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ε-shift radix systems and radix representations with shifted digit sets

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Abstract. Let $\varepsilon \in [0,1)$, $\mathbf{r} \in \mathbb{R}^d$ and define the mapping $\tau_{\mathbf{r},\varepsilon} : \mathbb{Z}^d \to \mathbb{Z}^d$ by

$$\tau_{\mathbf{r},\varepsilon}(\mathbf{z}) = (z_1, \dots, z_{d-1}, -\lfloor \mathbf{r}\mathbf{z} + \varepsilon \rfloor) \quad (\mathbf{z} = (z_0, \dots, z_{d-1})).$$

If for each $\mathbf{z} \in \mathbb{Z}^d$ there is a $k \in \mathbb{N}$ such that the k-th iterate of $\tau_{\mathbf{r},\varepsilon}$ satisfies $\tau^k_{\mathbf{r},\varepsilon}(\mathbf{z}) = \mathbf{0}$ we call $\tau_{\mathbf{r},\varepsilon}$ an *ε-shift radix system.*

In the present paper we unify classical shift radix systems ($\varepsilon = 0$) and symmetric shift radix systems ($\varepsilon = \frac{1}{2}$), which have already been studied in several papers and analyse the relation of ε-shift radix systems to β-expansions and canonical number systems with shifted digit sets. At the end we will state several characterisation results for the two dimensional case.

1. Introduction

The concept of shift radix systems was introduced by Akiyama *et al.* [1]. For an $\mathbf{r} \in \mathbb{R}^d$ define the function

$$\tau_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \qquad \mathbf{z} = (z_0, \dots, z_{d-1}) \mapsto (z_1, \dots, z_{d-1}, -\lfloor \mathbf{r}\mathbf{z} \rfloor),$$

where $\mathbf{r}\mathbf{z}$ is the scalar product of the vectors $\mathbf{r}$ and $\mathbf{z}$. The mapping $\tau_{\mathbf{r}}$ is called a shift radix system (SRS) if for any $\mathbf{z} \in \mathbb{Z}^d$ there exists a $k \in \mathbb{N}$ such that

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$\tau_{\mathbf{r}}^k(\mathbf{z}) = \mathbf{0}$. This motivates the definition of the two sets

$$\mathcal{D}_d := \{\mathbf{r} \in \mathbb{R}^d \mid (\tau_{\mathbf{r}}^k(\mathbf{z}))_{k\in\mathbb{N}} \text{ is ultimately periodic for all } \mathbf{z} \in \mathbb{Z}^d\} \quad \text{and}$$
$$\mathcal{D}_d^0 := \{\mathbf{r} \in \mathbb{R}^d \mid \tau_{\mathbf{r}} \text{ is an SRS}\}.$$

Shift radix systems have been studied in several articles and from different points of view. In [1], [5], [6] the relation to other dynamical systems have been analysed. It turned out that SRS can be seen as unifying generalisation of β-expansions as well as canonical number systems. Other researches (see [4], [20]) concentrated on characterising the shape of $\mathcal{D}_d^0$. Except for the trivial one dimensional case this turned out to be difficult. We obviously have $\mathcal{D}_d^0 \subseteq \mathcal{D}_d$, thus any analysis of $\mathcal{D}_d^0$ starts with $\mathcal{D}_d$. The interior of this set is equal to the set of all $\mathbf{r} = (r_0, \dots, r_{d-1}) \in \mathbb{R}^d$ for which the spectral radius of the matrix

$$R(\mathbf{r}) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -r_0 & -r_1 & \cdots & -r_{d-2} & -r_{d-1} \end{pmatrix} \in \mathbb{R}^{d\times d} \tag{1.1}$$

is strictly smaller than one. The boundary of $\mathcal{D}_d$ is hard to describe, partial results can be found in [2], [3]. $\mathcal{D}_d^0$ can be obtained by cutting out polyhedra from $\mathcal{D}_d$, where each polyhedron corresponds to a periodic point of $\tau_{\mathbf{r}}$. Akiyama *et al.* showed in [1, Corollary 7.7] that $\mathcal{D}_d^0$ for $d \geq 2$ cannot be constructed by cutting only finitely many polyhedra. In [4] they gave a first approximation of $\mathcal{D}_2^0$. The best approximation for $\mathcal{D}_2^0$ known up to now can be found in [20] (see Figure 1).

Akiyama and Scheicher [7] presented a slight modification of SRS, the so called symmetric shift radix systems. For an $\mathbf{r} \in \mathbb{R}^d$ define

$$\tilde{\tau}_{\mathbf{r}} : \mathbb{Z}^d \to \mathbb{Z}^d, \qquad \mathbf{z} = (z_0, \dots, z_{d-1}) \mapsto \left(z_1, \dots, z_{d-1}, -\left\lfloor \mathbf{r}\mathbf{z} + \frac{1}{2} \right\rfloor\right).$$

$\tilde{\tau}_{\mathbf{r}}$ is called a symmetric shift radix system (SSRS) if for each $\mathbf{z} \in \mathbb{Z}^d$ there exists a $k \in \mathbb{N}$ such that $\tilde{\tau}_{\mathbf{r}}^k(\mathbf{r}) = \mathbf{0}$. Analogously denote

$$\tilde{\mathcal{D}}_d := \{\mathbf{r} \in \mathbb{R}^d \mid (\tilde{\tau}_{\mathbf{r}}^k(\mathbf{z}))_{k\in\mathbb{N}} \text{ is ultimately periodic for all } \mathbf{z} \in \mathbb{Z}^d\} \quad \text{and}$$
$$\tilde{\mathcal{D}}_d^0 := \{\mathbf{r} \in \mathbb{R}^d \mid \tilde{\tau}_{\mathbf{r}} \text{ is an SSRS}\}.$$

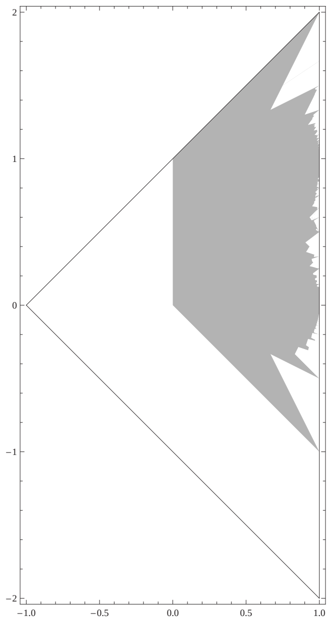

Figure 1. An approximation of $\mathcal{D}_2^0$

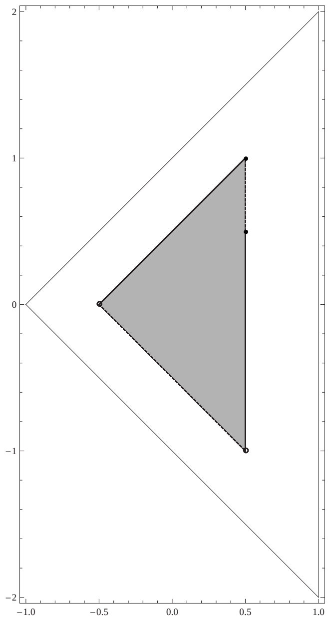

Figure 2. The set $\tilde{\mathcal{D}}_2^0$

The interior of $\tilde{\mathcal{D}}_d$ equals the interior of $\mathcal{D}_d$ and again, $\tilde{\mathcal{D}}_d^0$ can be constructed by cutting out polyhedra from $\tilde{\mathcal{D}}_d$. Surprisingly it turned out ([7, Theorem 5.2]) that $\tilde{\mathcal{D}}_2^0$ can be characterised completely (see Figure 2). Finally Huszti *et al.* [12] presented a full characterisation result of $\tilde{\mathcal{D}}_3^0$. For coloured and animated figures concerning this topic we also want to refer to the author's homepage [19].

These two very analogue definitions of SRS and SSRS suggest the construction of a new generalisation in the following way:

Definition 1.1. For an $\varepsilon \in [0, 1)$ and an $\mathbf{r} \in \mathbb{R}^d$ let

$$\tau_{\mathbf{r},\varepsilon} : \mathbb{Z}^d \to \mathbb{Z}^d, \qquad \mathbf{z} = (z_0, \dots, z_{d-1}) \mapsto (z_1, \dots, z_{d-1}, -\lfloor \mathbf{rz} + \varepsilon \rfloor).$$

$\tau_{\mathbf{r},\varepsilon}$ is called an *ε-shift radix system (ε-SRS)* if for any $\mathbf{z} \in \mathbb{Z}^d$ there exists a $k \in \mathbb{N}$ such that $\tau_{\mathbf{r},\varepsilon}^k(\mathbf{r}) = \mathbf{0}$. Further define

$$\mathcal{D}_{d,\varepsilon} := \{\mathbf{r} \in \mathbb{R}^d | (\tau_{\mathbf{r},\varepsilon}^n(\mathbf{z}))_{n \in \mathbb{N}} \text{ is ultimately periodic for all } \mathbf{z} \in \mathbb{Z}^d\} \quad \text{and}$$

$$\mathcal{D}_{d,\varepsilon}^0 := \{\mathbf{r} \in \mathbb{R}^d | \tau_{\mathbf{r},\varepsilon} \text{ is an } \varepsilon\text{-SRS}\}.$$

Note that defining modified ε-shift radix systems $\tau^*_{\mathbf{r},\varepsilon}$ with a ceiling function instead of the floor function does not yield a new dynamical system, since for each integer vector $\mathbf{z} \in \mathbb{Z}^d$ we had $\tau^*_{\mathbf{r},\varepsilon}(\mathbf{z}) = -\tau_{\mathbf{r},\varepsilon}(-\mathbf{z})$. An ε-SRS for $\varepsilon \notin [0,1)$ would not be meaningful any more since then $\tau_{\mathbf{r},\varepsilon}(\mathbf{0}) \neq \mathbf{0}$ and problems occurred in defining finiteness.

In Section 2 we will show that for fixed d the interior of $\mathcal{D}_{d,\varepsilon}$ is equal for all ε and $\mathcal{D}^0_{d,\varepsilon}$ is closely related to $\mathcal{D}^0_{d,1-\varepsilon}$ for $\varepsilon \in (0,1) \setminus \{\frac{1}{2}\}$. Further we will see that $\mathcal{D}^0_{d,\varepsilon}$ can be gained by cutting out polyhedra from $\mathcal{D}_{d,\varepsilon}$ and we will present a method to obtain these polyhedra. In Section 3 we will develop so called ε-β-expansions, *i.e.*, β-expansions with modified digit sets, and study their relation to ε-SRS. In Section 4 we make a similar approach for canonical number systems and define ε-canonical number systems. The main result of the paper is stated in Section 5 where we show that $\mathcal{D}^0_{2,\varepsilon}$ can be constructed by cutting out only finitely many polyhedra from $\mathcal{E}_2$ for any $\varepsilon \in (0,1)$. Finally, in the last Section, we will give the full characterisation of $\mathcal{D}^0_{d,\varepsilon}$ for two concrete values of ε.

