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On common fixed point of mappings and setvalued mappings with some weak conditions of commutativity

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Abstract. Some results on common fixed point of two set-valued and two single-valued mappings defined on a complete metric space with some weak commutativity conditions have been proved. Our work generalizes some earlier results due to Khan-Kubiaczyk, Chang, Singh-Whitefield and others.

1. Introduction

There exists an extensive literature on common fixed point of set-valued mappings satisfying contractive conditions controlled by a non-negative real-valued function from $[0,\infty)$ to $[0,\infty)$. In these results suitable conditions on the control function are crucial for the existence of fixed points. For this kind of work one can be referred to Singh-Meade [9], Barcz [1] and Khan-Kubiaczyk [6].

The purpose of this paper is to obtain some common fixed point theorems for two setvalued and two single valued mappings defined on a complete metric space employing some conditions weaker than commutativity. Our work generalizes several previously known results due to Khan-Kubiaczyk [6], Chang [2], Singh-Whitefield [8], Khan et al [5] and others.

2. Preliminaries and notations

Let (X,d) be a metric space, then following [7] we record

(i) $B(X)=\{A : A \text{ is a nonempty bounded subset of } X\}$

(ii) For $A,B\varepsilon B(X)$ we define

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$$D(A,B) = \inf\{d(a,b) : a\varepsilon A, b\varepsilon B\} \text{ and}$$
$$\delta(A,B) = \sup\{d(a,b) : a\varepsilon A, b\varepsilon B\}.$$

If $A = \{a\}$, then we write $\delta(A,B) = \delta(a,B)$ and if $B = \{b\}$ then $\delta(A,b) = d(a,b)$.

One can easily prove that for A, B, C in $B(X)$

$$\delta(A,B) = \delta(B,A) \geq 0,$$
$$\delta(A,B) \leq \delta(A,C) + \delta(C,B),$$
$$\delta(A,A) = \sup\{d(x,y) : x,y\varepsilon A\} = \text{ diam}\, A \text{ and}$$
$$\delta(A,B) = 0 \text{ implies that } A = B = \{a\}.$$

We require the following for future use:

Lemma 2.1 [3]. *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of (X,d) which converge to bounded subsets A and B respectively, then the sequence $\{\delta(A_n,B_n)\}$ converges to $\{\delta(A,B)\}$.*

Lemma 2.2 [4]. *Let $\{A_n\}$ be a sequence of nonempty bounded subsets of (X,d) and y be a point in X such that*

$$\lim_{n\to\infty} \delta(A_n, y) = 0.$$

Then the sequence $\{A_n\}$ converges to the set $\{y\}$.

Definition 2.3. Let $F : X \to B(X)$ be a set-valued mapping and $I : X \to X$ a single-valued mapping. Then, following [4,7], we say that the pair (F,I) is

(a) weakly commuting on X if for any x in X

$$\delta(FIx, IFx) \leq \max\{\delta(Ix,Fx),\ \text{diam}\, IFx\},$$

(aa) quasi-commuting on X if for any x in X

$$IFx \subseteq FIx,$$

(aaa) slightly commuting on X if for any x in X

$$\delta(FIx, IFx) \leq \max\{\delta(Ix,Fx),\ \text{diam}\, Fx\}.$$

Clearly two commuting mappings satisfy (a)–(aaa) but the converse may not be true. In [4] it is demonstrated by suitable examples that the foregoing three concepts are mutually independent and none of them implies the other two.

In accordance with [6], let Φ be the set of all realvalued functions $\phi : (\mathbb{R}^+)^5 \to \mathbb{R}^+$ which are upper semi-continuous from the right and non-decreasing in each of the co-ordinate variables such that $\phi(t,t,t,at,bt) < t$

for each $t \geq 0$, $a \geq 0$, $b \geq 0$, with $a+b \leq 4$. Also Ψ is the set of real valued functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ which are upper semicontinuous from the right and nondecreasing with $\psi(t) < t$ for $t > 0$.

