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On a class of projectively flat (α,β)-Finsler metrics

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Dedicated to Prof. Lajos Tamássy on the 85th anniversary of his birthday

Abstract. In this paper, we consider the Finsler metric $F=(\alpha+\beta)^{\lambda+1}/\alpha^{\lambda}$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_i(x)y^i$ is a 1-form and λ is a real number with $\lambda\neq -1,0,1$. We prove that this Finsler metric is locally projectively flat if and only if α is projectively flat and β is parallel with respect to α. Furthermore, F is locally projectively flat Finsler metric with constant flag curvature if and only if F is Minkowskian.

1. Introduction

Randers metrics are among the simplest Finsler metrics. They are expressed in the form $F=\alpha+\beta$, where $\alpha:=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric on a differentiable manifold M and $\beta:=b_i(x)y^i$ is a 1-form on M with $\|\beta\|_{\alpha}(x):=\sup_{y\in T_xM}|\beta(y)|/\alpha(y)<1$ for any point $x\in M$. Randers metrics were first used by the physicist G. Randers in 1941 from the standpoint of general relativity. The name was given by R. S. Ingarden, who applied it in his thesis [8] for studying the theory of the electron microscope. Up to now, many Finsler geometers have made great efforts in investigation on the geometric properties of Randers metrics

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and have obtained many important and interesting harvests (cf. [3] [4], [5], [6], [7], [21]).

In 1972, M. Matsumoto defined (α, β)-metrics [11] as a direct generalization of the Randers metric. Important applications of (α, β)-metrics in physics and biology(ecology) can be found [1], [2], [14]. The study of (α, β)-metrics can help us to better understand geometric properties of Finsler metrics in the general case. This is the reason for which it is worthy to study such metrics.

The study of (α, β)-metrics is more complicated than that of Randers metric. However, great progress about the study for (α, β)-metrics was made in recent years. Important results, full of geometric properties have been obtained(cf. [3], [9], [10], [12], [15], [18]).

One of the important problems in Finsler geometry is to study and characterize locally projectively flat Finsler metrics, which is the Hilbert's 4th problem in the regular case. According to the Beltrami Theorem, a Riemannian metric $\alpha := \sqrt{a_{ij}(x)y^iy^j}$ is projectively flat if and only if it is of constant sectional curvature. Further, it is known that a Randers metric $F = \alpha + \beta$ is projectively flat if and only if α is of constant sectional curvature and β is closed. In 1998, M. Matsumoto [10] discussed the Finsler spaces with (α, β)-metric of Douglas type and got a necessary and sufficient condition for that the Finsler metric $F = \alpha + \beta^2/\alpha$ with $b := \|\beta\|_\alpha \neq 0, 1$ is a Douglas metric. Further, in 2005 Z. Shen and G. Civi Yildirim obtained a necessary and sufficient condition that the Finsler metric $F = \alpha + \epsilon\beta + k\beta^2/\alpha$ is projectively flat, and they completely determined the local structure of the locally projectively flat Finsler metrics of the form $F = (\alpha+\beta)^2/\alpha$ with constant flag curvature [20]. Recently, Z. Shen found a sufficient condition for certain class of Finsler metrics in the form $F = \alpha\phi(\beta/\alpha)$ to be locally projectively flat [19]. Z. Shen, X. Mo and C. Yang also constructed a family of projectively flat Finsler metrics in the form $F = (\alpha+\beta)^2/\alpha$ with zero flag curvature, which is a slight generalization of Berwald's famous example [13].

In 1989 M. Matsumoto introduced an (α, β)-metric $F = \alpha^2/(\alpha - \beta)$ as a realization of P. Finsler's idea "a slope measure of a mountain with respect to a time measure". This metric is called later Matsumoto metric. M. Matsumoto proved that a Matsumoto space of dimension $n \geq 3$ with the metric $F = \alpha^2/(\alpha - \beta)$ is Douglas space if and only if β is parallel with respect to α [10]. We know that locally projectively flat Finsler metrics are always Douglas metrics. Thus, it is a natural problem to find the conditions for that a Matsumoto metric is locally projectively flat.