2. On the set $\mathcal{D}^0_{d,\varepsilon}$

Lots of basic properties and notations concerning $\mathcal{D}^0_{d,\varepsilon}$ and $\mathcal{D}_{d,\varepsilon}$ can be directly adopted from the well analysed cases $\varepsilon = 0$ and $\varepsilon = \frac{1}{2}$. Since $\mathcal{D}^0_{d,\varepsilon} \subseteq \mathcal{D}_{d,\varepsilon}$ we first study the structure of $\mathcal{D}_{d,\varepsilon}$. For a given matrix $M \in \mathbb{R}^{d\times d}$ denote by $\varrho(M)$ the spectral radius of M. For a $\lambda > \varrho(M)$ denote by $\|\cdot\|_\lambda$ the norm with the property

$$\|M\mathbf{x}\|_\lambda \leq \lambda \|\mathbf{x}\|_\lambda, \quad \forall x \in \mathbb{R}^d. \tag{2.1}$$

Denote also by $\|\cdot\|_\lambda$ a compatible matrix norm, *i.e.*,

$$\forall \mathbf{x} \in \mathbb{R}^d \, \forall B \in \mathbb{R}^{d\times d} : \|B\mathbf{x}\|_\lambda \leq \|B\|_\lambda \|\mathbf{x}\|_\lambda.$$

Such a norm always exists (see [13, Formula (3.2)]). Whenever we use another norm in the paper, we will explicitly mention this. Let $R(\mathbf{r})$ be as in (1.1) and

$$\mathcal{E}_d := \{\mathbf{r} \in \mathbb{R}^d \mid \varrho(R(\mathbf{r})) < 1\}.$$

Theorem 2.1.

$$\mathcal{E}_d \subseteq \mathcal{D}_{d,\varepsilon} \subseteq \overline{\mathcal{E}_d}, \qquad \partial\mathcal{D}_{d,\varepsilon} = \{\mathbf{r} \in \mathbb{R}^d \mid \varrho(R(\mathbf{r})) = 1\}.$$

Proof. For the proof see [1, Lemmas 4.1, 4.2, 4.3], where the statements have been proven for the classic SRS case ($\varepsilon = 0$). They can be transferred to other values of ε without difficulties. □

We can easily see that the interior of $\mathcal{D}_{d,\varepsilon}$ equals $\mathcal{E}_d$ and does not depend on ε. We do not expect $\mathcal{D}_{d,\varepsilon} \cap \partial\mathcal{D}_{d,\varepsilon} = \mathcal{D}_{d,\varepsilon} \setminus \mathcal{E}_d$ to be independent of ε, however, these difference sets seem to be very hard to describe and there exist only partial results for $\mathcal{D}_{2,0} \setminus \mathcal{E}_2$ (see [2], [3]). The set $\mathcal{E}_d$ was already studied in [1, Proposition 4.9]. Its characterisation is based on results of Schur [18] and Takagi [21]. For that reason $\mathcal{E}_d$ is sometimes referred to as the Schur–Takagi-region. $\mathcal{E}_d$ is bounded and characterised by several strict inequalities. It is a regular set, *i.e.*, $\mathcal{E}_d = \operatorname{int}(\overline{\mathcal{E}_d})$. We have

$$\begin{aligned}
\mathcal{E}_1 &= (-1,1),\\
\mathcal{E}_2 &= \{(x,y) \in \mathbb{R}^2 \mid |x| < 1 \wedge |y| < x+1\},\\
\mathcal{E}_3 &= \{(x,y,z) \in \mathbb{R}^3 \mid |x| < 1 \wedge |y - xz| < 1 - x^2 \wedge |x+z| < |y+1|\}. \qquad (2.2)
\end{aligned}$$

Note that the closure of $\mathcal{E}_d$ in general cannot be obtained by exchanging the strict inequalities to not strict ones. Of course, it works for $d = 2$ but already in the three dimensional case some problems occur (see [12, Theorem 2.5]).

In order to analyse the structure of $\mathcal{D}^0_{d,\varepsilon}$ we start with $\mathcal{D}_{d,\varepsilon}$ and have to remove all points $\mathbf{r}$ where $(\tau^k_{\mathbf{r},\varepsilon}(\mathbf{z}))_{k\in\mathbb{N}}$ is periodic for some $\mathbf{z} \in \mathbb{Z}^d$, $\mathbf{z} \neq \mathbf{0}$. In particular, $\mathbf{r} \notin \mathcal{D}^0_{d,\varepsilon}$ when there exist nonzero points $\mathbf{z}_0, \dots, \mathbf{z}_{l-1} \in \mathbb{Z}^d$ with

$$\mathbf{z}_0 \xmapsto{\tau_{\mathbf{r},\varepsilon}} \mathbf{z}_1 \xmapsto{\tau_{\mathbf{r},\varepsilon}} \cdots \xmapsto{\tau_{\mathbf{r},\varepsilon}} \mathbf{z}_{l-1} \xmapsto{\tau_{\mathbf{r},\varepsilon}} \mathbf{z}_0.$$

Since by definition an application of $\tau_{\mathbf{r},\varepsilon}$ keeps $d-1$ entries unchanged (only the position is shifted), it suffices to write down only the first entry of each vector. Hence the l vectors $\mathbf{z}_0, \dots, \mathbf{z}_{l-1}$ can be represented by l integers $z_0, \dots, z_{l-1}$ such that for each $i \in \{0, \dots, l-1\}$ we have

$$\mathbf{z}_i = (z_i, z_{i+1}, \dots, z_{i+d-1})$$

where the indices have to be taken modulo l. We will refer to such a sequence of integers apart from 0 as a cycle of $\tau_{\mathbf{r},\varepsilon}$ of period l or, more generally, a cycle of $\mathcal{D}_{d,\varepsilon}$ and will write it in the form $(z_0, \dots, z_{l-1})$.

We can turn the problem around and may ask: For which $\mathbf{r} = (r_0, \dots, r_{d-1})$ is some given $(z_0, \dots, z_{l-1}) \in \mathbb{Z}^l$ with $l \in \mathbb{N}$ a cycle of $\tau_{\mathbf{r},\varepsilon}$? To receive a satisfying answer, we have to look at the definition of $\tau_{\mathbf{r},\varepsilon}$ (take the indices of z modulo l in this paragraph). $(z_0, \dots, z_{d-1})$ is mapped to $(z_1, \dots, z_d)$ if and only if $\lfloor \sum_{i=0}^{d-1} r_i z_i + \varepsilon \rfloor = -z_d$ or, equivalently,

$$0 \leq r_0 z_0 + \cdots + r_{d-1} z_{d-1} + z_d + \varepsilon < 1.$$

Thus,

$$\tau_{\mathbf{r},\varepsilon} : (z_i, \dots, z_{i+d-1}) \mapsto (z_{i+1}, \dots, z_{i+d}), \quad \forall i \in (0, \dots, l-1)$$

if and only if

$$0 \leq r_0 z_i + \cdots + r_{d-1} z_{i+d-1} + z_{i+d} + \varepsilon < 1, \quad \forall i \in (0, \dots, l-1). \tag{2.3}$$

Hence, $(z_0, \dots, z_{l-1})$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for those $\mathbf{r}$ that satisfy the system of inequalities (2.3). Define

$$P_\varepsilon((z_0, \dots, z_{l-1})) := \{(r_0, \dots, r_{d-1}) \in \mathbb{R}^d \mid (r_0, \dots, r_{d-1}) \text{ satisfies (2.3)}\}.$$

In general $P_\varepsilon((z_0, \dots, z_{l-1}))$ is a polyhedron in $\mathbb{R}^d$. But the polyhedron can degenerate to a lower dimension or even be the empty set. $(z_0, \dots, z_{l-1})$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for some $\mathbf{r} \in \mathbb{R}^d$ if and only if $P_\varepsilon((z_0, \dots, z_{l-1})) \neq \emptyset$. According to [1], we call the cycle $(z_0, \dots, z_{l-1})$ degenerated, if we have $\dim P_\varepsilon((z_0, \dots, z_{l-1})) < d$. $(z_0, \dots, z_{l-1})$ is not a cycle if $P_\varepsilon((z_0, \dots, z_{l-1})) = \emptyset$. Note that, if π is a non degenerated cycle of $\mathcal{D}_{d,\varepsilon}$, analogously to the Lifting Theorem [1, Theorem 6.1], π is also a non-degenerated cycle of $\mathcal{D}_{d',\varepsilon}$ for all $d' > d$.

Let Π be the set of all integer vectors of finite dimension that are candidates for cycles. In particular, using the formalism of combinatorics on words (see, for instance, [8]), when we consider the words over $\mathbb{Z}$ and denote by N a set of representatives of the primitive necklaces of finite length without the word 0, we have $(z_0, \dots, z_{l-1}) \in \Pi$ if and only if $z_0 z_1 \dots z_{l-1} \in N$. Then it is easy to see that

$$\mathcal{D}^0_{d,\varepsilon} = \mathcal{D}_{d,\varepsilon} \setminus \bigcup_{\pi \in \Pi} P_\varepsilon(\pi).$$

The problem is that this representation is not very practicable, since Π is a countable infinite set. But we can see that it is much easier to recognise a subset of $\mathcal{D}_{d,\varepsilon}$ not to be a subset of $\mathcal{D}^0_{d,\varepsilon}$ than to be a subset.

For different values of ε the function P_ε satisfies

Lemma 2.2. *For all* $\varepsilon \in (0,1)$ *we have* $\operatorname{int}(P_\varepsilon((z_0, \dots, z_{l-1}))) = \operatorname{int}(P_{1-\varepsilon}((-z_0, \dots, -z_{l-1})))$.

