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Hahn–Mazurkiewicz revisited: A generalization

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Abstract. We generalize Hahn–Mazurkiewicz theorem from Peano continua to generalized Peano continua (locally connected, locally compact, connected, separable metric spaces) replacing the $[0,1]$ interval by the hedgehog. We also relate the number of "spines" of the hedgehog with compactifications with a finite remainder.

1. Introduction

One of the greatest achievements in General Topology was Hahn–Mazurkiewicz theorem (see [9] and [12]), which can be stated ([29, Theorem 31.5]): A Hausdorff space is a continuous image of the unit interval $[0,1]$ if and only if it is a Peano continuum (compact, connected, locally connected and metrizable). It characterizes which topological spaces are the continuous image of the unit interval, giving a generalization of the famous Peano curve. That is the reason why the spaces that appear in the characterization are known as Peano continua.

Since 1920, several generalizations of this important result have been obtained. But the generalizations have been mainly focused on replacing $[0,1]$ by a more general ordered continuum, as in [7], [10], [14]–[21], [24], [22], [23], [25]–[28]. Only a few attempts has been made in order to extend the theorem to non-continua (for example [5] and [13]).

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In this paper we generalize the Hahn–Mazurkiewicz theorem to a wider class of spaces that have every important property of Peano continua except compactness. That is, to locally compact, locally connected, connected, non-compact, separable metric spaces, called henceforth "*generalized Peano continua*". Generalized Peano continua include Euclidean spaces. A very interesting question is whether or not the Euclidean spaces can be filled by a curve of a certain kind.

Another important question one can ask about the Hahn–Mazurkiewicz theorem is which kind of spaces can be obtained as continuous images of $]0,1[$ or of $[0,1[$ (that is to say, of $\mathbb{R}$ or of $\mathbb{R}_0^+$). This is a fairly natural question, since those images correspond to the idea of "unbounded curves". But, since none of those spaces are compact, we think that it is more interesting to consider *perfect* images of $\mathbb{R}_0^+$ and $\mathbb{R}$ (note that Euclidean spaces of dimension greater than one are perfect images of $\mathbb{R}_0^+$ and of $\mathbb{R}$).

Surprisingly, there is a strong connection between both generalizations suggested. In fact, the continuous perfect images of $\mathbb{R}_0^+$ and $\mathbb{R}$ turn out to be certain classes of generalized Peano continua. That is, the natural generalization on one side of the theorem strictly corresponds to a natural generalization on the other side.

But we would like to obtain generalized Peano continua as a continuous perfect image of certain spaces that appear as generalizations (as natural as possible) of $[0,1]$, $\mathbb{R}_0^+$ and $\mathbb{R}$. These spaces are just the hedgehogs (and other ones that we call the spiders). The idea underlying the construction of those spaces is very simple: $\mathbb{R}_0^+$ has "one end" and $\mathbb{R}$ has "two ends" (or "spines" in the case of the hedgehog or "legs" in the case of spiders), so we consider spaces with as many "ends" as needed. We shall show that those spaces are the ones that do the job.

But there is still another surprising connection to explore: the number of legs (or spines) of the spider (or hedgehog) whose continuous perfect image is the space, is just the number of points that can be added to the space in order to obtain a finite compactification of it. In fact, this number becomes a classifying invariant for the class of generalized Peano continua under what we call "perfect equivalence".

It was proved in [5] that a topological space is a generalized Peano continuum if and only if it is the perfect image of the binary Cantor tree (which is a spider with infinite many legs). In this paper we focus on the finite case, by characterizing which spaces are the perfect image of spiders of a finite number of legs and how this is related with finite compactifications.

This paper is organized as follows. In Section 2 we introduce GF-spaces, a tool introduced by the authors in [2] which has proved to be very fruitful in showing connections among very diverse concepts like fractals, inverse limits, quasiuniformities or Wallman compactifications, to name a few. In Section 3 we define the hedgehogs and the spiders in the form we shall need later. In Section 4 we obtain the generalization of the Hahn–Mazurkiewicz theorem. In Section 5 we show the connections between the results obtained in Section 4 and the n-point compactifications introduced in [11]. Finally, in Section 6 we show that the role of the hedgehog and the spider in the preceding results can be performed by any other generalized Peano continua.

2. GF-spaces

We recall from [2] some definitions and introduce notation that will be useful in this paper.

Let $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ be a family of coverings. Recall that $\mathrm{St}(x, \Gamma_n) = \bigcup\{A_n \in \Gamma_n : x \in A_n\}$; we also define $U^{\mathbf{\Gamma}}_{xn} = X \setminus \bigcup\{A_n \in \Gamma_n : x \notin A_n\}$ which will be also noted by U_{xn} if there is no doubt about the family.

Let Γ be a covering of X. Γ is said to be *locally finite* if for all $x \in X$ there exists a neighborhood of x which meets only a finite number of elements of Γ. Γ is said to be a *tiling*, if all elements of Γ are regularly closed (a subset is regularly closed ([29, Problem 3.D]) if it is the closure of its interior) and they have disjoint interiors (see [1]).

Definition 2.1. Let X be a topological space. A pre-fractal structure over X is a family of coverings $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ such that $\{U^{\mathbf{\Gamma}}_{xn} : n \in \mathbb{N}\}$ is an open neighborhood base of x for all $x \in X$.

Furthermore, if Γ_{n+1} is a refinement of Γ_n, such that for all $x \in A_n$, with $A_n \in \Gamma_n$, there is $A_{n+1} \in \Gamma_{n+1}$ such that $x \in A_{n+1} \subseteq A_n$, we will say that $\mathbf{\Gamma}$ is a fractal structure over X.

If $\mathbf{\Gamma}$ is a (pre-) fractal structure over X, we will say that $(X, \mathbf{\Gamma})$ is a generalized (pre-) fractal space or simply a (pre-) GF-space. If there is no doubt about $\mathbf{\Gamma}$, then we will say that X is a (pre-) GF-space.

Note that if $\mathbf{\Gamma}$ is a pre-fractal structure then Γ_n is a closure preserving closed covering, for each $\Gamma_n \in \mathbf{\Gamma}$ (see [3, Proposition 2.4]).

If $\mathbf{\Gamma}$ is a fractal structure over X, and $\{\mathrm{St}(x, \Gamma_n) : n \in \mathbb{N}\}$ is a neighborhood base of x for all $x \in X$, we will call $(X, \mathbf{\Gamma})$ a starbase GF-space.

If Γ_n has the property P for all $n \in \mathbb{N}$, and $\mathbf{\Gamma}$ is a fractal structure over X, we will say that $\mathbf{\Gamma}$ is a fractal structure over X with the property P, and that X

is a GF-space with the property P. For example, if Γ_n is locally finite for all natural number n, and $\mathbf{\Gamma}$ is a fractal structure over X, we will say that $\mathbf{\Gamma}$ is a locally finite fractal structure over X, and that $(X, \mathbf{\Gamma})$ is a locally finite GF-space.

