} \begin{cases} f_{ij}^{**}(x) = (-1)^{j+1}q_2\left[\frac{1}{2}k_3(4x^3-3x^2+x)+ \right. \\ \qquad \left. + \left(k_4+(-1)^{i+1}b_2\right)x^4\right] \\ g(x) = b_2\left(x^4-(1-x)^4\right) + k_3\left(x^3+(1-x)^3\right) + \\ \qquad + k_4\left(x^4+(1-x)^4\right) \\ h(x) = q_2\left(x^4-(1-x)^4\right). \end{cases}$$

Group 14. The elements of this group are the quadruples (Ai,Bj,C2,D1) where the indices $i = 3, 4, 5$ and $j = 3, \ldots, 7$ and the parameters appearing in the solutions Ai and Bj should be choosen such that (Ai,Bj) be good pairs.

(A3,B3) is a good pair if and only if $a_1A_\alpha = s_1A_1$, i.e. if $\alpha = 1, s_1 = a_1$ (here we omitted the possibility $\alpha = 0, s_1 = 2a_1$ since by assumption $\alpha \neq 0$). From this we get the solution

$$\text{(E34)} \quad \begin{cases} f_{ij}^{**}(x) = p_1a_1x + (-1)^{j+1}a_1\left[i_0x + i_1\log x\right] \\ g(x) = a_1\left(x+(1-x)\right) \\ h(x) = p_1\left(x+(1-x)\right) + i_0\left(x-(1-x)\right) + \\ \qquad + 2i_1\left(\log x - \log(1-x)\right). \end{cases}$$

$(A3, B4)$ is a good pair if and only if $a_1A_\alpha = s_2A_\delta$ that is if $\delta = \alpha, a_1 = s_2$. Hence

$$\text{(E35)} \quad \begin{cases} f_{ij}^{**}(x) = a_1\left(p_1+(-1)^{j+1}i_2\right)x^\alpha \\ g(x) = a_1\left(x^\alpha+(1-x)^\alpha\right) \\ h(x) = p_1\left(x^\alpha+(1-x)^\alpha\right) + i_2\left(x^\alpha-(1-x)^\alpha\right). \end{cases}$$

(A3,B5) is a good pair exactly if $a_1A_\alpha = k_1A_1 + k_2A_2$. This holds (by the equality of the degrees of both sides) if $\alpha = 3$ and (by proposition 2)

$k_1 = -\frac{a_1}{2}$, $k_2 = \frac{3a_1}{2}$. From this we obtain the solution

$$\text{(E36)} \qquad \begin{cases} f_{ij}^{**}(x) = a_1 \left[p_1 x^3 + (-1)^{j+1} \frac{j_1}{2}(3x^2 - x)\right] \\ \quad g(x) = a_1 \left(x^3 + (1-x)^3\right) \\ \quad h(x) = p_1 \left(x^3 + (1-x)^3\right) + j_1 \left(x - (1-x)\right). \end{cases}$$

The pair (A3,B6) is good if $a_1 A_\alpha = s_3 A_3$ i.e. if $\alpha = 3, s_3 = a_1$. Hence

$$\text{(E37)} \qquad \begin{cases} f_{ij}^{**}(x) = a_1 \left[p_1 x^3 + (-1)^{j+1} \left(j_3 x^3 + \frac{j_2}{2}(3x^2 - x)\right)\right] \\ \quad g(x) = a_1 \left(x^3 + (1-x)^3\right) \\ \quad h(x) = p_1 \left(x^3 + (1-x)^3\right) + j_2 \left(x^2 - (1-x)^2\right) + \\ \qquad\qquad + j_3 \left(x^3 - (1-x)^3\right). \end{cases}$$

(A3,B7) is a bad pair since in the equality

$$a_1 A_\alpha = k_3 A_3 + k_4 A_4$$

the right hand side is of degree four thus $\alpha = 4$ or $\alpha = 5$. In the case $\alpha = 4$ we have (by proposition 1) $k_3 = 0, k_4 = a_1$ which contradicts to $k_3 \neq 0$. From $\alpha = 5$ it would follow that $a_1 = k_3 = k_4 = 0$ which is impossible.