The main purpose of this paper is to study and characterize a special class of locally projectively flat (α, β)-metrics

$$F = \frac{(\alpha+\beta)^{\lambda+1}}{\alpha^{\lambda}}, \quad \alpha = \sqrt{a_{ij}(x)y^i y^j}, \quad \beta = b_i(x)y^i, \tag{1}$$

where λ is an arbitrary real number. Obviously, this class of (α, β)-metrics contains Riemannian metric $F = \alpha(\lambda = -1)$, Randers metric $F = \alpha + \beta(\lambda = 0)$, and Shen metric $F = (\alpha + \beta)^2/\alpha(\lambda = 1)$. If we substitute β with $-\beta$ and take $\lambda = -2$, the resulting metric is just Matsumoto metric $F = \alpha^2/(\alpha - \beta)$. In the following, we always assume that $\lambda \neq -1, 0, 1$.

Theorem 1. *Let $F = (\alpha+\beta)^{\lambda+1}/\alpha^{\lambda}$ be an (α, β)-metric on a manifold M. Then F is locally projectively flat if and only if the following conditons hold:*

(i) *α is projectively flat, i.e. α is of constant curvature;*

(ii) *β is parallel with respect to α.*

If F is locally projectively flat metric with constant flag curvature, we have the following

Theorem 2. *Let $F = (\alpha+\beta)^{\lambda+1}/\alpha^{\lambda}$ be an (α, β)-metric on a manifold M. Then F is locally projectively flat Finsler metric with constant flag curvature if and only if α is flat and β is parallel with respect to α. In this case, F is locally Minkowskian.*

From Theorem 1 and Theorem 2, we have the following

Corollary 3. *A Matsumoto metric $F = \alpha^2/(\alpha - \beta)$ is locally projectively flat if and only if α is projectively flat (i.e. α is of constant curvature) and β is parallel with respect to α. Further, F is locally projectively flat with constant flag curvature if and only if α is flat and β is parallel with respect to α. In this case, F is locally Minkowskian.*

2. (α, β)-metric

An (α, β)-metric is expressed in the following form

$$F = \alpha\phi(s), \quad s = \beta/\alpha,$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-from, $\phi = \phi(s)$ is a C^∞ positive function on an open interval $(-b_o, b_o)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \le b < b_o. \tag{2}$$

It is well known that F is a Finsler metric if and only if $b := \|\beta_x\|_\alpha < b_o$ for any $x \in M$ [7].

Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),$$

$$s^i{}_j := a^{ih} s_{hj}, \quad s_j := b_i s^i{}_j = b^m s_{mj}, \quad e_{ij} := r_{ij} + b_i s_j + b_j s_i,$$

where "|" denotes the covariant derivative with respect to the Levi–Civita connection of α. We will denote $r_{00} := r_{ij} y^i y^j$, $s_0 := s_j y^j$, etc.

Let G^i and G^i_α be the spay coefficient of F and α respectively, given by

$$G^i = \frac{g^{il}}{4}\{[F^2]_{x^l y^l} y^k - [F^2]_{x^k}\}, \quad G^i_\alpha = \frac{a^{il}}{4}\{[\alpha^2]_{x^l y^l} y^k - [\alpha^2]_{x^k}\},$$

where $(g^{ij}) := \left(\frac{1}{2}[F^2]_{y^i y^j}\right)^{-1}$ and $(a^{ij}) := (a_{ij})^{-1}$. We have the following

Lemma 4 ([20]). *The geodesic coefficients G^i are related to G^i_α by*

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + T\big\{-2Q\alpha s_0 + r_{00}\big\}\frac{y^i}{\alpha} \\ + H\big\{-2Q\alpha s_0 + r_{00}\big\}\left\{b^i - s\frac{y^i}{\alpha}\right\}, \tag{3}$$

where

$$Q := \frac{\phi'}{\phi - s\phi'},$$

$$T := \frac{\phi'(\phi - s\phi')}{2\phi\big((\phi - s\phi') + (b^2 - s^2)\phi''\big)},$$

$$H := \frac{\phi''}{2\big((\phi - s\phi') + (b^2 - s^2)\phi''\big)},$$

where $s := \beta/\alpha$ and $b := \|\beta_x\|_\alpha$. The formula (3) *is given in* [7] *and* [16]. *A different version of* (3) *is given in* [10], [15].