Proof. $\mathbf{r} = (r_0, \dots, r_{d-1})$ is in $\operatorname{int}(P_\varepsilon((z_0, \dots, z_{l-1})))$ if and only if

$$0 < r_0 z_i + \cdots + r_{d-1} z_{i+d-1} + z_{i+d} + \varepsilon < 1, \quad \forall i \in (0, \dots, l-1)$$

(indices of z modulo l). Thus

$$-1 < r_0 z_i + \cdots + r_{d-1} z_{i+d-1} + z_{i+d} - 1 + \varepsilon < 0, \quad \forall i \in (0, \dots, l-1).$$

Multiplication with -1 yields

$$0 < r_0(-z_i) + \cdots + r_{d-1}(-z_{i+d-1}) - z_{i+d} + 1 - \varepsilon < 1, \quad \forall i \in (0, \ldots, l-1)$$

which is equivalent to $\mathbf{r} \in \operatorname{int}(P_{1-\varepsilon}((-z_0, \ldots, -z_{l-1})))$. $\square$

Corollary 2.3. *Let $\varepsilon \in (0,1)$ and denote the d-dimensional Lebesgue measure by λ_d. Then*

$$\lambda_d(\mathcal{D}^0_{d,\varepsilon} \triangle \mathcal{D}^0_{d,1-\varepsilon}) = 0$$

holds.

Our next task will be to find a way, how to obtain a set of cycles Π which have corresponding polyhedra intersecting with a given set $Q \subset \mathcal{D}_{d,\varepsilon}$ such that

$$\mathcal{D}^0_{d,\varepsilon} \cap Q = Q \setminus \bigcup_{\pi \in \Pi} P_\varepsilon(\pi).$$

Definition 2.4. For $\varepsilon \in [0,1)$ and an $\mathbf{r} \in \mathcal{D}_{d,\varepsilon}$ a point $\mathbf{x}$ is called a *periodic point* (with respect to $\tau_{\mathbf{r},\varepsilon}$) if there exists an $l \in \mathbb{N}$ such that $\tau^l_{\mathbf{r},\varepsilon}(\mathbf{x}) = \mathbf{x}$.

Lemma 2.5. *Let $\varepsilon \in [0,1)$ and $\mathbf{r} \in \mathcal{E}_d$. Then there exists only finitely many periodic points with respect to $\tau_{\mathbf{r},\varepsilon}$.*

Proof. The statement is a generalisation of [1, Lemma 4.2 (2)], the proof can be adapted without any difficulties. $\square$

The lemma immediately implies that $\tau_{\mathbf{r},\varepsilon}$ can have only finitely many cycles for $\mathbf{r} \in \operatorname{int}(\mathcal{D}_{d,\varepsilon})$. Now define the following set.

Definition 2.6. Let $\mathbf{r} \in \mathcal{E}_d$. A set $\mathcal{V} \subset \mathbb{Z}^d$ that satisfies

(1) $\pm(\delta_{1j}, \delta_{2j}, \ldots, \delta_{dj}) \in \mathcal{V} \quad (\forall j \in \{1, \ldots, d\})$,

(2) $\mathbf{z} \in \mathcal{V} \Rightarrow \tau_{\mathbf{r},0}(\mathbf{z}), -\tau_{\mathbf{r},0}(-\mathbf{z}) \in \mathcal{V}$

($\delta_{..}$ is the Kronecker delta) is called a *set of witnesses* of $\mathbf{r}$. The smallest set of witnesses is denoted by $\mathcal{V}(\mathbf{r})$.

A set of witnesses has nice properties concerning $\tau_{\mathbf{r},\varepsilon}$, although it does not depend on ε. We will see this in the next theorem.

Theorem 2.7. *Let $\varepsilon \in [0,1)$ and $\mathbf{r} \in \mathcal{E}_d$. $\mathbf{r} \in \mathcal{D}^0_{d,\varepsilon}$ if and only if a set of witnesses $\mathcal{V}$ does not contain nonzero periodic elements of $\tau_{\mathbf{r},\varepsilon}$.*

Proof. If $\mathcal{V}$ has a nonzero periodic element then $\mathbf{r} \notin \mathcal{D}_{d,\varepsilon}^0$ by the definition of $\mathcal{D}_{d,\varepsilon}^0$. The other implication is a little more tricky. It is based on the observation that for $a, b \in \mathbb{R}$ the floor function behaves as

$$\lfloor a+b \rfloor \in \{\lfloor a \rfloor + \lfloor b \rfloor,\ \lfloor a \rfloor + \lceil b \rceil = \lfloor a \rfloor - \lfloor -b \rfloor\}.$$

This implies that for any $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$ we have

$$\tau_{\mathbf{r},\varepsilon}(\mathbf{a}+\mathbf{b}) \in \{\tau_{\mathbf{r},\varepsilon}(\mathbf{a}) + \tau_{\mathbf{r},0}(\mathbf{b}), \tau_{\mathbf{r},\varepsilon}(\mathbf{a}) + (-\tau_{\mathbf{r},0}(-\mathbf{b}))\}.$$

Now suppose that $\mathbf{b} \in \mathcal{V}$. Then there exists a $\mathbf{c} \in \mathcal{V}$ such that

$$\tau_{\mathbf{r},\varepsilon}(\mathbf{a}+\mathbf{b}) = \tau_{\mathbf{r},\varepsilon}(\mathbf{a}) + \mathbf{c}$$

by the definition of $\mathcal{V}$. Thus, if $\mathbf{a}$ is not periodic, *i.e.*, if there is a $k \in \mathbb{N}$ such that $\tau_{\mathbf{r},\varepsilon}^k(\mathbf{a}) = \mathbf{0}$, we have that

$$\tau_{\mathbf{r},\varepsilon}^k(\mathbf{a}+\mathbf{b}) \in \mathcal{V}.$$

The fact that the positive and negative canonical unit vectors are included in $\mathcal{V}$ now implies that $\mathbf{r} \in \mathcal{D}_{d,\varepsilon}^0$ if $\mathcal{V}$ has no nonzero periodic elements. □

Since Theorem 2.7 does not contain any restrictions, which set of witnesses should be used, we agree on the use of $\mathcal{V}(\mathbf{r})$. Note that $\mathcal{V}(\mathbf{r})$ contains periodic elements when $\tau_{\mathbf{r},\varepsilon}$ is no ε-SRS but $\mathcal{V}(\mathbf{r})$ need not contain all periodic elements of $\tau_{\mathbf{r},\varepsilon}$. It is also possible to define a set of witnesses for a closed, convex $Q \subset \mathrm{int}(\mathcal{D}_{d,\varepsilon})$.

Definition 2.8. Let $Q \subset \mathcal{E}_d$ be closed and convex. A set $\mathcal{V} \subset \mathbb{Z}^d$ that satisfies

(1) $\pm(\delta_{1j}, \delta_{2j}, \dots, \delta_{dj}) \in \mathcal{V} \quad (\forall j \in \{1, \dots, d\})$,

(2) $\mathbf{z} = (z_0, \dots, z_{d-1}) \in \mathcal{V} \Rightarrow (z_1, \dots, z_{d-1}, j) \in \mathcal{V}$ for all

$$j \in \{\min_{\mathbf{s}\in Q} \lfloor -\mathbf{s}\mathbf{z} \rfloor, \dots, -\min_{\mathbf{s}\in Q} \lfloor \mathbf{s}\mathbf{z} \rfloor\}$$

is called a *set of witnesses* of Q. The smallest set of witnesses is denoted by $\mathcal{V}(Q)$.

We obviously have $\mathcal{V}(\{\mathbf{r}\}) = \mathcal{V}(\mathbf{r})$ and $\mathcal{V}(\mathbf{r}) \subseteq \mathcal{V}(Q)$ for $\mathbf{r} \in Q$ since Q is convex.

Lemma 2.9. *A set of witnesses $\mathcal{V}$ of Q is closed under the application of $\tau_{\mathbf{r},\varepsilon}$ for each $\mathbf{r} \in Q$ and $\varepsilon \in [0, 1)$.*

Proof. We first claim that for $a \in \mathbb{R}$ and $\varepsilon \in [0,1)$ we have

$$\lceil a \rceil \geq \lfloor a + \varepsilon \rfloor. \tag{2.4}$$

Indeed, if $a \in \mathbb{Z}$ then $\lceil a \rceil = \lfloor a \rfloor = \lfloor a + \varepsilon \rfloor$. Otherwise $\lceil a \rceil = \lfloor a \rfloor + 1 = \lfloor a + 1 \rfloor \geq \lfloor a + \varepsilon \rfloor$.

Now let $\mathbf{z} = (z_0, \dots, z_{d-1}) \in \mathcal{V}$. By definition $(z_1, \dots, z_{d-1}, j) \in \mathcal{V}$ for all

$$j \in I := \{\min_{\mathbf{s}\in Q} \lfloor -\mathbf{s}\mathbf{z} \rfloor, \dots, -\min_{\mathbf{s}\in Q} \lfloor \mathbf{s}\mathbf{z} \rfloor\}.$$

Then, using (2.4), we have

$$\begin{aligned}\min_{\mathbf{s}\in Q} \lfloor -\mathbf{s}\mathbf{z} \rfloor = \min_{\mathbf{s}\in Q}(-\lceil \mathbf{s}\mathbf{z} \rceil) \leq \min_{\mathbf{s}\in Q}(-\lfloor \mathbf{s}\mathbf{z} + \varepsilon \rfloor) \leq -\lfloor \mathbf{r}\mathbf{z} + \varepsilon \rfloor \\ \leq -\lfloor \mathbf{r}\mathbf{z} \rfloor \leq \max_{\mathbf{s}\in Q}(-\lfloor \mathbf{s}\mathbf{z} \rfloor) = -\min_{\mathbf{s}\in Q} \lfloor \mathbf{s}\mathbf{z} \rfloor.\end{aligned}$$

Therefore $-\lfloor \mathbf{r}\mathbf{z} + \varepsilon \rfloor \in I$ and $\tau_{\mathbf{r},\varepsilon}(\mathbf{z}) \in \mathcal{V}$. $\square$

Now we present a way how to find possible periods within a set of witnesses $\mathcal{V}$.

Definition 2.10. Let $\mathcal{V}$ be a finite set of witnesses for some closed, convex $Q \subset \mathcal{E}_d$. Let $G(\mathcal{V}, \varepsilon) = V \times E$ be the directed graph with vertices $V = \mathcal{V}$ and edges $E \subset V \times V$ such that

$$\begin{gathered}(\mathbf{x} = (x_0, \dots, x_{d-1}), (z_0, \dots, z_{d-1})) \in E \iff \\ z_0 = x_1, \dots, z_{d-2} = x_{d-1}, z_{d-1} \in \{-\max_{\mathbf{s}\in Q} \lfloor \mathbf{s}\mathbf{x} + \varepsilon \rfloor, \dots, -\min_{\mathbf{s}\in Q} \lfloor \mathbf{s}\mathbf{x} + \varepsilon \rfloor\}.\end{gathered}$$

The definition is meaningful because of Lemma 2.9. The finiteness condition on $\mathcal{V}$ assures the finiteness of the graph $G(\mathcal{V}, \varepsilon)$. We are interested in the (directed) cycles of this graph. To avoid confusion we will refer to cycles of graphs as graph-cycles. Although, we will see the two types of cycles to be closely related in this context. A graph-cycle of length l consists of l vectors in $\mathbb{Z}^d$. By the definition of the edges a graph-cycle has the shape

$$(x_0, \dots, x_{d-1}) \to (x_1, \dots, x_d) \to \dots \to (x_{l-1}, \dots, x_{d-2}) \to (x_0, \dots, x_{d-1}).$$

Similar to the construction of cycles, a graph-cycle is uniquely determined by the l integers that form the first entries of these vectors. By this consideration, let us make the convention that, when we speak of a graph-cycle of $G(\mathcal{V}, \varepsilon)$ for some set of witnesses $\mathcal{V}$, we mean the integer sequence of corresponding length that determines it. This sequence will also be written inside brackets.