(A4,B3) is a bad pair again since the equality

$$a_2 A_2 + a_3 A_3 = s_1 A_1$$

leads (by proposition 2) to

$$a_3 = -2s_1,\ a_2 = 3s_1,$$

hence $3a_3 + 2a_2 = 0$, which has been excluded.

The pair (A4,B5) gives a solution (it is a good pair) since

$$k_1 A_1 + (k_2 - a_2) A_2 - a_3 A_3 = 0$$

holds if and only if

$$k_1 = -\tfrac{1}{2} a_3, \quad k_2 = \tfrac{1}{2}(3a_3 + 2a_2),$$

hence we get

$$\text{(E38)} \qquad \begin{cases} f_{ij}^{**}(x) = p_2 \left[\frac{1}{6}(3a_3 + 2a_2)(2a_3 x^3 + 3a_2 x^2) - \frac{1}{24} a_2 a_3\right] + \\ \qquad\qquad + (-1)^{j+1} j_1 \left[\frac{1}{2}(3a_3 + 2a_2) x^2 - \frac{1}{2} a_3 x\right] \\ \quad g(x) = a_2 \left(x^2 + (1-x)^2\right) + a_3 \left(x^3 + (1-x)^3\right) \\ \quad h(x) = p_2 \left[a_2 \left(x^2 + (1-x)^2\right) + a_3 \left(x^3 + (1-x)^3\right)\right] + \\ \qquad\qquad + j_1 \left(x - (1-x)\right). \end{cases}$$

The pair (A4,B6) is bad since the equality $a_2A_2+a_3A_3 = s_3A_3$ leads to $a_3 = s_3, a_2 = 0$ contradicting to the condition $a_2 \neq 0$. Similarly, (A4,B7) is a bad pair since the equality $a_2A_2 + a_3A_3 = k_3A_3 + k_4A_4$ leads to $a_2 = a_3 = k_3 = k_4 = 0$ which is impossible. The pairs (A5,B3), (A5,B5), (A5,B6) are bad too since the degrees of G_1 and K_1 are different in the corresponding solutions. The pairs (A5,B4) and (A5,B7) would be good only if $\delta = 4, a_4 = s_1, a_5 = 0,$ or, if $\delta = 5, a_4 = 0, a_5 = s_2$ and $a_4 = a_5 = k_3 = k_4 = 0$ but these are not admissible values of the constants.

Group 15. The quadruples (Ai,B1,Ck,D2) ($i = 3, 4, 5$; $k = 3, \dots, 7$) form this group where (Ai,Ck) should be good pairs. The situation is similar to group 14 only the notation of the constants is different. Hence the good pairs and the corresponding parameter values are the following:

(A3,C3)	$\alpha = 1,$	$r_1 = p_1,$	
(A3,C4)	$\gamma = \alpha,$	$r_2 = p_1,$	
(A3,C5)	$\alpha = 3,$	$e_1 = -\frac{p_1}{2},$	$e_2 = \frac{3p_1}{2},$
(A3,C6)	$\alpha = 3,$	$r_3 = p_1,$	
(A4,C5)	$e_1 = -\frac{1}{2}p_2a_3,$	$e_2 = \frac{1}{2}p_2(3a_3 + 2a_2).$	

The corresponding solutions are:

$$\text{(E39)}\quad \begin{cases} f_{ij}^{**}(x) = p_1a_1x + (-1)^{i+1}p_1\left[c_0x + c_1\log x\right] \\ \quad g(x) = a_1\left(x + (1-x)\right) + c_0\left(x - (1-x)\right) + \\ \qquad\quad + 2c_1\left(\log x - \log(1-x)\right) \\ \quad h(x) = p_1\left(x + (1-x)\right), \end{cases}$$