It was proved in [2] that if $\mathbf{\Gamma}$ is a fractal structure over a topological space X then the family $\{U_n^{\mathbf{\Gamma}} : n \in \mathbb{N}\}$ is a transitive base for a quasi-uniformity on X, where $U_n^{\mathbf{\Gamma}} = \{(x, y) \in X \times X : y \in U_{xn}^{\mathbf{\Gamma}}\}$. We will write U_n for $U_n^{\mathbf{\Gamma}}$ and U_{xn}^{-1} for $(U_n^{\mathbf{\Gamma}})^{-1}(x)$ if there is no confusion on the fractal structure $\mathbf{\Gamma}$. We will also write U_{xn}^* for $(U_n^{\mathbf{\Gamma}})^*(x) = U_n^{\mathbf{\Gamma}}(x) \cap (U_n^{\mathbf{\Gamma}})^{-1}(x)$. We will adopt these notations for pre-fractal structures, too.

The following proposition has an easy proof and it is proved in [2, Proposition 3.2].

Proposition 2.2. *Let $(X, \mathbf{\Gamma})$ be a pre-GF-space. Then $U_{xn}^{-1} = \bigcap\{A_n \in \Gamma_n : x \in A_n\}$.*

The following definition was introduced in [4].

Definition 2.3. Let $\mathbf{\Gamma}$ be a pre-fractal structure over X. We say that Γ_n is connected, if for all $x, y \in X$, there exists a finite subfamily $\{A_n^i : 0 \leq i \leq k+1\}$ of Γ_n with $x \in A_n^0$, $y \in A_n^{k+1}$ and $A_n^i \cap A_n^j \neq \emptyset$ for all $|i - j| \leq 1$ (we call it a weak chain in Γ_n joining x and y). We say that $\mathbf{\Gamma}$ is connected if Γ_n is connected for all $n \in \mathbb{N}$.

3. The hedgehog and the spider

In this section we introduce the two classes of spaces that we shall show that can replace the unit interval for a generalized version of Hahn–Mazurkiewicz theorem. First, we show in the next example why $\mathbb{R}$ is not the perfect image of $[0, 1[$. We recall that $p : X \to Y$ is a perfect map if it is closed, continuous, surjective and $p^{-1}(y)$ is compact whenever $y \in Y$.

Example 3.1. It is easy to see that $[0, 1[$ is the perfect image of $\mathbb{R}$. We show that $\mathbb{R}$ is not the perfect image of $[0, 1[$.

Suppose that $f : [0, 1[\to \mathbb{R}$ is a perfect onto map. Then $f^{-1}(0)$ is a compact space in $[0, 1[$, let M be its maximum. Since $f(]M, 1[)$ is connected and does not meet 0, it follows that $f(]M, 1[) \subseteq \mathbb{R}^+$ or $f(]M, 1[) \subseteq \mathbb{R}^-$. Suppose the former. Then it is clear that $f^{-1}(\mathbb{R}_0^-) \subseteq [0, M]$ and it is closed, and hence it is compact, which contradicts that $\mathbb{R}_0^-$ is not compact.

Now one can guess that $\mathbb{R}$ is not the perfect image of $[0,1[$ because it has more ends (two) than $[0,1[$ (only one).

The following definition measures the number of ends of a space.

Definition 3.2. Let X be a topological space. We denote by $\operatorname{sdeg}(X)$ (the star-degree of X) the least integer $n \in \mathbb{N}$ such that for every compact subspace A of X there exists a continuum K with $A \subseteq K$ and such that $X \setminus K$ has at most n connected components.

Moreover, let $\operatorname{sdeg}(X) = \infty$ if there exists no such integer and $\operatorname{sdeg}(X) = 0$ if X is compact.

So let us construct a space with n ends. This space is similar to that known as the hedgehog (see [29, Problem 23A]), so we shall also call it hedgehog.

Definition 3.3. Let $n \in \mathbb{N}$ and $I_i = \mathbb{R}_0^+$ for $i = 1, \dots, n$. We define in $\bigoplus_{i=1}^n I_i$ the equivalence relation $x_i R y_j$ with $x_i \in I_i$ and $y_j \in I_j$ if and only if $x_i = y_j = 0$, with the distance $d(x_i, y_j)$ defined by $d_i(x_i, 0) + d_j(0, y_j)$ if $i \neq j$ and by $d_i(x_i, y_j)$ if $i = j$ (where d_i is the usual metric in I_i). This space will be denoted by H^n and will be called the large hedgehog with n spines (or simply the hedgehog with n spines).

Note that H^1 is homeomorphic to $\mathbb{R}_0^+$ and H^2 is homeomorphic to $\mathbb{R}$. Also note that H^n is a locally compact, locally connected, connected, separable metric space for every $n \in \mathbb{N}$. Now we extend the argument given in Example 3.1 to see the relation between the number of "spines" and perfect images.

Proposition 3.4. *H^n is not the perfect image of H^m for $m < n$, but H^m is the perfect image of H^n.*

Proof. It is easy to see that H^n is the perfect image of H^{n+1}. We need to show that H^{n+1} is not the perfect image of H^n.

Suppose that $f : H^n \to H^{n+1}$ is a perfect onto map. Then $f^{-1}(0)$ is a compact space in H^n, let M_i be maximum of it in I_i. Given $i \in \{1, \dots, n\}$, it is clear that $f(\,]M_i, \to [\,_i)$ is connected and does not meet 0, so it follows that there exists $j(i) \in \{1, \dots, n+1\}$ such that $f(\,]M_i, \to [\,_i) \subseteq \,]0, \to [\,_{j(i)}$. Let j_0 be such that $j_0 \neq j(i)$ for any $i \in \{1, \dots, n\}$ (note that it exists, since $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, n+1\}$). Then it is clear that $f^{-1}([\,0, \to [\,_{j_0}) \subseteq \bigcup_{k=1}^n [\,0, M_k]_k$ and it is closed, and hence it is compact, which contradicts that $[\,0, \to [\,_{j_0}$ is not compact. $\square$

Note that the previous result is a corollary of the more general Theorem 4.16.

Now we are going to construct another space that has n-ends. The reader may wonder if it is not enough with the hedgehog, but proofs will become easier using this new "animal" called the spider. In fact, we are only going to use the hedgehog in this section to see that for our purposes it is equivalent to use the hedgehog instead of the spider.

Definition 3.5. Let $n \in \mathbb{N}$, we define the spider with n legs D^n as follows. Let $d_\emptyset = \left(\frac{1}{2}, 0\right)$, $d_1 = \left(\frac{1}{2^2}, 1\right)$, $d_2 = \left(\frac{3}{2^2}, 1\right)$, and let $s(1) = 1$, $s(2) = 3$. In general let $d_{t_1 \ldots t_k} = \left(\frac{s(t_1 \ldots t_k)}{2^{k+1}}, k\right)$, where $s(t_1 \ldots t_k) = 2s(t_1 \ldots t_{k-1}) + (2t_k - 3)$ with $t_i \in \{1, 2\}$.

