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# On the Diophantine equation $X^2 - (2^{2m} + 1)Y^4 = -2^{2m}$

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*This paper is dedicated to Professor Paulo Ribenboim on the occasion of his 80th birthday*

**Abstract.** Using a recent result of Akhtari on quartic Thue equations, it is shown that the quartic equation $X^2 - (2^{2m} + 1)Y^4 = -2^{2m}$ has at most 12 solutions in odd positive integers $X$, $Y > 1$.

## 1. Introduction

In [3], LJUNGGREN proved that the quartic equation

$$X^2 - 2Y^4 = -1$$

has only the positive integer solutions $(x, y) = (1, 1), (239, 13)$. Also, in [4], LJUNGGREN proved that the only positive integer solutions to

$$X^2 - 5Y^4 = -4$$

are $(X, Y) = (1, 1)$. Since the case $m = 1$ was already studied, we need to consider $m \geq 2$.

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The goal of the present paper is to consider the family of equations

$$X^2 - (2^{2m} + 1)Y^4 = -2^{2m}.$$

It will first shown that a positive integer solution to this equation in positive integers gives rise to an integer solution to a certain Thue equation equation, and then apply a recent result of Akhtari [1] in order to deduce an upper bound for the number of integer solutions to $X^2-(2^{2m}+1)Y^4 = -2^{2m}$, which is independent of $m$. As the specific cases $m = 0$ and $m = 1$ have already been dealt with by Ljunggren, we state our main theorems only for $m \geq 2$.

**Theorem 1.1.** *For any integer $m \geq 2$, the Diophantine equation*

$$X^2 - (2^{2m} + 1)Y^4 = -2^{2m} \tag{1}$$

*has at most* 12 *solutions in odd positive integers $X$, $Y > 1$.*

This result is almost certainly not the best possible. An extensive computation has only found the integer solution $(X, Y) = (103, 5)$, with $m = 2$ and $X > 1$.

## 2. Proof of Theorem 1.1

All coprime integer solutions $(x, y)$ to the quadratic equation

$$x^2 - (2^{2m} + 1)y^2 = -2^{2m}$$

are given by

$$x + y\sqrt{2^{2m} + 1} = \pm\left(\pm 1 + \sqrt{2^{2m} + 1}\right)\left(2^m + \sqrt{2^{2m} + 1}\right)^{2i}$$

for some $i \geq 0$. We refer to Lemma 2 of [2] for this fact.

For brevity, let $a = 2^{m-1}$, and let

$$\alpha = T + U\sqrt{1 + 4a^2} = 2^m + \sqrt{2^{2m} + 1}.$$

For $i \geq 0$, define sequences $\{T_i\}$ and $\{U_i\}$ by

$$\alpha^i = T_i + U_i\sqrt{1 + 4a^2}.$$

Therefore, a positive integer solution to $X^2 - (2^{2m} + 1)Y^4 = -2^{2m}$ is equivalent to a solution to

$$Y^2 = T_{2k} \pm U_{2k}$$

for some $k \geq 0$. By the well known identities $T_{2k} = T_k^2 + (1 + 4a^2)U_k^2$ and

$U_{2k} = 2T_kU_k$, this gives

$$Y^2 = (T_k \pm U_k)^2 + (2aU_k)^2,$$

and it is evident that the terms involved in this equality are pairwise coprime. Thus, there are coprime non-negative integers $r$ and $s$, of opposite parity, for which

$$Y = r^2 + s^2,\ T_k \pm U_k = r^2 - s^2, \quad 2aU_k = 2rs.$$

We will assume that $r$ is even, as the argument for the other case is identical. Letting $R = r/a$, solving each of these expressions for $T_k$ and $U_k$, substituting the result into $T_k^2 - (1+4a^2)U_k^2 = \pm 1$, and then simplifying leads to the equation

$$s^4 \pm 2s^3R - 6a^2R^2s^2 \mp 2a^2R^3S + a^4R^4 = \pm 1.$$

Now putting $x = \pm s$ and $y = R$ gives the Thue equation

$$x^4 - 2x^3y - 6a^2x^2y^2 + 2a^2xy^3 + a^4y^4 = \pm 1.$$

There roots of the dehomogenized quartic polynomial

$$p_a(x) = x^4 - 2x^3 - 6a^2x^2 + 2a^2x + a^4, \tag{2}$$

are given explicitly by

$$\beta_1 = \frac{1}{2}\left(1 + \varepsilon + \sqrt{2}\sqrt{\varepsilon^2+\varepsilon}\right),$$
$$\beta_2 = \frac{1}{2}\left(1 + \varepsilon - \sqrt{2}\sqrt{\varepsilon^2+\varepsilon}\right),$$
$$\beta_3 = \frac{1}{2}\left(1 - \varepsilon + \sqrt{2}\sqrt{\varepsilon^2-\varepsilon}\right),$$
$$\beta_4 = \frac{1}{2}\left(1 - \varepsilon - \sqrt{2}\sqrt{\varepsilon^2-\varepsilon}\right),$$

where $\varepsilon = \sqrt{1+4a^2}$. We see therefore that $p_a(x)$ is irreducible, and that all four roots of the polynomial $p_a(x)$ are real.

The $j$-invariant of a quartic polynomial $a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, is defined to be the expression $j = 2a_2^3 - 9a_1a_2a_3 + 27a_1^2a_4 - 72a_0a_2a_4 + 27a_0a_3^2$, which happens to vanish in the case of $p_a(x)$.

We can now apply a recent result of Akhtari (see Theorem 1.1 of [1]), which that that if $F(x, y)$ is an irreducible homogeneous quartic polynomial with integer coefficients, whose roots are all real, and for which the j-invariant of the dehomogenized quartic of $F(x,y)$ vanishes, then the equation $|F(x,y)| = 1$ has at most 12 solutions in integers $(x,y)$, where the solution $(-x,-y)$ is identified with the solution $(x,y)$. In particular, the equation

$$x^4 - 2x^3y - 6a^2x^2y^2 + 2a^2xy^3 + a^4y^4 = \pm 1$$

has at most 12 solutions in integers $x$, $y$ (with $(-x,-y)$ identified with $(x,y)$), and we note that if a solution $(x,y)$ to this Thue equation gives rise to a positive integer solution to $Y^2 = T_{2k} \pm U_{2k}$, then $(-x,-y)$ gives rise to the same solution. This completes the proof of Theorem 1.1.

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