$$\text{(E40)}\quad \begin{cases} f_{ij}^{**}(x) = p_1\left[a_1 + (-1)^{i+1}c_2\right]x^\alpha \\ \quad g(x) = a_1\left(x^\alpha + (1-x)^\alpha\right) + c_2\left(x^\alpha - (1-x)^\alpha\right) \\ \quad h(x) = p_1\left(x^\alpha + (1-x)^\alpha\right), \end{cases}$$

$$\text{(E41)}\quad \begin{cases} f_{ij}^{**}(x) = p_1\left[a_1x^3 + (-1)^{i+1}\frac{d_1}{2}(3x^2 - x)\right] \\ \quad g(x) = a_1\left(x^3 + (1-x)^3\right) + d_1\left(x - (1-x)\right) \\ \quad h(x) = p_1\left(x^3 + (1-x)^3\right), \end{cases}$$

$$\text{(E42)}\quad \begin{cases} f_{ij}^{**}(x) = p_1\left[a_1x^3 + (-1)^{i+1}\left(d_3x^3 + \frac{d_2}{2}(3x^2 - x)\right)\right] \\ \quad g(x) = a_1\left(x^3 + (1-x)^3\right) + d_2\left(x^2 - (1-x)^2\right) + \\ \qquad\quad + d_3\left(x^3 - (1-x)^3\right) \\ \quad h(x) = p_1\left(x^3 + (1-x)^3\right) \end{cases}$$

$$\text{(E43)}\quad \begin{cases} f_{ij}^{**}(x) = p_2\left[\frac{1}{6}(3a_3+2a_2)(2a_3x^3+3a_2x^2) - \frac{1}{24}a_2a_3\right] + \\ \qquad + (-1)^{i+1}d_1p_2\left[\frac{1}{2}(3a_3+2a_2)x^2 - \frac{1}{2}a_3x\right] \\ g(x) = a_2\left(x^2+(1-x)^2\right) + a_3\left(x^3+(1-x)^3\right) + \\ \qquad + d_1\left(x-(1-x)\right) \\ h(x) = p_2\left[a_2\left(x^2+(1-x)^2\right) + a_3\left(x^3+(1-x)^3\right)\right]. \end{cases}$$

Group 16. This group contains 300 quadruples. Fortunately we do not have to study all of them since if a quadruple (Ai,Bj,Ck,Dl) in group 16 is admissible then the pairs (Ai,Bj), (Bj,Dl), (Dl,Ck), (Ck,Ai) have to be good. We have determined the possible good pairs i.e those pairs (Ai,Bj), (Bj,Dl), (Dl,Ck) and (Ck,Ai) (in groups 14, 13, 12 and 15 respectively) for which the parameters can be chosen such that they are good pairs). The list of these pairs is:

(A3,B3)	(B3,D3)	(D3,C3)	(C3,A3)
(A3,B4)	(B4,D4)	(D4,C4)	(C4,A3)
(A3,B5)	(B5,D4)	(D4,C5)	(C5,A3)
(A3,B6)	(B6,D4)	(D4,C6)	(C6,A3)
(A4,B5)	(B6,D5)	(D5,C6)	(C5,A4)
	(B7,D4)	(D4,C7)	

The next figure (on page 18) gives an overview on the quadruples to be investigated. We connected the solutions appearing in possible good pairs by arrows. The admissible quadruples determine paths consisting of four arrows such that at the beginning and at the end of the path we have the same solution (A3 or A4 in our case).

One can see that only 11 quadruples can be admissible in group 16. We shall study these quadruples one by one. We shall write down the apropriate part of the table of solutions and check if the condition of admissibility (for the parameters) can be satisfied or not.