We also require the following lemma due to Singh-Meade [9].

Lemma 2.4. *For $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$.*

3. Results

We prove the following

Theorem 3.1. *Let F, G be two set-valued mappings of a complete metric space (X, d) into $B(X)$, and I, J two self-mappings of X. Suppose that (F, I) and (G, J) are slightly commuting so that one of them is continuous, further*

$$F(X) \subseteq J(X), \quad G(X) \subseteq I(X)$$

and for all x, y in X and $\phi \varepsilon \Phi$

$$\text{(3.1.1)} \quad \delta(Fx, Gy) < \phi(\delta(Ix, Fx),\ \delta(Jy, Gy),\ \delta(Ix, Gy),\ \delta(Jy, Fx)\ d(Ix, Jy))$$

where for $\psi \varepsilon \Psi$ $t > 0$, $a \geq 0$, $b \geq 0$, $a+b \leq 4$

$$\psi(t) = \max \left\{ \begin{array}{l} \phi(t,t,t,at,bt), \phi(t,0,0,t,0),\ \phi(0,0,t,t,t), \\ \phi(0,t,t,0,0) \end{array} \right\} < t.$$

Then F, G, I and J have a unique common fixed point z such that $Iz = Jz = z$ and $Fz = Gz = \{z\}$. Also, z is the unique common fixed point of F and I, and of G and J.

Proof. Let $x_0 \varepsilon X$ and y_1 be an arbitrary point choosen in $X_1 = Fx_0$. Since $F(X) \subseteq J(X)$, we get a point $x_1 \varepsilon X$ such that $Jx_1 = y_1$. Now choose an arbitrary point y_2 in $X_2 = Gx_1$; as $G(X) \subseteq I(X)$, we get an $x_2 \varepsilon X$ with $Ix_2 = y_2$. Thus is general if we choose x_{2n} in X with $y_{2n+1} \varepsilon X_{2n+1} = Fx_{2n}$ then we always get some $x_{2n+1} \varepsilon X$ satisfying $Jx_{2n+1} = y_{2n+1}$. Again, let $y_{2n+2} \varepsilon X_{2n+2} = Gx_{2n+1}$ be arbitrary then there exists $x_{2n+2} \varepsilon X$ such that $Ix_{2n+2} = y_{2n+2}$ for $n = 0, 1, 2, \ldots$ Let us put $V_n = \delta(X_n, X_{n+1})$.

We distingush two cases:

Case 1. If $V_1 = 0$, then

$$V_1 = \delta(X_1, X_2) = \delta(Fx_0, Gx_1) = 0,$$

which means that $Fx_0 = y_1 = Jx_1 = Gx_1 = y_2 = Ix_2$. Since Gx_1 is a singleton, $\operatorname{diam} Gx_1 = 0$ and hence the slight commutativity of (G, J) gives

$$\text{(3.1.2)} \qquad GJx_1 = JGx_1 = GGx_1.$$

Now, using (3.1.1), we get

$$\begin{aligned}\delta(Fx_2, Gx_1) \leq& \phi(\delta(Fx_2, Gx_1), 0, 0, \delta(Fx_2, Gx_1), 0)\\ &\leq \psi(\delta(Fx_2, Gx_1) < \delta(Fx_2, Gx_1),\end{aligned}$$

getting thereby $Fx_2 = Gx_1$. Again, since Fx_2 is a singleton, $\operatorname{diam} Fx_2 = 0$, and the slight commutativity of (F, I) gives

$$IFx_2 = FIx_2 = FFx_2. \tag{3.1.3}$$

Applying (3.1.1) again we can have

$$\begin{aligned}\delta(FFx_2, Fx_2) &= \delta(FFx_2, Gx_1)\\ &\leq \phi(0, 0, \delta(FFx_2, Fx_2),\ \delta(FFx_2, Fx_2),\ \delta(FFx_2, Fx_2))\\ &\leq \psi(\delta(FFx_2, Fx_2)) < \delta(FFx_2, Fx_2),\end{aligned}$$