Theorem 2.11. *Let $\varepsilon \in [0,1)$, $Q \subset \mathcal{E}_d$ closed and convex and $\mathcal{V}$ a finite set of witnesses of Q. Further be Λ the set of graph-cycles of $G(\mathcal{V},\varepsilon)$ without the trivial one (0). Then*

$$\mathcal{D}^0_{d,\varepsilon} \cap Q = Q \setminus \bigcup_{\pi\in\Lambda} P_\varepsilon(\pi).$$

Proof. Let $\mathbf{r} \in Q$. Then $\mathcal{V}$ also includes a set of witnesses $\mathcal{V}_\mathbf{r}$ of $\mathbf{r}$. Let $\mathbf{z} \in \mathcal{V}_\mathbf{r}$. Then, by the construction of $G(\mathcal{V},\varepsilon)$, there exists an edge $(\mathbf{z}, \tau_{\mathbf{r},\varepsilon}(\mathbf{z}))$ of $G(\mathcal{V},\varepsilon)$. Hence, if $\mathbf{r} \notin \mathcal{D}^0_{d,\varepsilon}$, there exists a graph-cycle $\pi \in \Lambda$ which is a cycle of $\tau_{\mathbf{r},\varepsilon}$ by Theorem 2.7 and therefore $\mathbf{r} \in P_\varepsilon(\pi)$. This is true for all $\mathbf{r} \in Q$ which yields the desired result. □

Note that a graph-cycle in $G(\mathcal{V},\varepsilon)$ apart from the trivial one is always a cycle if $\mathcal{V}$ is a set of witnesses of a point, while for a larger sets it can contain graph-cycles that are no cycles.

Up to now we do not know anything about the construction of sets of witnesses and if there are finite ones at all. For the smallest ones $\mathcal{V}(\mathbf{r})$ and $\mathcal{V}(Q)$ there already exist excellent results. Note that for an $\mathbf{r} \in \mathcal{E}_d$ we have $\rho\,(R(\mathbf{r})) < 1$ and therefore λ in (2.1) can be chosen to be strictly smaller that 1.

Theorem 2.12. *Let $Q \subset \mathcal{E}_d$ be closed and convex. If there exists $\mathbf{r} \in Q, 1 > \lambda > \rho\,(R(\mathbf{r}))$ with $\max_{\mathbf{s}\in Q}(\|R(\mathbf{r}) - R(\mathbf{s})\|_\lambda) < 1 - \lambda$ then $\mathcal{V}(Q)$ is finite.*

Proof. See [20, Lemma 2.2]. For polyhedral Q, see also [1, Theorem 5.2] or [7, Theorem 4.2]. □

Theorem 2.12 implies that $\mathcal{V}(\mathbf{r})$ is finite for $\mathbf{r} \in \mathcal{E}_d$. In practice $\mathcal{V}(Q)$ for a given Q is calculated iteratively by starting with the positive and negative canonical unit vectors satisfying (1) of Definition 2.8 and then successively applying (2) of Definition 2.8. If Q is small enough, the procedure will stabilise, yielding $\mathcal{V}(Q)$. If there is no result after a certain number of steps, $\mathcal{V}(Q)$ could be infinite and the whole procedure is started with a smaller set Q. See [20, Algorithm 1] for this method. Once $\mathcal{V}(Q)$ is calculated the graph $G(\mathcal{V}(Q),\varepsilon)$ can be obtained easily by computing the edges as it is postulated in the definition. Afterwards the cyclic structure of the graph can be investigated. This algorithmic way to calculate $Q \cap \mathcal{D}^0_{d,\varepsilon}$ for a convex body $Q \subset \mathcal{E}_d$ was presented in [1, Theorem 5.2] for $\varepsilon = 0$ and [7, Theorem 4.2] for $\varepsilon = \frac{1}{2}$. A detailed discussion for more general closed and convex $Q \subset \mathcal{E}_d$ and $\varepsilon = 0$ can be found in [20, Algorithm 2]. It is straight forward to adapt these results for all $\varepsilon \in [0,1)$. Because the algorithm is based on an idea of Brunotte (see [10]), it is frequently referred to as Brunotte Algorithm.

3. ε-β-expansions

In 1957 Rényi [16] introduced the β-expansion for representing real numbers in some base $\beta > 1$ using the digit set $\mathcal{N} = [0, \beta) \cap \mathbb{Z}$. In the following the β-expansion has been studied excessively (for instance, see [9, 11, 14, 17]). Akiyama *et al.* [1] showed the close relation between β-expansions and shift radix systems. Akiyama and Scheicher [7] modified the β-expansion in that effect that they use a shifted digit set $\mathcal{N} = (-\frac{\beta+1}{2}, \frac{\beta+1}{2}) \cap \mathbb{Z}$ and showed that this so called symmetric β-expansion is analogously related to symmetric shift radix systems. We will develop an ε-β-expansion which unifies both previously mentioned theories and set it in relation to ε-SRS.

Definition 3.1. Let $\varepsilon \in [0, 1)$, $\beta > 1$ a real number and $\mathcal{N} = \mathcal{N}(\beta, \varepsilon) = (-1 + \varepsilon(1 - \beta), \beta + \varepsilon(1 - \beta)) \cap \mathbb{Z}$. For a $\gamma \in \mathbb{R}$ a sequence $(e_i)_{i \leq m} \in \mathcal{N}^{\mathbb{N}}$ is called an *ε-β-representation* of γ when

$$\gamma = \sum_{i \leq m} e_i \beta^i.$$

We say $(e_i)_{i \leq m} \in \mathcal{N}^{\mathbb{N}}$ to be *ε-admissible* when we have

$$\sum_{i \leq k} e_i \beta^i \in [-\varepsilon, 1 - \varepsilon)\beta^{k+1} \tag{3.1}$$

for all $k \leq m$. Then the corresponding representation

$$\gamma = \sum_{i \leq m} e_i \beta^i$$

is called the *ε-β-expansion* of γ.

Formula (3.1) is the generalisation of the β-expansion's greedy condition and its analogue for the symmetric β-expansion (see [7, Formula (3.1)]). For $\gamma \in [-\varepsilon, 1 - \varepsilon)$ the ε-β-representation can be obtained by applying the ε-β-shift

$$T_{\beta,\varepsilon} : [-\varepsilon, 1 - \varepsilon) \to [-\varepsilon, 1 - \varepsilon), \qquad \gamma \mapsto \beta\gamma - \lfloor \beta\gamma + \varepsilon \rfloor.$$

We denote a sequence $(e_i)_{i<0} \in \mathcal{N}^{\mathbb{N}}$ by $\delta_\varepsilon(\gamma)$ if it emerges from the successive application of the ε-β-shift on $\gamma \in [-\varepsilon, 1 - \varepsilon)$. We also define $\delta_\varepsilon(1 - \varepsilon)$ to be the sequence $e_{-1} + 1, e_{-2}, \ldots$ where $(e_i)_{i<0} = \delta_\varepsilon(1 - \varepsilon - \frac{1}{\beta})$. Note that $\delta_\varepsilon(1 - \varepsilon) \in \mathcal{N}^{\mathbb{N}}$. We immediately see that a sequence $(e_i) \in \mathcal{N}^{\mathbb{N}}$ is ε-admissible if and only if it equals $\delta_\varepsilon(\gamma)$ for some $\gamma \in [-\varepsilon, 1 - \varepsilon)$.

Each $\gamma \in \mathbb{R}$ has at least one ε-β-representation and exactly one ε-β-expansion except for the case $\varepsilon = 0$ where we have to restrict to $\gamma \in \mathbb{R} \cap [0, \infty)$. If $\gamma \notin [-\varepsilon, 1 - \varepsilon)$ there exists a $k \in \mathbb{N}$ with $\beta^{-k}\gamma \in [-\varepsilon, 1 - \varepsilon)$. Then the ε-β-shift can be applied to obtain the ε-β-expansion of $\beta^{-k}\gamma$. Multiplication with β^k then yields the ε-β-expansion of γ. For $\varepsilon = 0$ the ε-β-shift corresponds to the β-shift and for $\varepsilon = \frac{1}{2}$ the analogous mapping for the symmetric case (*cf.* [7, Section 3]).

It is well known that the 0-β-expansion satisfies the lexicographical order condition, *i.e.*, a sequence $(e_i)_{i\leq m} \in \mathcal{N}^{\mathbb{N}}$ is 0-admissible only if

$$(e_i)_{i\leq k} <_{\text{lex}} \delta_0(1), \quad \forall k \leq m.$$

[7, Theorem 3.1] provides an analogue result for $\varepsilon = \frac{1}{2}$. A generalisation of this can be easily obtained.

Lemma 3.2. *Let $\beta > 1$. A sequence $(e_i)_{i\leq m} \in \mathcal{N}^{\mathbb{N}}$ is ε-admissible only if for all $k \leq m$ we have*

$$\delta_\varepsilon(-\varepsilon) \leq_{\text{lex}} (e_i)_{i\leq k} <_{\text{lex}} \delta_\varepsilon(1-\varepsilon).$$

It seems to be more difficult to give a sufficient condition here.

For $\varepsilon \in [0, 1)$ and a real $\beta > 1$ define the sets

$$\mathbf{Per}(\varepsilon, \beta) := \{x \in \mathbb{R} \mid x \text{ has eventually periodic } \varepsilon\text{-}\beta\text{-expansion}\},$$

$$\mathbf{Fin}(\varepsilon, \beta) := \{x \in \mathbb{R} \mid x \text{ has finite } \varepsilon\text{-}\beta\text{-expansion}\}.$$

Observe that for $\varepsilon = 0$ the sets $\mathbf{Per}(\varepsilon, \beta)$ and $\mathbf{Fin}(\varepsilon, \beta)$ consist only of non-negative numbers.