The table for the quadruple (A3,B3,C3,D3):

	$G_1\,(K_1)$	$H_1\,(L_1)$	$G_2\,(K_2)$	$H_2\,(L_2)$
A3	a_1A_α	p_1A_α		
B3	s_1A_1			$i_0B_1+2i_1L_1$
C3		r_1A_1	$c_0B_1+2c_1L_1$	
D3			b_1L_1	q_1L_1

The quadruple (A3,B3,C3,D3) is admissible if and only if the functions in all the four columns of the above table are equal. Hence

$$\alpha = 1, s_1 = a_1; \quad r_1 = p_1; \quad c_0 = 0, c_1 = \tfrac{1}{2}b_1; \quad i_0 = 0, i_1 = \tfrac{1}{2}q_1$$

which gives the solution

$$\text{(E44)} \qquad \begin{cases} f_{ij}^{**}(x) = a_1 p_1 x + (-1)^{j+1}\tfrac{1}{2}a_1 q_1 \log x + \\ \qquad\qquad + (-1)^{i+1}\tfrac{1}{2}b_1 p_1 \log x + (-1)^{i+j+2}\tfrac{1}{2}b_1 q_1 \log^2 x \\ \quad g(x) = a_1\left(x + (1-x)\right) + b_1\left(\log x - \log(1-x)\right) \\ \quad h(x) = p_1\left(x + (1-x)\right) + q_1\left(\log x - \log(1-x)\right). \end{cases}$$

We make only one table to the quadruples (A3,B4,C4–6,D4) but the lines C4, C5, C6 will be studied one by one.

	$G_1\,(K_1)$	$H_1\,(L_1)$	$G_2\,(K_2)$	$H_2\,(L_2)$
A3	$a_1 A_\alpha$	$p_1 A_\alpha$		
B4	$s_2 A_\delta$			$i_2 B_\delta$
C4		$r_2 A_\gamma$	$c_2 B_\gamma$	
C5		$e_1 A_1 + e_2 A_2$	$d_1 B_1$	
C6		$r_3 A_3$	$d_2 B_2 + d_3 B_3$	
D4			$b_2 B_\beta$	$q_2 B_\beta$

In case of (A3,B4,C4,D4) the constants have to satisfy the conditions

$$\delta = \alpha,\ s_2 = a_1; \qquad \gamma = \alpha,\ r_2 = p_1;$$
$$\gamma = \beta,\ c_2 = b_2; \qquad \delta = \beta,\ i_2 = q_2$$

which gives the solution

$$\text{(E45)} \qquad \begin{cases} f_{ij}^{**}(x) = \big(a_1 + (-1)^{i+1} b_2\big)\big(p_1 + (-1)^{j+1} q_2\big)\, x^\alpha \\ g(x) = a_1\left(x^\alpha + (1-x)^\alpha\right) + b_2\left(x^\alpha - (1-x)^\alpha\right) \\ h(x) = p_1\left(x^\alpha + (1-x)^\alpha\right) + q_2\left(x^\alpha - (1-x)^\alpha\right). \end{cases}$$

The quadruples (A3,B4,C5,D4) and (A3,B4,C6,D4) do not determine solutions since the conditions for the constants

$$\delta = \alpha,\ s_2 = a_1; \qquad \alpha = 3,\ e_1 = -\tfrac{p_1}{2},\ e_2 = \tfrac{3p_1}{2};$$
$$\beta = 2,\ d_1 = b_2; \qquad \delta = \beta,\ i_2 = q_2$$

and

$$\delta = \alpha,\ s_2 = a_1; \qquad \alpha = 3,\ r_3 = p_1;$$
$$\beta = 4,\ d_2 = -c_2,\ d_3 = 2c_2; \qquad \delta = \beta,\ i_2 = q_2$$

lead to $3 = \alpha = \beta = 2$ and $3 = \alpha = \beta = 4$ which is not possible.

The parameters of the admissible quadruples (A3,B5,C4–6,D4) can be obtained from the table of page 20.

Only the quadruple (A3,B5,C5,D4) gives a solution provided that the constants satisfy the conditions

$$\alpha = 3,\ k_1 = -\tfrac{a_1}{2},\ k_2 = \tfrac{3a_1}{2}; \qquad e_1 = -\tfrac{p_1}{2},\ e_2 = \tfrac{3p_1}{2};$$
$$\beta = 2,\ d_1 = b_2; \qquad \beta = 2,\ j_1 = q_2$$