We denote $D^1_K = \{d_\emptyset\}$ and $S^1 = \{d_\emptyset\}$, $D^2_K = [d_\emptyset, d_1] \cup [d_\emptyset, d_2]$ and $S^2 = \{d_1, d_2\}$, $D^3_K = D^2_K \cup [d_1, d_{11}] \cup [d_1, d_{12}]$ and $S^3 = \{d_{11}, d_{12}, d_2\}$, $D^4_K = D^3_K \cup [d_2, d_{21}] \cup [d_2, d_{22}]$ and $S^4 = \{d_{11}, d_{12}, d_{21}, d_{22}\}$, and so on. Given $(x, y) \in \mathbb{R}^2$, let $r_{(x,y)} = \{(x', y') : x' = x;\ y' \geq y\}$. We define $D^n = D^n_K \cup \{r_z : z \in S^n\}$. D^n_K is called the body of the spider and $\{r_z : z \in S^n\}$ are the n legs of the spider.

Let $x, y \in D^n$. We say that y is "above" x if there exists a finite sequence $t_1, \ldots, t_m$ with $t_i \in \{1, 2\}$ such that $x \in [d_{t_1,\ldots,t_k}, d_{t_1,\ldots,t_{k+1}}]$ for some $k \in \{1, \ldots, m-1\}$ and $y \in [d_{t_1,\ldots,t_{m-1}}, d_{t_1,\ldots,t_m}]$, if $x, y \in [d_{t_1,\ldots,t_{m-1}}, d_{t_1,\ldots,t_m}]$ then $y_2 \geq x_2$ with $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Remark 3.6. Note that $\operatorname{sdeg}(D^n) = \operatorname{sdeg}(H^n) = n$.

Note moreover that D^1 is homeomorphic to $[0, 1[$ and D^2 is homeomorphic to $\mathbb{R}$. Also note that D^n is a locally compact, locally connected, connected, non-compact separable metric space for every $n \in \mathbb{N}$. It holds that D^n is not the perfect image of D^m for $m < n$, but D^m is the perfect image of D^n. In fact we have the next proposition.

Proposition 3.7. *Let $n \in \mathbb{N}$. Then H^n is the perfect image of D^n and D^n is the perfect image of H^n.*

Proof. Let H^n be $\frac{\bigoplus_{i=1}^n I_i}{R}$ with the relation R as defined in 3.3 (where $I_i = \mathbb{R}^+_0$). Let $K = \frac{\bigoplus_{i=1}^n K_i}{R}$ with $K_i = [0, 1]$. It is easy to construct an onto continuous map $f_0 : K \to D^n_K$ with $f_0(1_i) = x_i$, where $S^n = \{x_1, \ldots, x_n\}$ and $1_i = 1 \in K_i$. If $1 \leq k \leq n$, then we can define $f_k : [1, \to [_k \to r_{x_k}$ a homeomorphism with $f_k(1) = x_k$. Then it is clear that the map $f : H^n \to D^n$ defined by $f_0(x)$ if $x \in K$ and by $f_k(x)$ if $x \in [1, \to [_k \subseteq I_k$ is an onto perfect mapping (note that f_0 is perfect and f_k is a homeomorphism).

The converse is similar. $\square$

Note that H^n and D^n are not homeomorphic for $n \geq 3$. The aim of the next definition and the next two propositions is to introduce standard fractal structures in the spider and the hedgehog from the standard fractal structure of the legs and the spines.

Definition 3.8. Let X be a topological space and let $\{F_i : i \in I\}$ be a locally finite closed covering of X. For each $i \in I$, let $\mathbf{\Gamma}^i$ be a pre-fractal structure over F_i. Let $\Gamma_n = \bigcup_{i\in I} \Gamma_n^i$ and let $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$. $\mathbf{\Gamma}$ is called the 'weak fractal structure' induced on X by the family $\{(F_i, \mathbf{\Gamma}^i) : i \in I\}$.

Proposition 3.9. *Let X be a topological space and let $\{F_i : i \in I\}$ be a locally finite closed covering of X. For each $i \in I$, let $\mathbf{\Gamma}^i$ be a locally finite pre-fractal structure over F_i, and let $\mathbf{\Gamma}$ be the 'weak fractal structure' induced on X by the family $\{(F_i, \mathbf{\Gamma}^i) : i \in I\}$. Then $\mathbf{\Gamma}$ is, in fact, a locally finite pre-fractal structure over X. If $\mathbf{\Gamma}^i$ is a fractal structure for each $i \in I$ then so is $\mathbf{\Gamma}$. If $\mathbf{\Gamma}^i$ is starbase for each $i \in I$ then so is $\mathbf{\Gamma}$.*

Proof. It is clear that Γ_n is a closed locally finite covering of X. On the other hand, by using Proposition 2.2 , it is straightforward to check that $U^{\mathbf{\Gamma}}_{xn} = \bigcap_{i\in I} U^{\mathbf{\Gamma}^i}_{xn}$, and since $\{F_i : i \in I\}$ is locally finite, it follows that $\mathbf{\Gamma}$ is a pre-fractal structure over X. The rest of the proposition is easy to prove. $\square$

The proof of the next proposition is straightforward. The elements of a tiling fractal structure can be compared with the concept of partitioning of Bing ([6]).

Proposition 3.10. *Let X be a topological space and let $\{F_i : i \in I\}$ be a finite closed covering of X. For each $i \in I$, let $\mathbf{\Gamma}^i$ be a pre-fractal structure over F_i, and let $\mathbf{\Gamma}$ be the 'weak fractal structure' induced on X by the family $\{(F_i, \mathbf{\Gamma}^i) : i \in I\}$. Then $\mathbf{\Gamma}$ is a pre-fractal structure over X. If $\mathbf{\Gamma}^i$ is a fractal structure for each $i \in I$ then so is $\mathbf{\Gamma}$. If $\mathbf{\Gamma}^i$ is starbase, finite or locally finite for each $i \in I$ then so is $\mathbf{\Gamma}$. If $\{F_i : i \in I\}$ is a tiling and $\mathbf{\Gamma}^i$ are tilings, then so is $\mathbf{\Gamma}$.*

Definition 3.11. We define the usual finite fractal structure over the interval $[0, 1]$ as $\Gamma_n = \left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] : 0 \leq k \leq 2^n - 1\right\}$.

We define the usual finite fractal structure over $\mathbb{R}_0^+$ (isometrically in r_x) as $\Gamma_n = \left\{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] : 0 \leq k \leq n2^n - 1\right\} \cup \{[n, \rightarrow[\,\}$.

In D^n we define the usual fractal structure as the weak fractal structure induced by the family $\{[d_{t_1,\ldots,t_k}, d_{t_1,\ldots,t_{k+1}}] : t_i \in \{1, 2\}; d_{t_1,\ldots,t_{k+1}} \in D^n_K\} \cup \{r_x : x \in S^n\}$ with their usual fractal structures.