It is well-known that a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathrm{R}^n$ is projectively flat if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{4}$$

This is due to G. Hamel. By (4), we have the following

Lemma 5 ([20]). *An (α, β)-metric $F = \alpha\phi(s)$ is projectively flat on an open subset $\mathcal{U} \subset \mathrm{R}^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0, \tag{5}$$

where $y_m := a_{im} y^i$.

3. Proof of Theorem 1

In this section, we consider the (α,β)-metric in the following form:

$$F=\frac{(\alpha+\beta)^{\lambda+1}}{\alpha^{\lambda}}=\alpha\phi(s),\quad \phi=(1+s)^{\lambda+1},\quad s=\frac{\beta}{\alpha}.$$

Fixing a positive constant $b_o>0$, let λ satisfy the following conditions

$$(1+s)^{\lambda+1}>0,\quad (1+s)(1-s\lambda)+\lambda(\lambda+1)(b^2-s^2)>0,\quad |s|\le b<b_o. \tag{6}$$

Then $F=(\alpha+\beta)^{\lambda+1}/\alpha^{\lambda}$ is a Finsler metric if and only if β satisfies $b:=\|\beta_x\|_\alpha<b_o$. For $F=(\alpha+\beta)^{\lambda+1}/\alpha^{\lambda}$, by Lemma 4, we have

$$Q=\frac{\lambda+1}{1-\lambda s},\qquad H=\frac{\lambda(\lambda+1)}{2[1+\lambda(\lambda+1)b^2-(\lambda-1)s-\lambda(\lambda+2)s^2]} \tag{7}$$

Proof of Theorem 1. Assume F is projectively flat, then (5) holds. Substituting (7) into (5) yields

$$\begin{aligned}&(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^3\frac{\lambda+1}{1-\lambda s}s_{l0}\\&+\frac{\lambda(\lambda+1)\alpha}{2[1+\lambda(\lambda+1)b^2-(\lambda-1)s-\lambda(\lambda+2)s^2]}\left\{\frac{-2(\lambda+1)\alpha}{1-\lambda s}s_0+r_{00}\right\}[b_l\alpha-sy_l]=0.\end{aligned} \tag{8}$$

Multiplying (8) by $2(1-\lambda s)[1+\lambda(\lambda+1)b^2-(\lambda-1)s-\lambda(\lambda+2)s^2]$ yields

$$\begin{aligned}&2[1-\lambda s]\big[1+\lambda(\lambda+1)b^2-(\lambda-1)s-\lambda(\lambda+2)s^2\big](a_{ml}\alpha^2-y_my_l)G^m_\alpha\\&+2\alpha^3(\lambda+1)\big[1+\lambda(\lambda+1)b^2-(\lambda-1)s-\lambda(\lambda+2)s^2\big]s_{l0}\\&+\lambda(\lambda+1)\alpha\big\{-2(\lambda+1)\alpha s_0+r_{00}(1-\lambda s)\big\}[b_l\alpha-sy_l]=0.\end{aligned} \tag{9}$$

Contracting (9) with b^l yields

$$\begin{aligned}&2[1-\lambda s]\big[1+\lambda(\lambda+1)b^2-(\lambda-1)s-\lambda(\lambda+2)s^2\big](b_m\alpha^2-y_m\beta)G^m_\alpha\\&+(\lambda+1)\alpha\big\{2\alpha^2[1-(\lambda-1)s-\lambda s^2]s_0+\lambda r_{00}(1-\lambda s)[b^2\alpha-s\beta]\big\}=0\end{aligned} \tag{10}$$