Definition 3.3. Let $\varepsilon \in (0, 1)$. An algebraic integer $\beta > 1$ is said to have property (ε-F) if $\mathbf{Fin}(\varepsilon, \beta) = \mathbb{Z}[\beta^{-1}]$. A real number $\beta > 1$ is said to have property (0-F) if $\mathbf{Fin}(\varepsilon, \beta) = \mathbb{Z}[\beta^{-1}] \cap [0, \infty)$.

Property (0-F) is obviously equal to the well known property (F).

For algebraic integers β the ε-β-expansion is closely related to ε-SRS. Let $\beta > 1$ be an algebraic integer with minimal polynomial $A(x) = x^{d+1} + a_d x^d + \cdots + a_1 x + a_0$ and fix

$$\mathbf{r} = (r_0, \ldots, r_{d-1}) \qquad \text{with} \qquad \begin{aligned} &r_d = 1, \\ &r_j = a_{j+1} + \beta r_{j+1} \qquad (0 \leq j \leq d-1). \end{aligned} \tag{3.2}$$

Theorem 3.4. *Let $\varepsilon \in [0, 1)$ and β be a positive algebraic integer with minimal polynomial $A(x) = x^{d+1} + a_d x^d + \cdots + a_1 x + a_0$ and $\mathbf{r}$ be defined as in (3.2). Then β has property (ε-F) if and only if $\mathbf{r} \in \mathcal{D}^0_{d,\varepsilon}$.*

Proof. For $\varepsilon = 0$ this has already been shown in [1, Theorem 2.1.]. One can easily generalise this proof to all values of ε by showing that the dynamical systems $T_{\beta,\varepsilon}$ and $\tau_{\mathbf{r},\varepsilon}$ are conjugate. □

Corollary 3.5. *Let $\varepsilon \in [0,1)$ and β be a positive algebraic integer with minimal polynomial $A(x) = x^{d+1} + a_d x^d + \cdots + a_1 x + a_0$ and $\mathbf{r}$ be defined as in* (3.2). *Then $\mathbb{Z}[\beta^{-1}] \subset \boldsymbol{Per}(\varepsilon,\beta)$ ($\mathbb{Z}[\beta^{-1}] \cap [0,\infty) \subset \boldsymbol{Per}(\varepsilon,\beta)$ for $\varepsilon = 0$) if and only if $\mathbf{r} \in \mathcal{D}_{d,\varepsilon}$.*

For the classical β-expansion ($\varepsilon = 0$) we already know that β has property (F) only if it is a Pisot number (see [11]). We see that this is still true for other values of ε.

4. ε-CNS

The most general definition of canonical number systems (CNS) can be found in [15]. Let $A(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with $|a_0| \geq 2$, $\mathcal{R} := \mathbb{Z}[x]/(\mathbb{Z}[x]A(x))$ the factor ring, $X \in \mathcal{R}$ the image of x under the canonical epimorphism and the set of digits $\mathcal{N} := \{0,\ldots,|a_0|-1\}$. The pair $(A,\mathcal{N})$ is called a canonical number system (CNS) if each $P \in \mathcal{R}$ can be represented as

$$P = \sum_{i=0}^{n} e_i X^i \quad \text{with} \quad e_i \in \mathcal{N}.$$

In [1] the close relation of CNS and SRS was treated. A modification of the set of digits leads us to the so-called symmetric canonical number systems (SCNS) which have been introduced by Akiyama and Scheicher [7]: the pair $(A,\mathcal{N})$ is an SCNS if each $P \in \mathcal{R}$ can be represented as

$$P = \sum_{i=0}^{n} e_i X^i \quad \text{with} \quad e_i \in \mathcal{N} := \left[-\frac{|a_0|}{2}, \frac{|a_0|}{2}\right) \cap \mathbb{Z}.$$

In that paper the relation between SCNS and SSRS were analysed. This motivates a generalisation that includes both types of canonical number systems.

Definition 4.1. Let $\varepsilon \in [0,1)$, $A(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ with $|a_0| \geq 2$, $\mathcal{R} := \mathbb{Z}[x]/(\mathbb{Z}[x]A(x))$ the factor ring, $X \in \mathcal{R}$ the image of x under the canonical epimorphism and the set of digits $\mathcal{N} := [-\varepsilon|a_0|, (1-\varepsilon)|a_0|) \cap \mathbb{Z}$. The pair $(A,\mathcal{N})$ is called an ε-canonical number system (an ε-CNS) if each $P \in \mathcal{R}$ allows a representation as

$$P = \sum_{i=0}^{n} e_i X^i \text{ with } e_i \in \mathcal{N}.$$

It is easy to see that the case $\varepsilon = 0$ corresponds to CNS while $\varepsilon = \frac{1}{2}$ corresponds to the SCNS. Let $P = P_0 \in \mathcal{R}$. P_0 has a unique representation of the shape $P_0 = \sum_{i=0}^{d-1} p_i^{(0)} X^i$. The ε-CNS representation of P can be obtained by applying a backward division algorithm (*cf.* [1, Section 3]). For $k \geq 0$ we inductively calculate $P_{k+1} := \sum_{i=0}^{d-1} p_i^{(k+1)} X^i$ by $P_{k+1} = \sum_{i=0}^{d-1} (p_{i+1}^{(k)} + q_k a_{i+1}) X^i$ where $p_d^{(k)} = 0$ and q_k is the unique integer that satisfies $p_0^{(k)} = e_k + q_k a_0$ with $e_k \in \mathcal{N}$. For each $l \in \mathbb{N}$ we then have

$$P = \sum_{i=0}^{l-1} e_i X^i + P_l X^l.$$

Thus, $(A,\mathcal{N})$ is an ε-CNS exactly if the backward division algorithm ends up in 0 for each $P \in \mathcal{R}$.

Theorem 4.2. *Let $\varepsilon \in [0,1)$, $A(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$ and $\mathcal{N} := [-\varepsilon|a_0|, (1-\varepsilon)|a_0|) \cap \mathbb{Z}$. $(A,\mathcal{N})$ is an ε-CNS if and only if $(\frac{1}{a_0}, \frac{a_{d-1}}{a_0}, \ldots, \frac{a_1}{a_0}) \in \mathcal{D}_{d,\varepsilon}^0$.*

Proof. The proof runs analogously to the proof of [1, Theorem 3.1]. □

5. About $\mathcal{D}_{2,\varepsilon}^0$

In the following we will concentrate on the two dimensional case and show that $\mathcal{D}_{2,\varepsilon}^0$ for $\varepsilon \in (0,1)$ can be completely characterised by cutting out finitely many polyhedra from $\mathcal{E}_2$. The result is based on the following lemma.

Lemma 5.1. *If there exists a closed set $D \subset \mathcal{E}_d$ with $\mathcal{D}_{d,\varepsilon}^0 \subset D$ then $\mathcal{D}_{d,\varepsilon}^0$ can be obtained from D by cutting out finitely many polyhedra.*

Proof. This was already stated in [1, Corollary 5.4] for $\varepsilon = 0$ and can be generalised easily. □

In order to prove the possibility of a finite representation of $\mathcal{D}_{2,\varepsilon}^0$ we will show the existence of a closed set D with $\mathcal{D}_{2,\varepsilon}^0 \subset D \subset \mathcal{E}_2$ as postulated in the above lemma. We first show this for $\varepsilon \in (0, \frac{1}{2})$ and then use the symmetry described in Lemma 2.2 to obtain a similar result for $\varepsilon \in (\frac{1}{2}, 1)$. For $\varepsilon = \frac{1}{2}$ [7, Theorem 5.2] already provides a full characterisation of $\mathcal{D}^0_{2,\frac{1}{2}}$.

Theorem 5.2. *For $\varepsilon \in (0, \frac{1}{2})$ the set $\mathcal{D}^0_{2,\varepsilon}$ can be completely characterised by cutting out finitely many polyhedra.*

Proof. For showing the theorem we will prove for each $\varepsilon \in (0, \frac{1}{2})$ the existence of a set D which satisfies the conditions made in Lemma 5.1.

Fix an $\varepsilon \in (0, \frac{1}{2})$. From Theorem 2.1 and (2.2) we know that $\overline{\mathcal{D}_{2,\varepsilon}}$ equals the triangle $\{(x, y) \in \mathbb{R}^2 | |x| \le 1, |y| \le x + 1\}$. Denote by $\square(Q_1, \dots, Q_k)$ the convex hull of the point $Q_1, \dots, Q_k$ and define the following sets:

$$
\begin{aligned}
T_1 &:= \square(Q_1, Q_2, Q_3, Q_4) \quad \text{with} \\
&\quad Q_1 = (-1, 0),\ Q_2 = \left(\frac{\varepsilon}{2} - 1, -\frac{\varepsilon}{2}\right),\ Q_3 = (1, 2 - \varepsilon), Q_4 = (1, 2), \\
T_2 &:= \square(Q_1, Q_2, Q_3, Q_4) \setminus \overline{Q_1Q_4} \quad \text{with} \\
&\quad Q_1 = (-1, 0),\ Q_2 = (1, -2),\ Q_3 = (1, -1 - \varepsilon),\ Q_4 = \left(\frac{-\varepsilon - 1}{2}, \frac{-\varepsilon + 1}{2}\right), \\
T_3 &:= \square(Q_1, Q_2, Q_3, Q_4) \quad \text{with} \\
&\quad Q_1 = (1 - \varepsilon, -1 + \varepsilon),\ Q_2 = (1 - \varepsilon, -1),\ Q_3 = (1, -1 - \varepsilon),\ Q_4 = (1, -1 + \varepsilon), \\
T_4 &:= \square(Q_1, Q_2, Q_3, Q_4) \setminus (\overline{Q_2Q_3} \cup \overline{Q_1Q_4}) \quad \text{with} \\
&\quad Q_1 = (1 - \varepsilon, -\varepsilon),\ Q_2 = (1 - \varepsilon, -1 + \varepsilon),\ Q_3 = (1, -1 + \varepsilon),\ Q_4 = (1, -\varepsilon), \\
T_5 &:= \square(Q_1, Q_2, Q_3, Q_4) \quad \text{with} \\
&\quad Q_1 = (1 - \varepsilon, \varepsilon),\ Q_2 = (1 - \varepsilon, -\varepsilon),\ Q_3 = (1, -\varepsilon),\ Q_4 = (1, \varepsilon), \\
T_6 &:= \{(x, y) \in \mathbb{R}^2 \mid 1 - \varepsilon \le x \le 1,\ \varepsilon < y < x - \varepsilon\}, \\
T_7 &:= \{(x, y) \in \mathbb{R}^2 \mid 1 - \varepsilon \le x \le 1,\ \varepsilon < y \le 1 + \varepsilon,\ x - \varepsilon \le y < x + 1 - \varepsilon\}, \\
T_8 &:= \{(x, y) \in \mathbb{R}^2 \mid 1 - \varepsilon \le x \le 1,\ 1 + \varepsilon < y < 1 + x - \varepsilon\}.
\end{aligned}
$$