It is easy to see that the usual fractal structure $\mathbf{\Gamma}$ of D^n is a finite connected starbase fractal structure and such that A_n is connected for all $A_n \in \Gamma_n$.

4. Generalized Peano continua

The other main concept of the paper is that of a generalized Peano continuum. In this section we will introduce it and study its main properties as GF-spaces. First we recall the following result from [4].

Proposition 4.1 ([4, Proposition 3.5])**.** *Let $\mathbf{\Gamma}$ be a fractal structure over a connected space X. Then $\mathbf{\Gamma}$ is connected.*

We name as "generalized Peano continua" those spaces that satisfy all the interesting properties of Peano continua except being compact.

Definition 4.2. A topological space X is said to be a generalized Peano continuum (see [5]) if it is a locally compact, locally connected, connected metrizable space.

Note that a generalized Peano continuum is separable ([5]).

As first examples of generalized Peano continua we have D^n, H^n, as well as $\mathbb{R}^n$.

Lemma 4.3 ([4, Lemma 3.7])**.** *Let $\mathbf{\Gamma}$ be a starbase fractal structure over X, K be a compact subset of X and F be a closed subset of X disjoint from K. Then there exists $n \in \mathbb{N}$ such that $\mathrm{St}(K, \Gamma_n) \cap F = \emptyset$.*

Perfect onto mappings can be used to induce a fractal structure on the image.

Proposition 4.4 ([4, Proposition 4.12])**.** *Let $(X, \mathbf{\Gamma})$ be a starbase GF-space and Y a topological space. Let $f : X \to Y$ be an onto perfect mapping. Let $\mathbf{\Delta} = f(\mathbf{\Gamma})$ be defined by $\Delta_n = \{f(A_n) : A_n \in \Gamma_n\}$ for all $n \in \mathbb{N}$. Then $\mathbf{\Delta}$ is a starbase fractal structure over Y. If $\mathbf{\Gamma}$ is finite, so is $\mathbf{\Delta}$.*

Lemma 4.5. *Let X be a locally connected, connected metrizable space, let A be a compact subspace of X and let M be a connected component of $X \setminus A$. Then $A \cap \overline{M} \neq \emptyset$.*

Proof. Let d be a compatible metric for X and let us suppose that $A \cap \overline{M} = \emptyset$. Since A is compact, $\varepsilon = d(A, \overline{M}) > 0$. Let $x \in A$ and $y \in M$. Since X is connected and locally connected, there exists a chain $(C_i)_{i=1,\dots,n}$ of connected subspaces of X with diameter less than ε which joins x and y. It is clear that if $C_i \cap A = \emptyset$ and $C_i \cap M \neq \emptyset$, then $C_i \subseteq M$, and hence there exists C_i with $A \cap C_i \neq \emptyset$ and $C_i \cap M \neq \emptyset$. But then $d(A, \overline{M}) < \varepsilon$, a contradiction. □

The following lemma gives a clue for a classification of generalized Peano continua.

Lemma 4.6. *Let X be a generalized Peano continuum. Then for every compact subspace A of X, there exists a continuum K with $A \subseteq K$ and such that $X \setminus K$ has a finite number of connected components.*

Proof. Let A be a continuum subspace of X (we do not say subcontinuum, because X is not a continuum. Note that we can suppose that A is a continuum, since we can always get a continuum in X that contains a given compact subspace of X). Let $X^* = X \cup \{o\}$ be the one-point compactification of X, and d a metric for X^* (note that X^* is metrizable, since X is separable). Let $\delta = d(o, A) > 0$ and $\{M_i : i \in I\}$ the countable number (since X is separable, and M_i is open) of connected components of $X \setminus A$ which are not included in $B_c\big(A, \frac{\delta}{4}\big) = \{x \in X : d(a, x) \leq \frac{\delta}{4}$ for some $a \in A\}$.

Let $J = \{i \in I : M_i$ is not included in $B_c(A, \frac{\delta}{3})\}$. Analogously to the proof of Lemma 4.5, it holds that $M_j \cap B_c\big(A, \frac{\delta}{2}\big) \setminus B_c\big(A, \frac{\delta}{3}\big) \neq \emptyset$ whenever $j \in J$. For each $j \in J$, let $x_j \in M_j \cap B_c\big(A, \frac{\delta}{2}\big) \setminus B_c\big(A, \frac{\delta}{3}\big)$. Then (x_j) has an adherent point x in X^*. It is clear by construction that $x \notin B_c\big(A, \frac{\delta}{4}\big)$ and that $x \in X$, and hence, there exists a connected component M of $X \setminus A$ such that $x \in M$. Since M is open, J must be finite (note that (x_j) is adherent to x, but $x_j \notin M$ for any $j \in J$ except pherhaps one of them).

Let $K = A \cup \bigcup\{M \subseteq B_c(A, \frac{\delta}{3}) : M$ is a connected component of $X \setminus A\}$. Then $X \setminus K$ has a finite number of connected components. Note that $K = A \cup \bigcup\{Cl_{X^*}(M) : M \subseteq B_c(A, \frac{\delta}{3})$ is a connected component of $X \setminus A\}$. Since, by Lemma 4.5, $Cl_X(M) \cap A \neq \emptyset$ for any connected component M of $X \setminus A$, K is connected, and since $K \subseteq B_c(A, \frac{\delta}{3})$, K is compact. Therefore, K verifies the thesis of the lemma. □

The next result provides a characterization of generalized Peano continua in terms of the one point compactification.

Corollary 4.7. *Let X be a connected locally compact non-compact Hausdorff space. Then X is a generalized Peano continuum if and only if its one point compactification is a Peano continuum.*

Proof. We will only show that, if the space is a generalized Peano continuum, the infinity point of the one point compactification has a neighborhood base of connected subsets.

Let X be a generalized Peano continuum, and $X^* = X \cup \{o\}$ the one point compactification of X. Let $X^* \setminus A$, with A compact, be a neighborhood of o. By Lemma 4.6, there exists a continuum K with $A \subseteq K$ and such that $X \setminus K$ has a finite number of connected components $C_1, \ldots, C_n$.

We can assume that $o \in \overline{C_i}$ for each $i = 1, \dots n$, where the closure is taken in X^*. Indeed, if $o \notin \overline{C_i}$ for some $i = 1, \dots, n$, we can define $K' = K \cup \bigcup\{\overline{C_j} : o \notin \overline{C_j}\}$. It holds that K' is a compact subspace of X containing K and, since $C_k \cap \overline{C_j} = \emptyset$ for $j \neq k$ (note that each C_k is open in X^*), $X \setminus K' = \bigcup\{C_j : o \in \overline{C_j}\}$, so we can continue with the argument replacing K with K'.