Multiplying (10) by α^3 yields

$$\begin{aligned}&2[\alpha-\lambda\beta]\big[(1+\lambda(\lambda+1)b^2)\alpha^2-(\lambda-1)\alpha\beta-\lambda(\lambda+2)\beta^2\big](b_m\alpha^2-y_m\beta)G^m_\alpha\\&+(\lambda+1)\alpha^2\big\{2\alpha^2[\alpha^2-(\lambda-1)\alpha\beta-\lambda\beta^2]s_0+\lambda r_{00}(\alpha-\lambda\beta)[b^2\alpha^2-\beta^2]\big\}=0.\end{aligned} \tag{11}$$

We note that the coefficients of α must be zero (because α^{even} is a polynomial in y^i). Then (11) is equivalent to the following two equations

$$2\beta\big\{(\lambda-1)\alpha^2+\lambda[(1+\lambda(\lambda+1)b^2)\alpha^2-\lambda(\lambda+2)\beta^2]\big\}(b_m\alpha^2-y_m\beta)G^m_\alpha \\ =(\lambda+1)\alpha^2\big\{2\alpha^2(\alpha^2-\lambda\beta^2)s_0-\lambda^2r_{00}\beta[b^2\alpha^2-\beta^2]\big\} \quad (12)$$

and

$$2\big\{(1+\lambda(\lambda+1)b^2)\alpha^2-3\lambda\beta^2\big\}(b_m\alpha^2-y_m\beta)G^m_\alpha \\ =(\lambda+1)\alpha^2\big\{2(\lambda-1)\alpha^2\beta s_0-\lambda r_{00}[b^2\alpha^2-\beta^2]\big\} \quad (13)$$

Note that $\lambda\neq-1$. Eliminating on $2(b_m\alpha^2-y_m\beta)G^m_\alpha$ and $(\lambda+1)\alpha^2$ from (12) and (13) yields

$$\beta\big\{(\lambda-1)\alpha^2+\lambda[(1+\lambda(\lambda+1)b^2)\alpha^2-\lambda(\lambda+2)\beta^2]\big\}\{2(\lambda-1)\alpha^2\beta s_0-\lambda r_{00}[b^2\alpha^2-\beta^2]\} \\ =\big\{(1+\lambda(\lambda+1)b^2)\alpha^2-3\lambda\beta^2\big\}\{2\alpha^2(\alpha^2-\lambda\beta^2)s_0-\lambda^2r_{00}\beta[b^2\alpha^2-\beta^2]\big\}. \quad (14)$$

Let us rewrite (14) in the following

$$A+B\alpha^2+C\alpha^4+D\alpha^6=0, \quad (15)$$

where

$$A:=\lambda^3(\lambda-1)r_{00}\beta^5,$$
$$B:=2\lambda^2(1+\lambda+\lambda^2)s_0\beta^4-\lambda(\lambda-1)(1+\lambda^2b^2)r_{00}\beta^3$$
$$C:=-2[(1+\lambda)(1+\lambda^3b^2)+2\lambda^2]s_0\beta^2+\lambda(\lambda-1)b^2r_{00}\beta,$$
$$D:=2[1+\lambda(\lambda+1)b^2]s_0$$

In (15), A can be divided by α^2, but β^5 can not. Then there is a scalar function $\tau:=\tau(x)$, such that $r_{00}=\tau\alpha^2$. Then A and B become

$$A=\lambda^3(\lambda-1)\tau\alpha^2\beta^5,$$
$$B=2\lambda^2(1+\lambda+\lambda^2)s_0\beta^4-\lambda(\lambda-1)(1+\lambda^2b^2)\tau\alpha^2\beta^3.$$