obtaining thereby $FFx_2 = Fx_2$. Thus Fx_2 is a fixed point of F. It follows from (3.1.3) that Fx_2 is also a fixed point of I. Since $Fx_2 = Gx_1$, we can get

$$\begin{aligned}\delta(Gx_1, GGx_1) &= \delta(Fx_2, GGx_1)\\ &\leq \phi(0, 0, \delta(Gx_1, GGx_1),\ \delta(Gx_1, GGx_1),\ \delta(Gx_1, GGx_1))\\ &\leq \psi(\delta(Gx_1, GGx_1) < \delta(Gx_1, GGx_1),\end{aligned}$$

which gives that $GGx_1 = Gx_1$. Thus $Fx_2 = Gx_1$ is a fixed point of G and from (3.1.2) it follows that $Fx_2 = Gx_1$ is also a fixed point of J. Thus $Fx_2 = y_1 = Jx_1 = Gx_1 = y_2 = Ix_2 = Fx_2$ is a common fixed point of F, G I and J.

Case II. Suppose that $V_n > 0$, $n = 1, 2, \ldots$, then

$$\begin{aligned}V_{2n+1} &= \delta(X_{2n+1}, X_{2n+2}) = \delta(Fx_{2n}, Gx_{2n+1})\\ &\leq \phi(V_{2n}, V_{2n+1}, V_{2n} + V_{2n+1}, 2V_{2n}, V_{2n+1}).\end{aligned}$$

Let us assume that $V_{2n+1} > V_{2n}$, then

$$V_{2n+1} \leq \phi(V_{2n+1}, V_{2n+1}, 2V_{2n+1}, 2V_{2n+1}, V_{2n+1}) \leq \psi(V_{2n+1}) < V_{2n+1},$$

which is a contradiction. Hence $V_{2n+1} \leq V_{2n}$. Similarly one can show that $V_{2n+2} \leq V_{2n+1}$. Then $\{V_n\}$ is a decreasing sequence. Now, since

$$V_2 \leq \phi(V_1, V_1, V_1, 2V_1, 2V_1) \leq \psi(V_1),$$

it follows by induction that

$$V_{2n+1} \leq \overset{2n}{\psi}(V_1)$$

and hence Lemma 2.4 gives that

$$\lim_{n\to\infty} V_n = 0.$$

We now show that $\{y_n\}$ is a Cauchy sequence. For this it is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence. Suppose $\{y_{2n}\}$ is not Cauchy sequence. Then there is an $\varepsilon > 0$ such that for an even integer $2k$ there exists even integers $2m(k) > 2n(k) > 2k$ such that

$$d(y_{2n(k)},\ y_{2m(k)}) > \varepsilon. \tag{3.1.4}$$

For every even integer $2k$, let $2m(k)$ be the least positive integer exceeding $2n(k)$ satisfying (3.1.4) and such that

$$d(y_{2n(k)},\ y_{2m(k)-2}) < \varepsilon. \tag{3.1.5}$$

Now

$$\varepsilon \le d(y_{2n(k)},\ y_{2m(k)}) \le d(y_{2n(k)},\ y_{2m(k)-2}) + V_{2m(k)-2} + V_{2m(k)-1}.$$

Then by (3.1.4) and (3.1.5) it follows that

$$\lim_{k\to\infty} d(y_{2n(k)},\ y_{2m(k)}) = \varepsilon. \tag{3.1.6}$$

Also, by the triangle inequality, we have

$$|d(y_{2n(k)},\ y_{2n(k)-1}) - d(y_{2n(k)},\ y_{2m(k)})| < V_{2m(k)-1}$$

and

$$|d(y_{2n(k)+1},\ y_{2m(k)-1}) - d(y_{2n(k)},\ y_{2m(k)})| < V_{2m(k)-1} + V_{2n(k)}.$$