	$G_1\,(K_1)$	$H_1\,(L_1)$	$G_2\,(K_2)$	$H_2\,(L_2)$
A3	a_1A_α	p_1A_α		
B5	$k_1A_1+k_2A_2$			j_1B_1
C4		r_2A_γ	c_2B_γ	
C5		$e_1A_1+e_2A_2$	d_1B_1	
C6		r_3A_3	$d_2B_2+d_3B_3$	
D4			b_2B_β	q_2B_β

hence

$$\text{(E46)}\qquad \begin{cases} f_{ij}^{**}(x)=a_1p_1x^3+(-1)^{j+1}\frac{a_1q_2}{2}(3x^2-x)+\\ \qquad\qquad +(-1)^{i+1}\frac{b_2p_1}{2}(3x^2-x)+(-1)^{i+j+2}b_2q_2x^2\\ g(x)=a_1\left(x^3+(1-x)^3\right)+b_2\left(x^2-(1-x)^2\right)\\ h(x)=p_1\left(x^3+(1-x)^3\right)+q_2\left(x^2-(1-x)^2\right). \end{cases}$$

The quadruples (A3,B5,C4,D4) and (A3,B5,C6,D4) are not admissible since the conditions for the constants

$$\begin{aligned} &\alpha=3,\ k_1=-\tfrac{a_1}{2},\ k_2=\tfrac{3a_1}{2}; && \gamma=\beta,\ c_2=b_2;\\ &\gamma=\beta,\ c_2=b_2; && \beta=2,\ j_1=q_2 \end{aligned}$$

or

$$\begin{aligned} &\alpha=3,\ k_1=-\tfrac{a_1}{2},\ k_2=\tfrac{3a_1}{2}; && \gamma=\beta,\ c_2=b_2;\\ &\gamma=1,\ \beta=2,\ c_2=b_2; && \beta=2,\ j_1=q_2 \end{aligned}$$

and

$$\begin{aligned} &\alpha=3,\ k_1=-\tfrac{a_1}{2},\ k_2=\tfrac{3a_1}{2}; && \alpha=3,\ r_3=p_1;\\ &\beta=4,\ d_2=-b_2,\ d_3=2b_2; && \beta=2,\ j_1=q_2 \end{aligned}$$

lead to $3=\alpha=\gamma=\beta=2$ or $3=\alpha=\gamma=1$ and $4=\beta=2$ which are not allowed values of the constants.

Out of the quadruples (A3,B6,C4–6,D4) only (A3,B6,C6,D4) will be admissible the other two lead to contradiction.

(A3,B6,C6,D4) is admissible if and only if

$$\begin{aligned} &\alpha=3,\ s_3=a_1; && \alpha=3,\ r_3=p_1;\\ &\beta=4,\ d_2=-b_2,\ d_3=2b_2; && \beta=4,\ j_2=-q_2,\ j_3=2q_2 \end{aligned}$$

	$G_1\,(K_1)$	$H_1\,(L_1)$	$G_2\,(K_2)$	$H_2\,(L_2)$
A3	a_1A_α	p_1A_α		
B6	s_3A_3			$j_2B_2+j_3B_3$
C4		r_2A_γ	c_2B_γ	
C5		$e_1A_1+e_2A_2$	d_1B_1	
C6		r_3A_3	$d_2B_2+d_3B_3$	
D4			b_2B_β	q_2B_β

supplying the solution

$$\text{(E47)}\quad \begin{cases} f^{**}_{ij}(x) = a_1p_1x^3 + (-1)^{j+1}\frac{a_1q_2}{2}(4x^3-3x^2+x)+ \\ \qquad\qquad + (-1)^{i+1}\frac{p_1b_2}{2}(4x^3-3x^2+x) + (-1)^{i+j+2}b_2q_2x^4 \\ g(x) = a_1\left(x^3+(1-x)^3\right) + b_2\left(x^4-(1-x)^4\right) \\ h(x) = p_1\left(x^3+(1-x)^3\right) + q_2\left(x^4-(1-x)^4\right). \end{cases}$$