Replacing them in (15) yields

$$\lambda^2\beta^4[\lambda(\lambda-1)\tau\beta+2(1+\lambda+\lambda^2)s_0]+[C-\lambda(\lambda-1)(1+\lambda^2b^2)\tau\beta^3]\alpha^2+D\alpha^4=0. \quad (16)$$

In (16), $\beta^4[\lambda(\lambda-1)\tau\beta+2(1+\lambda+\lambda^2)s_0]$ can be divided by α^2, but β^4 can not. Thus, $\lambda(\lambda-1)\tau\beta+2(1+\lambda+\lambda^2)s_0$ can be divided by α^2. This is impossible unless

$$\lambda(\lambda-1)\tau\beta+2(1+\lambda+\lambda^2)s_0=0. \tag{17}$$

That is

$$\lambda(\lambda-1)\tau b_i+2(1+\lambda+\lambda^2)s_i=0. \tag{18}$$

Contracting (18) with b^i yields

$$\lambda(\lambda-1)\tau b^2=0.$$

Since $b^2\neq 0$ and $\lambda\neq 0,1$, then $\tau=0$. From $r_{00}=\tau\alpha^2$ and (17), we can see that

$$r_{00}=0, \quad s_0=0. \tag{19}$$

Substituting (19) into (8) yields

$$(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^3\frac{\lambda+1}{1-\lambda s}s_{l0}=0,$$

that is

$$(\alpha-\lambda\beta)(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^4(\lambda+1)s_{l0}=0. \tag{20}$$

The coefficients of α must be zero (note: α^{even} is a polynomial in y^i). Then (20) is equivalent to

$$\begin{gathered}\alpha(a_{ml}\alpha^2-y_my_l)G^m_\alpha=0,\\ -\lambda\beta(a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^4(\lambda+1)s_{l0}=0.\end{gathered} \tag{21}$$

From (21) we can see $s_{l0}=0$. By (19) β is parallel with respect to α. Putting $s_{l0}=0$ and $r_{00}=0$ into (3) yields

$$G^i=G^i_\alpha. \tag{22}$$

Since F is projectively flat there is a function $P=P(x,y)$ such that $G^i=Py^i$, where P is homogeneous in y of degree one. Then $G^i_\alpha=Py^i$, that is α is projectively flat.

On the other hand, it is clear that the converse is also true by (3). This completes the proof. □

4. Proof of Theorem 2

Assume that the (α, β)-metric in the form (1) is a locally projectively flat Finsler metric with constant flag curvature $c = constant$. By Theorem 1, α is projectively flat, i.e. α has constant sectional curvature μ. Hence, up to a scaling, α is locally isometric to

$$\alpha = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x,y\rangle^2}}{1+\mu|x|^2}$$

Further, from (22), $G^i = G^i_\alpha = Py^i$, where

$$P = \frac{-\mu\langle x,y\rangle}{1+\mu|x|^2}.$$

Using $G^i_\alpha = Py^i$, we first have the following

$$\mu = \frac{P^2 - P_{x^k}y^k}{\alpha^2},$$

and then,

$$P^2 - P_{x^k}y^k = \mu\alpha^2. \tag{23}$$

Using $G^i = Py^i$ and (23), we have

$$c = \frac{P^2 - P_{x^k}y^k}{F^2} = \mu\frac{\alpha^{2(\lambda+1)}}{(\alpha+\beta)^{2(\lambda+1)}}.$$

From this we obtain

$$\mu\alpha^{2(\lambda+1)} = c(\alpha+\beta)^{2(\lambda+1)}.$$

We note that $\lambda + 1 \neq 0$, so

$$\mu^{\frac{1}{\lambda+1}}\alpha^2 = c^{\frac{1}{\lambda+1}}(\alpha+\beta)^2. \tag{24}$$

The left side of (24) is purely quadratic, but the right side is not, thus we obtain $c = 0$ and $\mu = 0$. So, we obtain that α is flat and β is parallel with respect to α. In this case, F is locally Minkowskian. The converse is obvious. This proves Theorem 2.

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