Figure 3 shows the position and shape of these sets for two different values of ε. In Lemma 5.3 to Lemma 5.10 we show that for all $i \in \{1, \dots, 8\}$ the set T_i is not contained in $\mathcal{D}^0_{2,\varepsilon}$. Now set

$$D := \overline{\mathcal{D}_{2,\varepsilon} \setminus \bigcup_{i=1}^{8} T_i} = \{(x, y) \in \mathbb{R}^2 \mid -x - \varepsilon \le y \le x + 1 - \varepsilon,\ x \le 1 - \varepsilon\} \subset \mathcal{E}_2$$

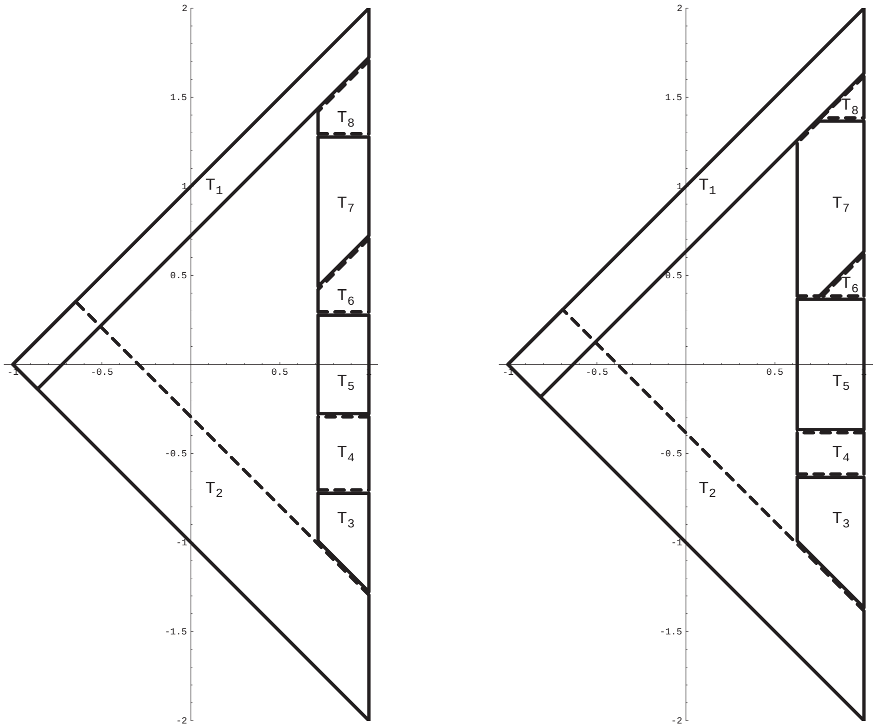

Figure 3. The sets T_i for $\varepsilon = \frac{2}{7}$ (left) and $\varepsilon = \frac{3}{8}$ (right)

and observe that $\mathcal{D}^0_{2,\varepsilon} \subset D$. Thus D satisfies the conditions of Lemma 5.1 which proves the theorem. $\square$

Lemma 5.3. *$\pi := (1, -1)$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for all $\mathbf{r} \in T_1$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. The set T_1 is a closed polygon. Since $P_\varepsilon(\pi)$ is a polygon, too, it is enough to show that $Q_i \in P_\varepsilon(\pi)$ for $i \in \{1, 2, 3, 4\}$. $P_\varepsilon(\pi)$ is defined by the inequalities

$$-\varepsilon \leq x - y + 1 < 1 - \varepsilon,$$
$$-\varepsilon \leq -x + y - 1 < 1 - \varepsilon.$$

It is easily checked that all four points really satisfy this system of inequalities which proves the lemma. $\square$

Lemma 5.4. *$\pi := (1)$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for all $\mathbf{r} \in T_2$ and $\varepsilon \in (0, \frac{1}{2})$.*

PROOF. The proof runs analogously to the proof of Lemma 5.6 [1]. $\square$

Lemma 5.5. *$\pi := (1, 1, 0, -1, -1, 0)$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for all $\mathbf{r} \in T_3$ and $\varepsilon \in (0, \frac{1}{2})$.*

PROOF. The proof runs analogously to the proof of Lemma 5.3. $\square$

Lemma 5.6. *$\pi := (1, 1, 0, -1, 0)$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for all $\mathbf{r} \in T_4$ and $\varepsilon \in (0, \frac{1}{2})$.*

PROOF. The set T_4 is a rectangle where the lines $\overline{Q_2Q_3} \subset \{(x, y) \in \mathbb{R}^2 | y = -1 + \varepsilon\}$ and $\overline{Q_1Q_4} \subset \{(x, y) \in \mathbb{R}^2 | y = -\varepsilon\}$ are not included. Note that $\square(Q_1, Q_2, Q_3, Q_4)$ is a polygon, thus

$$\begin{aligned} T_4 = \square(Q_1, Q_2, Q_3, Q_4) \setminus (\{(x, y) \in \mathbb{R}^2 \mid y \\ = -1 + \varepsilon\} \cup \{(x, y) \in \mathbb{R}^2 \mid y = -\varepsilon\}). \end{aligned} \tag{5.1}$$

$P_\varepsilon(\pi)$ is defined by the inequalities

$$\begin{aligned} &-\varepsilon \le x + y < 1 - \varepsilon, \\ &-\varepsilon \le x - 1 < 1 - \varepsilon, \\ &-\varepsilon \le -y <^* 1 - \varepsilon, \\ &-\varepsilon \le -x + 1 < 1 - \varepsilon, \\ &-\varepsilon \le y + 1 <^* 1 - \varepsilon. \end{aligned}$$

Two strict $<$ are tagged with *. Exchange them by non strict $\le$ and leave the other inequalities unchanged. This modified system of inequalities defines another polygon, let us call it $P_\varepsilon(\pi)^*$. Obviously

$$P_\varepsilon(\pi) = P_\varepsilon(\pi)^* \setminus (\{(x, y) \in \mathbb{R}^2 \mid y = -1 + \varepsilon\} \cup \{(x, y) \in \mathbb{R}^2 | y = -\varepsilon\}). \tag{5.2}$$

Now $\square(Q_1, Q_2, Q_3, Q_4) \subset P_\varepsilon(\pi)^*$ since all four points satisfy all inequalities of $P_\varepsilon(\pi)^*$. Observing (5.1) and (5.2) then immediately yields $T_4 \subset P_\varepsilon(\pi)$, which proves the lemma. $\square$

Lemma 5.7. *$\pi := (1, 0, -1, 0)$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for all $\mathbf{r} \in T_5$ and $\varepsilon \in (0, \frac{1}{2})$.*

PROOF. The proof runs analogously to the proof of Lemma 5.3. $\square$

[1] In Lemma 5.4 the cycle has period 1 and we only have one double inequality. Thus the situation is easier than in Lemma 5.6.

Lemma 5.8. *$\pi := (1,0,-1,1,1,-1,0)$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for all $\mathbf{r} \in T_6$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. For $\varepsilon \in (0, \frac{1}{3})$ we have $T_6 = \square(Q_1, Q_2, Q_3, Q_4) \setminus (\overline{Q_2Q_3} \cup \overline{Q_1Q_4})$ with vertices $Q_1 = (1-\varepsilon, 1-2\varepsilon)$, $Q_2 = (1-\varepsilon, \varepsilon)$, $Q_3 = (1, \varepsilon)$ and $Q_4 = (1, 1-\varepsilon)$. For $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$ we have $T_6 = \square(Q_1, Q_2, Q_3) \setminus (\overline{Q_1Q_2} \cup \overline{Q_1Q_3})$ with $Q_1 = (2\varepsilon, \varepsilon)$, $Q_2 = (1, \varepsilon)$ and $Q_3 = (1, 1-\varepsilon)$. However, one can easily prove that $T_6 \subset P_\varepsilon(\pi)$ in an analogous way as it was done in Lemma 5.6. $\square$

Lemma 5.9. *$\pi := (1,0,-1)$ is a cycle of $\tau_{\mathbf{r},\varepsilon}$ for all $\mathbf{r} \in T_7$ and $\varepsilon \in (0, \frac{1}{2})$.*

Proof. For $\varepsilon \in \left(0, \frac{1}{3}\right)$ we have $T_7 = \square(Q_1, Q_2, Q_3, Q_4)$ with $Q_1 = (1-\varepsilon, 1+\varepsilon)$, $Q_2 = (1-\varepsilon, 1-2\varepsilon)$, $Q_3 = (1, 1-\varepsilon)$ and $Q_4 = (1, 1+\varepsilon)$. $T_7 \subset P_\varepsilon(\pi)$ can be shown analogously to Lemma 5.3. For $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$ we have $T_7 = \square(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6) \setminus (\overline{Q_1Q_2} \cup \overline{Q_3Q_4})$ with $Q_1 = (2\varepsilon, 1+\varepsilon)$, $Q_2 = (1-\varepsilon, 2-2\varepsilon)$, $Q_3 = (1-\varepsilon, \varepsilon)$, $Q_4 = (2\varepsilon, \varepsilon)$, $Q_5 = (1, 1-\varepsilon)$ and $Q_6 = (1, 1+\varepsilon)$. For $\varepsilon = \frac{1}{3}$ the points Q_1 and Q_2 as well as the points Q_3 and Q_4 coincide giving a quadrangle with the points $Q_1 = Q_2 = (\frac{2}{3}, \frac{4}{3})$ and $Q_3 = Q_4 = (\frac{2}{3}, \frac{1}{3})$ missing. Analogously to Lemma 5.6 it can be proved that $T_7 \subset P_\varepsilon(\pi)$. $\square$

Lemma 5.10. *$T_8 \cap \mathcal{D}^0_{2,\varepsilon} = \emptyset$ for $\varepsilon \in (0, \frac{1}{2})$.*

Proof. First suppose that $\varepsilon \in [\frac{1}{3}, \frac{1}{2})$. Then $T_8 = \square(Q_1, Q_2, Q_3) \setminus (\overline{Q_1Q_2} \cup \overline{Q_1Q_3})$ with $Q_1 = (2\varepsilon, 1+\varepsilon)$, $Q_2 = (1, 1+\varepsilon)$ and $Q_3 = (1, 2-\varepsilon)$. Let $Q_4 := (\frac{1+2\varepsilon}{2}, 1+\varepsilon) \in \overline{Q_1Q_2}$ and subdivide T_8 into the triangles

$$T_8^{(1)} := \square(Q_1, Q_3, Q_4) \setminus (\overline{Q_1Q_4} \cup \overline{Q_1Q_3}),$$
$$T_8^{(2)} := \square(Q_2, Q_3, Q_4) \setminus (\overline{Q_2Q_4} \cup \overline{Q_3Q_4}).$$

See the rightmost example $\varepsilon = \frac{3}{8}$ in Figure 4. Consider the cycles $\pi_1 := (1,0,-1,2,-2)$ and $\pi_2 := (1,0,-1,2,-2,1,1,-2,2,-1,0,1,-1)$. Analogously to Lemma 5.6 it can be shown that $T_8^{(1)} \subset P_\varepsilon(\pi_1)$ and $T_8^{(2)} \subset P_\varepsilon(\pi_2)$.