For (A3,B6,C4,D4) and (A3,B6,C5,D4) the conditions of admissibility are:

$$\begin{aligned} &\alpha=3,\ s_3=a_1; && \gamma=\alpha,\ r_2=p_1; \\ &\gamma=\beta,\ c_2=b_2; && \beta=4,\ j_2=-q_2,\ j_3=2q_2 \end{aligned}$$

or

$$\begin{aligned} &\alpha=3,\ s_3=a_1; && \gamma=\alpha,\ r_2=p_1; \\ &\gamma=1,\ \beta=2,\ c_2=b_2; && \beta=4,\ j_2=-q_2,\ j_3=2q_2 \end{aligned}$$

and

$$\begin{aligned} &\alpha=3,\ s_3=a_1; && \alpha=3,\ e_1=-\tfrac{p_1}{2},\ e_2=\tfrac{3p_1}{2}; \\ &\beta=2,\ d_1=b_2; && \beta=4,\ j_2=-q_2,\ j_3=2q_2 \end{aligned}$$

leading to contradictions $3=\alpha=\gamma=\beta=4$ or $3=\alpha=\gamma=14$ and $2=\beta=4$.

The table of the last quadruple (A4,B5,C5,D4) is on page 22.

Here the constants have to satisfy the conditions

$$\begin{aligned} &k_1=-\tfrac{a_3}{2},\ k_2=\tfrac{1}{2}(3a_3+2a_2); && e_1=-\tfrac{1}{2}p_2a_3,\ e_2=\tfrac{1}{2}p_2(3a_3+2a_2); \\ &\beta=2,\ d_1=b_2; && \beta=2,\ j_1=q_2 \end{aligned}$$

	$G_1\,(K_1)$	$H_1\,(L_1)$	$G_2\,(K_2)$	$H_2\,(L_2)$
A4	$a_2A_2+a_3A_3$	$p_2a_2A_2+p_2a_3A_3$		
B5	$k_1A_1+k_2A_2$			j_1B_1
C5		$e_1A_1+e_2A_2$	d_1B_1	
D4			b_2B_β	q_2B_β

giving the solution

$$\text{(E48)}\qquad \begin{cases} f_{ij}^{**}(x) = p_2\left[\frac{1}{6}(3a_3+2a_2)(2a_3x^3+3a_2x^2) - \dfrac{a_2a_3}{24}\right] + \\ \qquad + \frac{1}{2}\left[(-1)^{j+1}q_2 + (-1)^{i+1}b_2p_2\right]\cdot \\ \qquad \cdot\left[(3a_3+2a_2)x^2 - a_3x\right] + (-1)^{i+j+2}b_2q_2x^2 \\ g(x) = a_2\left(x^2+(1-x)^2\right) + a_3\left(x^3+(1-x)^3\right) + \\ \qquad + b_2\left(x^2-(1-x)^2\right) \\ h(x) = p_2\left[a_2\left(x^2+(1-x)^2\right) + a_3\left(x^3+(1-x)^3\right)\right] + \\ \qquad + q_2\left(x^2-(1-x)^2\right). \end{cases}$$

Theorem. *Suppose that* $f_{ij}, g, h : \,]0,1[\, \to \mathbf{C}$ $(i,j=1,2)$ *and* f_{ij} $(i,j=1,2)$ *are measurable on* $]0,1[$*. Then all solutions of the functional equation*

$$\text{(E)}\quad f_{11}(xy) + f_{12}(x(1-y)) + f_{21}((1-x)y) + \\ + f_{22}((1-x)(1-y)) = g(x)h(y) \qquad (x,y\in\,]0,1[\,)$$

are given by

$$f_{ij}(x) = f_{ij}^{*}(x) + f_{ij}^{**}(x) \quad (i,j=1,2;\ x\in\,]0,1[\,)$$

where

$$f_{ij}^{*}(x) = a\left(x-\frac{1}{4}\right) + (-1)^{j+1}f + \\ + (-1)^{i+1}e + (-1)^{i+j}\left[b + c\log x + d(x^2-x)\right] \quad (x\in\,]0,1[\,)$$

and the functions f_{ij}^{**}, g, h *are given by the formulae* (E1), (E2), (E6), (E19), (E23), (E25), (E26), (E30), (E32), (E33), (E34), (E37), (E39), (E42), (E44), (E45), (E46), (E47), (E48) *where* $a,b,c,d,e,f;$