Since $X^* \setminus K = \bigcup\{C_i \cup \{o\} : i = 1, \dots, n\}$, and $C_i \cup \{o\}$ are connected subsets (note that $C_i \subseteq C_i \cup \{o\} \subseteq \overline{C_i}$) with the point o in common, it follows that $X^* \setminus K$ is a connected neighborhood of o contained in $X^* \setminus A$. □

In a Peano continuum, we can get a fractal structure with very good properties.

Lemma 4.8. *Let X be a Peano continuum. Then there exists a finite connected starbase fractal structure over X with A_n a continuum for each $A_n \in \Gamma_n$ and each $n \in \mathbb{N}$.*

Proof. Let $f : [0,1] \to X$ be a continuous onto mapping. Let $\Gamma_n = \{[\frac{k}{2^n}, \frac{k+1}{2^n}] : k = 0, \dots, 2^n - 1\}$ and $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$. Then $\mathbf{\Gamma}$ is the usual finite fractal structure over $[0,1]$. Let $\mathbf{\Delta} = f(\mathbf{\Gamma})$ (see Proposition 4.4). By Proposition 4.4, $\mathbf{\Delta}$ is a finite starbase fractal structure over X. Since X is connected, then $\mathbf{\Delta}$ is connected, and it is clear that $f([\frac{k}{2^n}, \frac{k+1}{2^n}])$ is a continuum for each $k = 0, \dots, 2^n - 1$ and $n \in \mathbb{N}$. □

Definition 4.9. Let $\mathbf{\Gamma}$ be a fractal structure, $\Gamma_n \in \mathbf{\Gamma}$ and $A_n \in \Gamma_n$. We will write $\Gamma_m(A_n) = \{A_{n+m} \in \Gamma_{n+m} : A_{n+m} \subseteq A_n\}$.

Using the previous results we get a fractal structure with good properties for any generalized Peano continuum.

Lemma 4.10. *Let X be a generalized Peano continuum. Then there exists a locally finite connected starbase fractal structure $\mathbf{\Gamma} = \{\Gamma_n : n \in \mathbb{N}\}$ over X with A_n a continuum and $\Gamma_1(A_n)$ finite for each $A_n \in \Gamma_n$ and each $n \in \mathbb{N}$ and such that Γ_n is countable for all $n \in \mathbb{N}$.*

Proof. By Corollary 4.7, the one point compactification $X^* = X \cup \{o\}$ is a Peano continuum. By Lemma 4.8, there exists a finite connected starbase fractal structure $\mathbf{\Gamma}$ over X^* with A_n a continuum for each $A_n \in \Gamma_n$ and $n \in \mathbb{N}$.

Let $\Gamma'_n = \{A_n \in \Gamma_n : o \notin A_n\} \cup \{A_m \in \Gamma_m : m > n; o \notin A_m;$ and $A_m \nsubseteq \bigcup\{B \in \Gamma_{m-1} : o \notin B\}\}$. It is clear that Γ'_n is a covering of X for each $n \in \mathbb{N}$ and that A'_n is a continuum for each $A'_n \in \Gamma'_n$ and $n \in \mathbb{N}$. Let $\mathbf{\Gamma}' = \{\Gamma'_n : n \in \mathbb{N}\}$.

Let $n \in \mathbb{N}$, and let us prove that Γ'_n is locally finite. Let $x \in X$. Since X is open in X^* and $\mathbf{\Gamma}$ is starbase, there exists $m \in \mathbb{N}$ such that $o \notin \mathrm{St}(x, \Gamma_m)$. If

$x \in A_k$ with $A_k \in \Gamma_k$, $k > m$, $o \notin A_k$ and $A_k \nsubseteq \bigcup\{B \in \Gamma_{k-1} : o \notin B\}$, then there exists $A_{k-1} \in \Gamma_{k-1}$ with $A_k \subseteq A_{k-1}$, and hence $o \in A_{k-1} \subseteq \mathrm{St}(x, \Gamma_{k-1}) \subseteq \mathrm{St}(x, \Gamma_m)$, a contradiction. Then given $A'_n \in \Gamma'_n$, $U^{\mathbf{\Gamma}'}_{xm} \cap A'_n \neq \emptyset$ if and only if $A'_n \in \big\{B_n \in \Gamma_n : o \notin B_n\} \cup \{A_k \in \Gamma_k : n < k \leq m;\ o \notin A_k;$ and $A_k \nsubseteq \bigcup\{B \in \Gamma_{k-1} : o \notin B\}\big\}$, which is finite, since Γ_i is finite for each $i \in \mathbb{N}$. Therefore Γ'_n is locally finite. It is clear that Γ'_n is countable.

Let $x \in X$ and $m \in \mathbb{N}$ with $o \notin \mathrm{St}(x, \Gamma_m)$. By the previous paragraph, $\mathrm{St}(x, \Gamma_m) = \mathrm{St}(x, \Gamma'_m)$, and hence $\mathbf{\Gamma}'$ is a starbase fractal structure over X. By Proposition 4.1, $\mathbf{\Gamma}'$ is connected.

Finally, given $A'_n \in \Gamma'_n$, then $A'_n = A_k$ for some $A_k \in \Gamma_k$ with $k \geq n$ and $o \notin A_k$. Let $A_i \in \Gamma_i$ with $i > k$ and such that $A_i \subseteq A_k$. Since $o \notin A_k$ then $A_i \notin \Gamma'_{k+1} \setminus \Gamma_{k+1}$. It follows that $\Gamma_1(A'_n)$ is finite for each $A'_n \in \Gamma_n$ and $n \in \mathbb{N}$. □

Remark 4.11. Let X be a generalized Peano continuum, and let $\mathbf{\Gamma}$ be as in the the previous lemma. Note that if A is connected then $\mathrm{St}(A, \Gamma_n)$ is connected and also note that if A is a continuum in X then $\mathrm{St}(A, \Gamma_n)$ is also a Peano continuum (see the next remark).

Remark 4.12. Let $\mathbf{\Gamma}$ be a locally finite starbase fractal structure over X such that A_n is a continuum whenever $A_n \in \Gamma_n$ and $n \in \mathbb{N}$. Then it is clear that A_n is locally connected, since B_m is connected for every $B_m \in \Gamma_m(A_n)$. Therefore $\bigcup_{i \in I} A^i_n$ is locally connected and locally compact for any subfamily $\{A^i_n : i \in I\}$ of Γ_n and any $n \in \mathbb{N}$.

Remark 4.13. Note that since $\mathbf{\Gamma}$ is starbase, it follows from the previous remark that any generalized Peano continuum has a neighborhood base of Peano continua.

A step further in the line of Lemma 4.6: if there is only one connected component, the space is the perfect image of $\mathbb{R}^+_0$.