By using (3.1.6) we get $d(y_{2n(k)}, y_{2m(k)-1}) \to \varepsilon$ and $d(y_{2n(k)+1}, y_{2m(k)-1}) \to \varepsilon$ as $k \to \infty$. Now by (3.1.1) we get

$$\begin{aligned} d(y_{2n(k)},\ y_{2m(k)}) &\le V_{2n(k)} + \delta(Fx_{2n(k)},\ Gx_{2m(k)-1}) \\ &\le V_{2n(k)} + \phi(V_{2n(k)},\ V_{2m(k)-1},\ d(y_{2m(k)},\ y_{2m(k)-1}) \\ &+V_{2m(k)-1},\ d(y_{2m(k)-1},\ y_{2n(k)+1}) + V_{2n(k)}, d(y_{2n(k)},\ y_{2m(k)-1})) \end{aligned}$$

which on letting $k \to \infty$ reduces to

$$\varepsilon < \phi(0, 0, \varepsilon, \varepsilon, \varepsilon,) < \varepsilon,$$

giving a contradiction. Thus $\{y_{2n}\}$ is a Cauchy sequence and converges to a point z in X. Thus the sequences $\{y_{2n}\} = \{Ix_{2n}\}$ and $\{y_{2n+1}\} = \{Jx_{2n+1}\}$ converge to z whereas the sequences of sets $\{Fx_{2n}\}$ and $\{Gx_{2n+1}\}$ converge to the set $\{z\}$.

Since (F, I) slightly commute, we have

$$\delta(FIx_{2n},\ IFx_{2n}) \le \max\{\delta(Ix_{2n},\ Fx_{2n}),\ \delta(Fx_{2n}, Fx_{2n})\}$$

which on letting $n \to \infty$ gives (by Lemma 2.1)

$$\lim_{n\to\infty} \delta(FIx_{2n},\ IFx_{2n}) = d(z,z) = 0.$$

Let us assume that I is continuous, then the sequence $\{Iy_{2n}\} = \{I^2x_{2n}\}$ converges to Iz. Thus

$$\begin{aligned} d(Iy_{2n+1},\ y_{2n+2}) &\le\ \delta(IFx_{2n},\ Gx_{2n+1}) \\ &\le\ \delta(IFx_{2n},\ FIx_{2n}) + \delta(FIx_{2n},\ Gx_{2n+1}) \\ \le\ \delta(IFx_{2n},\ &FIx_{2n}) + \phi(\{Iy_{2n},\ Iy_{2n+1}) + \delta(Iy_{2n+1},\ IFx_{2n}) \\ &+\delta(IFx_{2n},\ FIx_{2n})\},\ \delta(y_{2n+1},\ Gx_{2n+1}),\ \delta(Iy_{2n},\ Gx_{2n+1}), \\ &\{\delta(y_{2n+1},\ IFx_{2n}) + \delta(IFx_{2n},\ FIx_{2n})\},\ d(Iy_{2n},\ y_{2n+1})). \end{aligned}$$

Letting $n \to \infty$ and using Lemma 2.1 and Lemma 2.2 we obtain

$$d(Iz,z) \le \phi(0,0,d(Iz,z),\ d(Iz,z),\ d(Iz,z)) \le \psi(d(Iz,z)) < d(Iz,z)$$

which gives that $Iz = z$.

Similarly, applying condition (3.1.1) to $\delta(Fz, y_{2n+2}) \le \delta(Fz, Gx_{2n+1})$ and making $n \to \infty$, we can prove that $Fz = \{z\}$ which means that z is in the range of F. Since $F(X) \subseteq J(X)$, there exists a point z' in X such that $Jz' = z$. Now

$$\begin{aligned} \delta(z, Gz') = \delta(Fz, Gz') &\le \phi(0, \delta(z,\ Gz'),\ \delta(z,\ Gz'),\ 0, 0) \\ &\le \psi(\delta(z, Gz')) < \delta(z, Gz'), \end{aligned}$$

which gives that $Gz' = \{z\}$.