Now suppose $\varepsilon \in (0, \frac{1}{3})$. Let $Q_1 = (1-\varepsilon, 2-2\varepsilon)$, $Q_2 = (1-\varepsilon, 1+\varepsilon)$, $Q_3 = (1, 1+\varepsilon)$ and $Q_4 = (1, 2-\varepsilon)$. Then $T_8 = \square(Q_1, Q_2, Q_3, Q_4) \setminus (\overline{Q_2Q_3} \cup \overline{Q_1Q_4})$. Again we have to do some subdivision. Define $Q_5 = (1-\varepsilon, \frac{3-\varepsilon}{2})$ and $Q_6 = (1, \frac{3+\varepsilon}{2})$. Note that $Q_5 \in \overline{Q_1Q_2}$ and $Q_6 \in \overline{Q_3Q_4}$. Let

$$\pi_1 := (1,0,-1,2,-2,2,-1,0,1,-1),$$
$$\pi_2 := (1,0,-1,2,-1,0,1,-1),$$

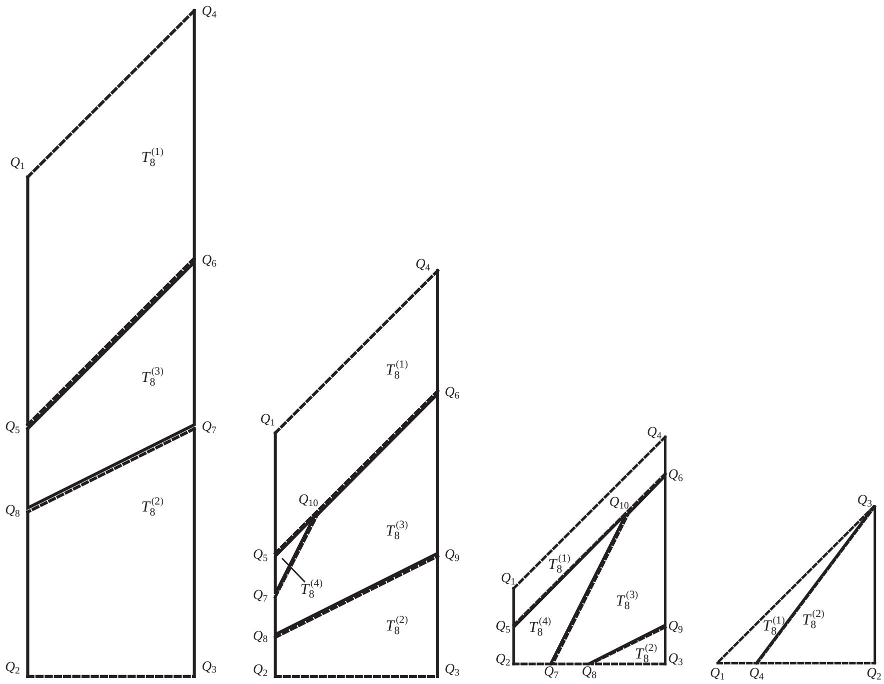

Figure 4. Partition of the set T_8 for $\varepsilon = \frac{1}{6}$, $\varepsilon = \frac{2}{9}$, $\varepsilon = \frac{2}{7}$, $\varepsilon = \frac{3}{8}$ (from left to right)

$$\pi_3 := (1, 0, -1, 2, -2, 1, 1, -2, 2, -1, 0, 1, -1),$$
$$\pi_4 := (1, 0, -1, 2, -2).$$

In the same manner as in the proof of Lemma 5.6 one can show that

$$T_8^{(1)} := \square(Q_1, Q_5, Q_6, Q_4) \setminus (\overline{Q_5Q_6} \cup \overline{Q_1Q_4})$$

is fully contained in $P_\varepsilon(\pi_1)$. Set

$$T_8^* := T_8 \setminus T_8^{(1)} = \square(Q_2, Q_3, Q_6, Q_5) \setminus \overline{Q_2Q_3}.$$

We have to distinguish several cases.

$\varepsilon \in \left(\frac{1}{4}, \frac{1}{3}\right)$**:** Define $Q_7 = \left(\frac{1+2\varepsilon}{2}, 1+\varepsilon\right)$, $Q_8 = (3\varepsilon, 1+\varepsilon)$, $Q_9 = (1, \frac{3-\varepsilon}{2})$ and $Q_{10} = (\frac{1+3\varepsilon}{2}, 1+2\varepsilon)$. We have $Q_7, Q_8 \in \overline{Q_2Q_3}$, $Q_9 \in \overline{Q_3Q_6}$ and $Q_{10} \in \overline{Q_5Q_6}$. Thus the sets

$$T_8^{(2)} := \square(Q_8, Q_3, Q_9) \setminus (\overline{Q_3Q_8} \cup \overline{Q_8Q_9}),$$

$$T_8^{(3)} := \square(Q_7, Q_8, Q_9, Q_6, Q_{10}) \setminus (\overline{Q_7Q_8} \cup \overline{Q_7Q_{10}}),$$
$$T_8^{(4)} := \square(Q_2, Q_7, Q_{10}, Q_5) \setminus \overline{Q_2Q_7}$$

form a partition of T_8^* (see third sketch in Figure 4). In the style of Lemma 5.6 we can now show that $T_8^{(i)} \subset P_\varepsilon(\pi_i)$ for $i \in \{2, 3, 4\}$.

$\varepsilon = \frac{1}{4}$: Runs similar to the previous case with the only difference that here the points Q_7 and Q_8 coincide with Q_2. Thus $T_8^{(3)}$ is a quadrangle and $T_8^{(4)}$ is a triangle (closed, just with the point Q_2 missing).

$\varepsilon \in \left(\frac{1}{5}, \frac{1}{4}\right)$: Define $Q_7 = (1-\varepsilon, 2-3\varepsilon)$, $Q_8 = (1-\varepsilon, \frac{3-2\varepsilon}{2})$, $Q_9 = \left(1, \frac{3-\varepsilon}{2}\right)$ and $Q_{10} = \left(\frac{1+3\varepsilon}{2}, 1+2\varepsilon\right)$. We have $Q_7, Q_8 \in \overline{Q_2Q_5}$, $Q_9 \in \overline{Q_3Q_6}$ and $Q_{10} \in \overline{Q_5Q_6}$. Again the sets

$$T_8^{(2)} := \square(Q_2, Q_3, Q_9, Q_8) \setminus (\overline{Q_2Q_3} \cup \overline{Q_8Q_9}),$$
$$T_8^{(3)} := \square(Q_7, Q_8, Q_9, Q_6, Q_{10}) \setminus \overline{Q_7Q_{10}},$$
$$T_8^{(4)} := \square(Q_5, Q_7, Q_{10})$$

form a partition of T_8^* (see second sketch in Figure 4) and as before we have $T_8^{(i)} \subset P_\varepsilon(\pi_i)$ for $i \in \{2, 3, 4\}$.

$\varepsilon = \frac{1}{5}$: The situation is comparable to the previous case with the difference that $Q_7 = Q_{10} = Q_5 = \left(\frac{4}{5}, \frac{7}{5}\right)$. Thus $T_8^{(3)}$ is a closed quadrangle with the only exception that Q_5 is missing and $T_8^{(4)}$ consists only of the point Q_5.

$\varepsilon \in \left(0, \frac{1}{5}\right)$: Let $Q_7 = \left(1, \frac{3-\varepsilon}{2}\right)$ and $Q_8 = \left(1-\varepsilon, \frac{3-2\varepsilon}{2}\right)$. $Q_7 \in \overline{Q_3Q_6}$ and $Q_8 \in \overline{Q_2Q_5}$. The sets

$$T_8^{(2)} := \square(Q_2, Q_3, Q_7, Q_8) \setminus (\overline{Q_2Q_3} \cup \overline{Q_7Q_8}),$$
$$T_8^{(3)} := \square(Q_5, Q_8, Q_7, Q_6)$$

partition T_8^* (see leftmost sketch in Figure 4). Further $T_8^{(i)} \subset P_\varepsilon(\pi_i)$ for $i \in \{2, 3\}$. □

We now turn to the case $\varepsilon \in (\frac{1}{2}, 1)$ and prove

Theorem 5.11. *For $\varepsilon \in (0, \frac{1}{2})$ the set $\mathcal{D}^0_{2,1-\varepsilon}$ can be fully characterised by cutting out finitely many polyhedra.*