Lemma 4.14. *Let X be a noncompact generalized Peano continuum with* $\mathrm{sdeg}(X) = 1$. *Then X is the perfect image of $H^1 = \mathbb{R}^+_0$.*

Proof. Let $\mathbf{\Gamma}$ be the fractal structure given by Lemma 4.10. Let $B_1 \in \Gamma_1$ and let K be a continuum in X which contains $\mathrm{St}(B_1, \Gamma_1)$ and such that $X \setminus K$ has only one connected component M_1. Let $L_1 = \mathrm{St}(M_1, \Gamma_1)$, $K_1 = \mathrm{St}(K, \Gamma_1)$ and let $x_1 \in K_1 \cap L_1$ (note that since K_1 and L_1 are closed, $X = K_1 \cup L_1$ and X is connected, then it follows that $K_1 \cap L_1 \neq \emptyset$). By Remark 4.11 it follows that K_1 is a Peano continuum and hence there exists a continuous onto map $f_1 : [0, 1] \to K_1$ with $f_1(1) = x_1$.

On the other hand it holds that L_1 is a locally connected (by Remark 4.12), locally compact (by Remark 4.12), connected (by Remark 4.11) noncompact (if L_1 is compact then $X = K_1 \cup L_1$ is compact, which is a contradiction) separable metric space (that is, a generalized Peano continuum) with $\mathrm{sdeg}(L_1) = 1$.

Recursively we construct a sequence (K_n) of Peano continua with $K_n \cap K_{n+1} \neq \emptyset$, $X = \bigcup_{n\in\mathbb{N}} K_n$ and such that given $x \in X$ there exists $n \in \mathbb{N}$ with $x \notin K_m$ for $m \geq n$; a sequence (L_n) of generalized Peano continua with $\mathrm{sdeg}(L_n) = 1$, $K_n \cap L_n \subseteq K_{n+1} \subseteq L_n$ and $K_n \cap L_{n+1} = \emptyset$, and there exist a sequence $x_n \in K_n \cap K_{n+1}$ and a sequence of continuous onto mappings $f_n : [n-1, n] \to K_n$ with $f_n(n-1) = x_{n-1}$ and $f_n(n) = x_n$. Therefore the map $f : \mathbb{R}_0^+ \to X$ defined as $f(x) = f_n(x)$ if $x \in [n-1, n]$ is a continuous mapping from $H^1 = \mathbb{R}_0^+$ onto X.

Let see that f is a perfect mapping. Given $x \in X$, there exists $n \in \mathbb{N}$ such that $x \notin K_m$ for $m \geq n$ and hence $f^{-1}(x) \subseteq [0, n]$ so it is compact; on the other hand, given F a closed set in H^1 it is clear that $f_n(F \cap [n-1, n])$ is closed and hence $f(F) = f\big(\bigcup_{n\in\mathbb{N}} F \cap [n-1, n]\big) = \bigcup_{n\in\mathbb{N}} f_n(F \cap [n-1, n])$. Since the family $\{K_n : n \in \mathbb{N}\}$ is locally finite (given $x \in X$, there exists $n \in \mathbb{N}$ such that $x \notin K_m$ for $m \geq n$ and hence x only meets K_i for $i \in \{1, \ldots, n\}$), then $\{f_n(F \cap [n-1, n]) : n \in \mathbb{N}\}$ is a locally finite closed family and hence $f(F)$ is closed and then f is a perfect mapping. $\square$

The previous result gives the clue: the number of connected components is just the number of "ends" of the space, so we call the spider to help us. First, we need the following proposition.

Proposition 4.15. *Let X and Y be generalized Peano continua and $f : X \to Y$ an onto perfect map. Then* $\mathrm{sdeg}(Y) \leq \mathrm{sdeg}(X)$.

Proof. Let $n = \mathrm{sdeg}(X)$ and A a compact subspace of Y. Then $f^{-1}(A)$ is a compact subspace of X, so there exists a continuum K' in X such that $f^{-1}(A) \subseteq K'$ and $X \setminus K'$ has at most n connected components.

Let K'' be a continuum in X with $f^{-1}(f(K')) \subseteq K''$ and such that $X \setminus K''$ has at most n connected components $N_1, \ldots, N_k$ (with $k \leq n$). Given $i \in \{1, \ldots, k\}$, there exists a connected component M_i of $Y \setminus f(K')$ such that $f(N_i) \subseteq M_i$. Let $K = Y \setminus \bigcup_{i=1}^k M_i$. Since K is closed, $f^{-1}(K)$ is closed. On the other hand, $f^{-1}(K) \subseteq K''$, and hence it is compact (indeed, let $x \in f^{-1}(K)$ and suppose that $x \notin K''$. Then there exists $i \in \{1, \ldots, k\}$ with $x \in N_i$. But then $f(x) \in M_i \subseteq Y \setminus K$, a contradiction). Therefore K is compact. Furthermore, $K = f(K') \cup \bigcup\{\overline{M} : M$ is a connected component of $Y \setminus f(K')$ different from M_i for $i \in \{1, \ldots, k\}\}$.

Since $f(K')$ is connected, the closure of each component $\overline{M}$ is connected and, by Lemma 4.5, $f(K') \cap \overline{M} \neq \emptyset$ for each component M of $Y \setminus f(K')$ different from M_i for $i \in \{1, \ldots, k\}$, it follows that K is connected. Therefore K is a continuum which contains A and $Y \setminus K$ has k connected components. We conclude that $\mathrm{sdeg}(Y) \leq n = \mathrm{sdeg}(X)$. □

Theorem 4.16. *Let $n \in \mathbb{N}$. The following statements are equivalent:*

(1) *X is the perfect image of D^n (equivalently, of H^n).*

(2) *X is a generalized Peano continuum with $\mathrm{sdeg}(X) \leq n$.*

Proof. 1) implies 2). Suppose that X is the perfect image of D^n. By [5, Lemma 1.1], X is a generalized Peano continuum, and by Proposition 4.15 and Remark 3.6, $\mathrm{sdeg}(X) \leq n$.

2) implies 1). In order to prove the converse, let X be a generalized Peano continuum with $\mathrm{sdeg}(X) \leq n$.

Let $\mathbf{\Gamma}$ be the fractal structure of Lemma 4.10.

For each compact subspace H of X, we define $n(H) = \min\{i \in \mathbb{N}$: there exists a continuum $K(H)$ with $H \subseteq K(H)$ and such that $X \setminus H(K)$ has exactly i connected component$\}$, and let $m = \max\{n(H) : H$ is compact$\}$. Note that $m \leq n$. Let A be a compact subspace of X such that $m = n(A)$, and let K be a continuum subspace of X with $A \subseteq K$ and such that $X \setminus K$ has exactly m connected component $M_1, \ldots, M_m$. It is clear that M_i is not relatively compact for $i = 1, \ldots, m$ (if it is, then $K \cup \overline{M_i}$ is a continuum containing A and such that $X \setminus (K \cup \overline{M_i})$ has only $m-1$ connected components, which contradicts that $m = n(A)$).