α; $a_1, \dots, a_5$; $b_2, \dots, b_6$; $c_0, c_1; d_2, d_3$; $e_1, \dots, e_4$; $i_0, i_1; j_2, j_3$; $k_1, \dots, k_4$; p_1, p_2, p_3; q_2, q_3; r_3; s_3 *are arbitrary complex constants.*

Proof. We have seen that all measurable solutions of (E) are given by (E1)–(E48). These solution classes are pairwise disjoint if the constants satisfy the conditions given in Section 2. To complete the proof we show that from the 19 solutions listed in the theorem we can obtain all solutions (E1)–(E48) by special choice of the constants. In the next table we give how this can be done.

The function systems (Ei) listed in our theorem give solutions of equation (E) for all (even 0) values of the parameters since these functions depend continuously from the parameters. □

We remark that the inhomogeneous parts of the solutions contain 3,4 or 5 arbitrary parameters.The only 5 parameter family of solutions is (E48). The solutions have the following symmetry property. If we exchange the indices i and j on the right hand side of f_{ij} and exchange the functions g and h as well we get a solution of (E) again. f_{ij}^* is symmetric in the indices i, j (if we exchange the notation of the constants e, f). Corresponding to this symmetry the inhomogeneous parts f_{ij}^{**} of the solution and g, h of (E2), (E30), (E32), (E33), (E39), (E42) can be obtained from the solutions (E1), (E23), (E25), (E26), (E34), (E37) respectively (by exchanging i and j and g and h and in some cases by introducing new constants). The remaining solutions (E6), (E19), (E44), (E45), (E46), (E47), (E48) also have this symmetry property but these transform into theirselves provided that the constants are redenoted in a suitable way.

Solution	Special case of the solution	By which values of the parameters
E3	E44	$a_1 = p_1 = 0$
E4	E45	$a_1 = p_1 = 0, \alpha = \beta$
E5	E32	$s_3 = 0$
E7	E39	$a_1 = 0, p_1 = r_1$
E8	E45	$a_1 = q_2 = 0, p_1 = r_2, b_2 = c_2, \alpha = \gamma$
E9	E23	$b_2 = d_1, q_2 = 0$
E10	E42	$a_1 = 0, p_1 = r_3$
E11	E26	$b_2 = d_4, q_2 = 0$
E12	E34	$a_1 = s_1, p_1 = 0$
E13	E45	$a_1 = s_2, b_2 = p_1 = 0, q_2 = i_2, \alpha = \delta$
E14	E30	$b_2 = 0, q_2 = j_1$

E15	E37	$a_1 = s_3, p_1 = 0$
E16	E33	$b_2 = 0, q_2 = j_4$
E17	E45	$b_2 = q_2 = 0$
E18	E48	$b_2 = q_2 = 0$
E20	E44	$a_1 = 0, p_1 = r_1$
E21	E45	$a_1 = 0, p_1 = r_2, \alpha = \beta$
E22	E45	$a_1 = 0, p_1 = r_2, \alpha = 1$
E24	E47	$a_1 = 0, p_1 = r_3$
E27	E44	$a_1 = s_1, p_1 = 0$
E28	E45	$a_1 = s_2, p_1 = 0, \alpha = \beta$
E29	E45	$a_1 = s_2, p_1 = 0, \alpha = 1$
E31	E47	$a_1 = s_3, p_1 = 0$
E35	E45	$b_2 = 0, q_2 = i_2$
E36	E48	$a_2 = b_2 = 0, a_3 = a_1, q_2 = j_1, p_2 a_3 = p_1$
E38	E48	$b_2 = 0, q_2 = j_1$
E40	E45	$b_2 = c_2, q_2 = 0$
E41	E46	$b_2 = d_1, q_2 = 0$
E43	E48	$b_2 = d_1, q_2 = 0$

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(Received March 24, 1992)