Proof. We will show this very analogously to Theorem 5.2. Since by Lemma 2.2 for some cycle π we have $\mathrm{int}(P_\varepsilon(\pi)) = \mathrm{int}(P_{1-\varepsilon}(-\pi))$ for $\varepsilon \in (0,1)$ we can expect that we can use the negative cycles and quasi the same sets. Only the boundaries will change a little. In particular, we define

$$\tilde{T}_1 := \square(Q_1,Q_2,Q_3,Q_4) \setminus \overline{Q_2Q_3} \quad \text{with}$$
$$Q_1 = (-1,0),\ Q_2 = \left(\frac{\varepsilon}{2}-1,-\frac{\varepsilon}{2}\right),\ Q_3 = (1,2-\varepsilon),\ Q_4 = (1,2),$$
$$\tilde{T}_2 := \square(Q_1,Q_2,Q_3,Q_4) \quad \text{with}$$
$$Q_1 = (-1,0),\ Q_2 = (1,-2),\ Q_3 = (1,-1-\varepsilon),\ Q_4 = \left(\frac{-\varepsilon-1}{2},\frac{-\varepsilon+1}{2}\right),$$
$$\tilde{T}_3 := \square(Q_1,Q_2,Q_3,Q_4) \setminus (\overline{Q_1Q_2} \cup \overline{Q_2Q_3} \cup \overline{Q_1Q_4}) \quad \text{with}$$
$$Q_1 = (1-\varepsilon,-1+\varepsilon),\ Q_2 = (1-\varepsilon,-1),\ Q_3 = (1,-1-\varepsilon),\ Q_4 = (1,-1+\varepsilon),$$
$$\tilde{T}_4 := \square(Q_1,Q_2,Q_3,Q_4) \setminus \overline{Q_1Q_2} \quad \text{with}$$
$$Q_1 = (1-\varepsilon,-\varepsilon),\ Q_2 = (1-\varepsilon,-1+\varepsilon),\ Q_3 = (1,-1+\varepsilon), Q_4 = (1,-\varepsilon),$$
$$\tilde{T}_5 := \square(Q_1,Q_2,Q_3,Q_4) \setminus (\overline{Q_1Q_2} \cup \overline{Q_2Q_3} \cup \overline{Q_1Q_4}) \quad \text{with}$$
$$Q_1 = (1-\varepsilon,\varepsilon),\ Q_2 = (1-\varepsilon,-\varepsilon),\ Q_3 = (1,-\varepsilon), Q_4 = (1,\varepsilon),$$
$$\tilde{T}_6 := \{(x,y) \in \mathbb{R}^2 | 1-\varepsilon < x \le 1,\ \varepsilon \le y \le x-\varepsilon\},$$
$$\tilde{T}_7 := \{(x,y) \in \mathbb{R}^2 | 1-\varepsilon < x \le 1,\ \varepsilon \le y < 1+\varepsilon, x-\varepsilon < y \le x+1-\varepsilon\},$$
$$\tilde{T}_8 := \{(x,y) \in \mathbb{R}^2 | 1-\varepsilon < x \le 1,\ 1+\varepsilon \le y \le 1+x-\varepsilon\}.$$

In the same style as in Lemma 5.3 to Lemma 5.10 we can now show that

$$\tilde{T}_1 \subset P_{1-\varepsilon}\left((1,-1)\right),$$
$$\tilde{T}_2 \subset P_{1-\varepsilon}\left((-1)\right),$$
$$\tilde{T}_3 \subset P_{1-\varepsilon}\left((1,1,0,-1,-1,0)\right),$$
$$\tilde{T}_4 \subset P_{1-\varepsilon}\left((1,0,-1,-1,0)\right),$$
$$\tilde{T}_5 \subset P_{1-\varepsilon}\left((1,0,-1,0)\right),$$
$$\tilde{T}_6 \subset P_{1-\varepsilon}\left((1,0,-1,0,1,-1,-1)\right),$$
$$\tilde{T}_7 \subset P_{1-\varepsilon}\left((1,-1,0)\right),$$

$$\begin{aligned}\tilde{T}_8 \subset (&P_{1-\varepsilon}\left((1,0,-1,2,-2,2,-1,0,1,-1)\right) \cup P_{1-\varepsilon}\left((1,0,-1,2,-1,0,1,-1)\right)\\ &\cup P_{1-\varepsilon}\left((1,0,-1,2,-2,1,1,-2,2,-1,0,1,-1)\right) \cup P_{1-\varepsilon}\left((1,0,-1,2,-2)\right),\end{aligned}$$

where in the last case not necessarily all of the four integer vectors are cycles. This depends on ε, analogously to Lemma 5.10. Thus we have

$$D := \overline{\mathcal{D}_{2,1-\varepsilon} \setminus \bigcup_{i=1}^{8} \tilde{T}_i} = \{(x,y) \in \mathbb{R}^2 \mid -x-\varepsilon \le y \le x+1-\varepsilon,\ x \le 1-\varepsilon\} \subset \mathcal{E}_2.$$

This set D is exactly the same as in Theorem 5.2 and we again have that $\mathcal{D}^0_{2,1-\varepsilon} \subset D$. Thus D satisfies the conditions of Lemma 5.1 which proves the theorem. □

6. Examples

Finally we will use the results of Section 2 and Section 5 to give explicit characterisations of $\mathcal{D}^0_{2,\varepsilon}$ for $\varepsilon = \frac{1}{5}$ and $\varepsilon = \frac{1}{10}$. From the proofs of Theorem 5.2 and Theorem 5.11 we know

$$\mathcal{D}^0_{2,\varepsilon} \subset D^*(\varepsilon) := \begin{cases} \{(x,y) \in \mathbb{R}^2 \mid -x-\varepsilon \le y < x+1-\varepsilon,\ x < 1-\varepsilon\} \\ \qquad \text{for } \varepsilon \in \left(0, \dfrac{1}{2}\right) \\ \{(x,y) \in \mathbb{R}^2 \mid -x-1+\varepsilon < y \le x+\varepsilon,\ x \le \varepsilon\} \\ \qquad \text{for } \varepsilon \in \left(\dfrac{1}{2}, 1\right) \end{cases}$$

and that we can characterise $\mathcal{D}^0_{2,\varepsilon}$ completely for $\varepsilon \in (0,1)$. Thus we have to apply the algorithm described after Theorem 2.11 on $\overline{D^*(\varepsilon)}$. The sets $\mathcal{D}^0_{2,\frac{1}{5}}$ and $\mathcal{D}^0_{2,\frac{1}{10}}$ are shown as grey areas in Figure 5 and Figure 6. The dashed lines do not belong to them. We will state these two characterisation results as theorems.

Theorem 6.1. *The set $D^0_{2,\frac{1}{5}}$ equals the set $D^*\left(\frac{1}{5}\right)$ where the polyhedra $P_{\frac{1}{5}}(\pi_i)$, $i \in \{1,2,3\}$, with*

$$\pi_1 = (0,1), \quad \pi_2 = (-1,0,1), \quad \pi_3 = (-1,-1,1,2,1)$$

are cut out.

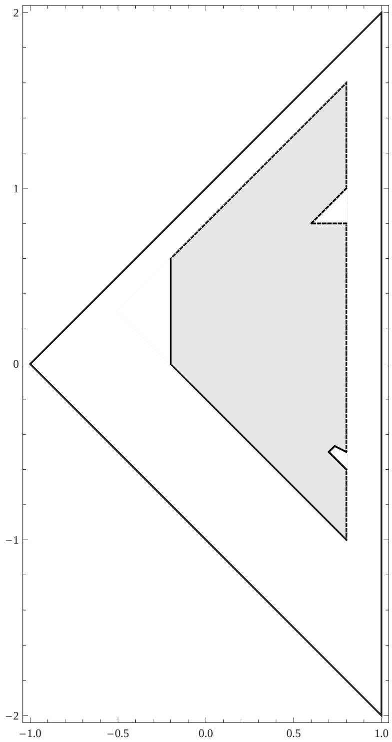

Figure 5. The set $\mathcal{D}^0_{2,\frac{1}{5}}$

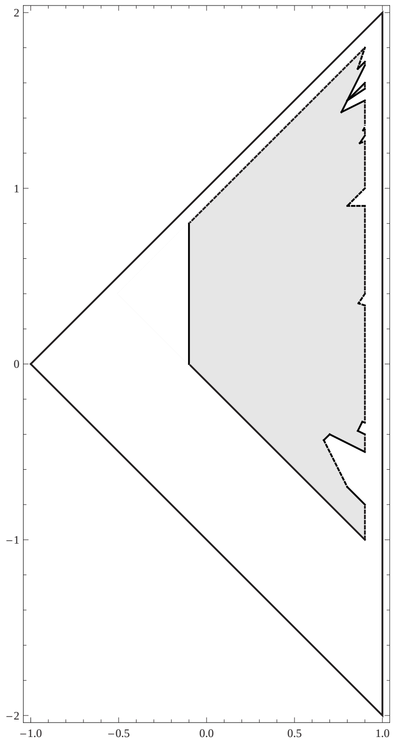

Figure 6. The set $\mathcal{D}^0_{2,\frac{1}{10}}$

Theorem 6.2. *The set $D^0_{2,\frac{1}{10}}$ equals the set $D^*\left(\frac{1}{10}\right)$ where the polyhedra $P_{\frac{1}{10}}(\zeta_i)$, $i \in \{1,\dots,10\}$, with*

$$\zeta_1 = (0,1), \qquad \zeta_2 = (-3,1,3,-2,-2,3,1),$$
$$\zeta_3 = (-4,2,1,-3,4,-2,-1,4), \qquad \zeta_4 = (-1,0,1),$$
$$\zeta_5 = (-5,5,-4,3,-1,-1,3,-4,5), \qquad \zeta_6 = (-2,1,1,-2,3),$$
$$\zeta_7 = (-1,-1,1,2,1), \qquad \zeta_8 = (-3,3,-2,1,1,-2,3),$$
$$\zeta_9 = (-3,2,1,-3,3,-1,-1,3), \qquad \zeta_{10} = (-2,-1,2,2,-1,-2,1,3,1)$$

are cut out.

Note that the algorithm returns more cycles but the above characterisations have been minimised. There is no polygon which is covered by others. When we compare these results with $\mathcal{D}^0_{2,0}$ and $\mathcal{D}^0_{2,\frac{1}{2}}$, which are shown in the introduction in Figure 1 and Figure 2, we can conjecture that for ε approaching 0 more and more peaks "grow".

From Corollary 2.3 we know that $\mathcal{D}^0_{2,\frac{1}{5}}$ and $\mathcal{D}^0_{2,\frac{4}{5}}$ and $\mathcal{D}^0_{2,\frac{1}{10}}$ and $\mathcal{D}^0_{2,\frac{9}{10}}$, respectively, only differ by a set of measure 0. Indeed, $\mathcal{D}^0_{2,\frac{4}{5}}$ and $\mathcal{D}^0_{2,\frac{9}{10}}$ have quasi the same shape as $\mathcal{D}^0_{2,\frac{1}{5}}$ and $\mathcal{D}^0_{2,\frac{1}{10}}$ and we have

Theorem 6.3. *Let π_1, π_2 and π_3 as in Theorem 6.1 and $\zeta_1, \dots, \zeta_{10}$ as in Theorem 6.2. Then*

$$\mathcal{D}^0_{2,\frac{4}{5}} = D^*\left(\frac{4}{5}\right) \setminus \bigcup_{i=1}^{3} P_{\frac{4}{5}}(-\pi_i), \qquad \mathcal{D}^0_{2,\frac{9}{10}} = D^*\left(\frac{9}{10}\right) \setminus \bigcup_{i=1}^{10} P_{\frac{9}{10}}(-\zeta_i).$$

We omit a detailed illustration of $\mathcal{D}_{2,\frac{4}{5}}$ and $\mathcal{D}_{2,\frac{9}{10}}$ here since they equal the sets $\mathcal{D}_{2,\frac{1}{5}}$ and $\mathcal{D}_{2,\frac{1}{10}}$, only the boundaries are reversed.

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