Let $K_i = \mathrm{St}(M_i, \Gamma_1)$. It is clear that K_i is a locally connected (by Remark 4.12), locally compact (by Remark 4.12), connected (by Remark 4.11), noncompact (since $M_i \subseteq K_i$, if K_i is compact then M_i will be relatively compact, but it is not) separable metric space.

Claim: Let us see that $\mathrm{sdeg}(K_i) = 1$.

Let B be a compact subspace of K_i. By Lemma 4.6 there exists a continuum subspace C of K_i containing B and such that $K_i \setminus C$ has a finite number $N_1, \ldots, N_k$ of connected components in K_i. It follows that only one of them is not relatively compact. To see this, since $K_i = C \cup \bigcup_{j=1}^k \overline{N_j}$, and K_i is not compact, then it follows that N_j is not relatively compact for at least one $j \in \{1, \ldots, k\}$. On the other hand, if N_1, N_2 are not relatively compact, let C' be a continuum (in X) containing $K \cup C$. Then it is clear that $X \setminus C' \subseteq X \setminus (K \cup C) \subseteq (\bigcup_{j \neq i} M_j) \cup (\bigcup_{j=1}^k M_i \cap N_j)$. Let $x_j \in M_j \setminus C'$ (note it is nonempty, since M_j is not relatively compact) for $j \neq i$, and let $y_j \in M_i \cap N_j \setminus C'$ for $j = 1, 2$ (note it is nonempty, since $M_i \cap N_j$ is not relatively compact. For, note that since $K_i \setminus \mathrm{St}(K, \Gamma_1) \subseteq M_i$, then $N_j = (M_i \cap N_j) \cup (N_j \cap \mathrm{St}(K, \Gamma_1))$, and since the latter part of the union is relatively compact, then it follows that the former is not). Let L_j be a connected component of $X \setminus C'$ that contains x_j for $j \neq i$, and let L'_j be a connected component of $X \setminus C'$ that contains y_j for $j = 1, 2$. Then $\{L_j : j \neq i;\ 1 \leq j \leq m\} \cup \{L'_1, L'_2\}$ is a family of $m + 1$ different (note that $L_j \subseteq M_j$ for $j \neq i$ and $L'_j \subseteq M_i \cap N_j$ for $j = 1, 2$) connected components of $X \setminus C'$, and hence $n(K \cup B) \geq m + 1$ which contradicts the choice of m.

Now, suppose that N_1 is not relatively compact. It is easy to see that if we define $K' = K_i \setminus N_1 = C \cup \bigcup_{j>1}^{k} N_j = C \cup \bigcup_{j>1}^{k} \overline{N_j}$, it follows that K' is a continuum containing B and such that $K_i \setminus K' = N_1$ has only one connected component, what proves the claim.

Now, it is clear that K_i verifies the hypotheses of Lemma 4.14. Therefore there exist perfect onto mappings $f_i : r_{t_i} \to K_i$ with $f(t_i) = z_i$ for $1 \leq i \leq m$, for all $t_i \in S^m$ and for some $z_i \in X$. On the other hand, it is easy to construct (by Hahn–Mazurkiewicz theorem) a (perfect) onto mapping $g : D^m_K \to \mathrm{St}(K, \Gamma_1)$ (since $\mathrm{St}(K, \Gamma_1)$ is a Peano continuum) such that $g(t_i) = z_i$ for $1 \leq i \leq m$, for all $t_i \in S^m$. It is clear that the map $f : D^m \to X$ defined by $f_i(z)$ if $z \in r_{t_i}$ with $1 \leq i \leq m$ and $t_i \in S^m$ and by $g(z)$ if $z \in D^m_K$ is a perfect onto map from D^m onto X. Therefore X is the perfect image of D^m and then of D^n. □

5. Finite compactifications

In this section we give another turn of the screw: the second condition in Theorem 4.16 resembles a similar one for finite compactifications. Could we find a relation? The next definition and the corollary show that we can.

Definition 5.1. Let X be a locally compact space. We denote by $\mathrm{fcdeg}(X)$ (the finite compactification degree of X) the greatest integer $n \in \mathbb{N}$ such that X has an n-point compactification (that is, a compactification with a remainder of n points). Moreover, let $\mathrm{fcdeg}(X) = \infty$ if X has an n-point compactification for all $n \in \mathbb{N}$ and $\mathrm{fcdeg}(X) = 0$ if X is compact.

Note from the definitions that $\mathrm{fcdeg}(X)$ is defined as the maximum of some integers while $\mathrm{sdeg}(X)$ is defined as the minimum of another set of integers. It would be surprising that both were equal. However that is exactly what happens, as shown in the next theorem, which is in the spirit of results of [11].

Theorem 5.2. *Let X be a generalized Peano continua. Then* $\mathrm{sdeg}(X) = \mathrm{fcdeg}(X)$.

Proof. The case when X is compact is clear, so let us suppose that X is not compact. Suppose that $\mathrm{fcdeg}(X) = n$. Let A be a compact subspace of X, and let K be any continuum in X containing A such that its complementary has $k = \mathrm{sdeg}(X)$ (or a finite number k if $\mathrm{sdeg}(X) = \infty$) connected components. Let $M_1, \ldots, M_k$ be the connected components of $X \setminus K$. Note that M_i is not relatively compact, since otherwise $K \cup \overline{M_i}$ is a continuum (note that $K \cap \overline{M_i} \neq \emptyset$ by Lemma 4.5) containing A and such that its complementary has only $k-1$ connected components, which contradicts that $k = \mathrm{sdeg}(X)$.

Then $M_1, \ldots, M_k$ are open subsets of X with $M_i \cap M_j = \emptyset$, $X \setminus \bigcup_{i=1}^{k} M_i = K$ is compact, and $X \setminus \bigcup_{i \neq j} M_i = K \cup M_j$ is not compact (since M_j is not relatively compact) for $j = 1, \ldots, k$. Therefore X has a k-point compactification by [11], and hence $n = \mathrm{fcdeg}(X) \geq k$. Therefore $\mathrm{sdeg}(X) \leq \mathrm{fcdeg}(X)$.

Now, suppose that $\mathrm{sdeg}(X) = k$ and let $n \in \mathbb{N}$ with $k < n \leq \mathrm{fcdeg}(X)$. Since X has an n-point compactification, by [11] there exist open subsets G_i, with $i = 1, \ldots, n$, of X such that $G_i \cap G_j = \emptyset$ for $i \neq j$, $X \setminus \bigcup_{j \neq i} G_j$ is not compact (and hence G_i is not relatively compact for any $i \in \{1, \ldots, n\}$) and $B = X \setminus \bigcup_{i=1}^{n} G_i$ is a compact subset of X. Let K' be a continuum containing B and such that $X \setminus K'$ has l (with $l \leq k < n$) connected components $N_1, \ldots N_l$. Suppose that $N_i \cap G_j \neq \emptyset$. Then $G_j \cap N_i$ is a clopen subset of N_i, and hence $N_i \subseteq G_j$. We can suppose that $N_i \subseteq G_i$ for $i = 1, \ldots, l$, then $G_{l+1} \cap N_i = \emptyset$ for every $1 \leq i \leq l$, and hence $G_{l+1} \subseteq K'$, which contradicts that G_{l+1} is not relatively compact. Therefore $n \leq k$, that is, $\mathrm{sdeg}(X) \geq \mathrm{fcdeg}(X)$.

We conclude that $\mathrm{sdeg}(X) = \mathrm{fcdeg}(X)$. □

We present a very neat characterization of perfect images of $[0, 1[$.

Corollary 5.3. *A generalized Peano continua is the perfect image of* $[0, 1[$ *if and only if its only finite compactification is the one-point compactification.*

6. Perfect equivalence

In this final section we prove that the role of the hedgehog or the spider in the characterization and classification of generalized Peano continua can be played by any generalized Peano continuum of the same class.

We first begin checking that the role of $[0, 1]$ in the classical Hahn–Mazurkiewicz theorem can be played by any other Peano continua.

Remark 6.1 ([29, Problem 31.A]). Let X, Y be Peano continua with more than one point. Then X is the image of Y and Y is the image of X.

The next two lemmas are of technical nature, but they are needed for the characterization theorem.

Lemma 6.2. *Let X be a noncompact generalized Peano continuum, and let A be any compact subspace of X. Then there exist $n \in \mathbb{N}$ ($n = \mathrm{sdeg}(X)$ if $\mathrm{sdeg}(X)$ is finite), P a Peano continuum containing A and K_i generalized Peano continua with $\mathrm{sdeg}(K_i) = 1$ for $i = 1, \ldots, n$, such that $X = P \cup \bigcup_{i=1}^{n} K_i$ and $K_i \cap K_j \subseteq P$ for $i \neq j$.*

Proof. It follows from a reasoning similar to the proof of Theorem 4.16. Just take $P = \mathrm{St}(K, \Gamma_1)$. □

Lemma 6.3. *Let X be a generalized Peano continuum with $\mathrm{sdeg}(X) = 1$. Then there exists a sequence of continua C_k and a sequence of Peano continua H_k such that $X = \bigcup_{k\in\mathbb{N}} C_k$, $C_k \cap C_l \neq \emptyset$ if and only if $|k-l| \leq 1$ and $H_k \subseteq C_k \setminus \bigcup_{l\neq k} C_l$ for $k \in \mathbb{N}$.*

Proof. Let K_n be a sequence of compact subspaces of X with $K_n \subseteq K_{n+1}^{\circ}$ for $n \in \mathbb{N}$ and $X = \bigcup_{n\in\mathbb{N}} K_n$ (see [8, Example 3.8.C.b]).

Let $x \in K_1^{\circ}$. By Remark 4.13, there exists a Peano continuum H_1 with $x \in H_1 \subseteq K_1^{\circ}$. Let $C_1 = K_1$.

Suppose that we have constructed C_n and H_n and let us go to construct C_{n+1} and H_{n+1}. For, let $y \in K_{n+1}^{\circ} \setminus K_n$. By Remark 4.13, there exists a Peano continuum H_{n+1} with $y \in H_{n+1} \subseteq K_{n+1}^{\circ} \setminus K_n$. Let $C_{n+1} = K_{n+1} \setminus K_n^{\circ}$.

Since $\bigcup_{i=1}^{n} C_i = K_n$, then $\bigcup_{i\in\mathbb{N}} C_i = X$. If $z \in C_n \cap C_{n+2}$ then $z \in K_n \cap (K_{n+2} \setminus K_{n+1}^{\circ}) \subseteq K_n \setminus K_{n+1}^{\circ} = \emptyset$, a contradiction, and hence $C_k \cap C_l \neq \emptyset$ if and only if $|k-l| \leq 1$. Finally, it is clear that $H_n \cap C_{n-1} \subseteq H_n \cap K_{n-1} = \emptyset$ and $H_n \cap C_{n+1} \subseteq K_n^{\circ} \cap C_{n+1} = \emptyset$, and hence $H_k \subseteq C_k \setminus \bigcup_{l\neq k} C_l$ for $k \in \mathbb{N}$. □

In the next proof we will use the hedgehog instead of the spider.

Theorem 6.4. *Let X, Y be generalized Peano continua with $\mathrm{sdeg}(X) = \mathrm{sdeg}(Y)$ finite. Then X is the perfect image of Y and Y is the perfect image of X.*

Proof. Clearly, by Theorem 4.16, it is enough to show that if X is a generalized Peano continuum then $H^{\mathrm{sdeg}(X)}$ is the perfect image of X.

Suppose that $n = 1$. By Lemma 6.3, there exist a sequence C_k of continua such that $X = \bigcup_{k\in\mathbb{N}} C_k$, $C_k \cap C_l \neq \emptyset$ if and only if $|k-l|s \leq 1$ and a sequence of Peano continua H_k with $H_k \subseteq C_k \setminus \bigcup_{l\neq k} C_l$. By Remark 6.1, there exist

$f_k : H_k \to [k-1, k]$ onto mappings for each $k \in \mathbb{N}$.

Let $g_k : H_k \cup (C_k \cap C_{k-1}) \cup (C_k \cap C_{k+1}) \to [k, k+1]$ be the continuous map defined by $g_k(x) = f_k(x)$ if $x \in H_k$, $g_k(x) = k-1$ if $x \in C_k \cap C_{k-1}$ and $g_k(x) = n$ if $x \in C_k \cap C_{k+1}$.

By Tietze extension Theorem, there exists a continuous extension $F_k : C_k \to [k-1, k]$ which is onto, since g_k is. Then it is clear that $F : X \to \mathbb{R}_0^+ = H^1$ defined by $F_k(x)$ if $x \in C_k$ is a perfect (note that $F^{-1}(x) \subseteq C_{k-1} \cup C_k \cup C_{k+1}$ for $x \in [k-1, k]$) onto (continuous) mapping.

Suppose that $n > 1$. Let A be any compact subspace of X and let P and K_i with $i = 1, \ldots, n$ be as in Lemma 6.2. Let H_i^1 $(i = 1, \ldots, n)$ be the n spines of H^n and let o be the intersection of them. Let $f_i : P \cap K_i \to H_i^1$ be the constant map $f_i(x) = o$ for any $x \in P \cap K_i$. Since $P \cap K_i$ is compact, following the proof of case $n = 1$, there exists an onto perfect extension $F_i : K_i \to H_i^1$. Now, the map $F : X \to H^n$ defined by $F_i(x)$ if $x \in K_i$ and o if $x \in P$ is an onto perfect map. $\square$

This final result can be understood as a classifying result: among the class of generalized Peano continua, the equivalence relation "X is *perfectly equivalent* to Y if and only if X is the perfect image of Y and vice versa" is completely determined by a numerical invariant, sdeg. It is interesting to ask if this equivalence relation can be also determined by a numerical invariant outside the class of generalized Peano continua.

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