Since (G, J) is slightly commuting, we can have

$$\delta(Gz, Jz) = \delta(GJz',\ JGz') < \delta(Jz',\ Gz') = 0,$$

obtaining thereby $Gz = Jz$ and so

$$\begin{aligned} \delta(z, Gz) = \delta(Fz, Gz) &\le \phi(0, 0, \delta(z,\ Gz),\ \delta(z,\ Gz), \delta(z, Gz)) \\ &\le \psi(\delta(z, Gz)) < \delta(z, Gz), \end{aligned}$$

which implies that $Gz = \{z\} = Jz$. Thus we have shown that $Iz = Jz = Fz = Gz = \{z\}$, hence z is a common fixed point of F, G, I and J.

If we now assume that F is continuous, then the sequence $\{Fy_{2n}\} = \{FIx_{2n}\}$ converges to Fz. Since $Iy_{2n+1} \varepsilon IFx_{2n}$, the inequality (3.1.1) yields

$$\begin{aligned} \delta(Gx_{2n+1}, Fy_{2n+1}) \le \phi(\{\delta(Fx_{2n}, FIx_{2n}) + \delta(FIx_{2n}, IFx_{2n})\}, \\ \delta(Fx_{2n}, FFx_{2n}),\ \{\delta(IFx_{2n}, FIx_{2n}) + \delta(FIx_{2n}, Fx_{2n})\} \\ \delta(Jx_{2n+1}, Gx_{2n+1}), \{\delta(IFx_{2n}, FIx_{2n}) + \delta(FIx_{2n}, FFx_{2n})\}). \end{aligned}$$

Making $n \to \infty$, we obtain

$$\begin{aligned}\delta(z,Fz) &\le \phi(\delta(z,Fz),\ \delta(z,Fz),\ \delta(z,Fz),\ 0,2\ \delta(z,Fz))\\ &\le \psi(\delta(z,Fz) < \delta(z,Fz),\end{aligned}$$

which gives that $Fz = \{z\}$. Since $F(X) \subseteq J(X)$, there exists a point z' in X such that $Jz' = z$.

Similarly, using (3.1.1) on $\delta(Gz', Fx_{2n})$ and making $n \to \infty$ one can prove that $Gz' = \{z\}$. Now, by the slight commutativity of (G, J) we find

$$\delta(Gz, Jz) \le \delta(GJz', JGz') \le \delta(Jz', Gz') = 0,$$

which gives that $Gz = Jz$. Further, applying (3.1.1) to $\delta(Fx_{2n}, Gz)$ and letting $n \to \infty$, we can show that $Gz = \{z\}$. Thus it is established that $Jz = Gz = \{z\}$.

Since $G(X) \subseteq I(X)$ there exists a point z'' in X such that $Iz'' = z$. Thus

$$\begin{gathered}\delta(Fz'', z) = \delta(Fz'', Gz)\\ \le \phi(\delta(Fz'', z), 0, 0, \delta(Fz'', z), 0) \le \psi(\delta Fz'', z) < \delta(Fz'', z),\end{gathered}$$

implying thereby $Fz'' = \{z\}$.

By the slight commutativity of (F, I), we can have

$$\delta(Fz, Iz) = \delta(FIz'', IFz'') < \delta(Iz'', Fz'') = 0,$$

which yields that $Fz = Iz$. Thus we have shown that

$$Fz = Gz = Iz = Jz = \{z\}.$$

If we assume the mapping J (or G) to be continuous instead of I (or F), then the proof is similar, hence it is omitted.

For uniqueness, let w be another fixed point of (F, I), then

$$\begin{aligned}d(w,z) = d(Fw, Gz) &\le \phi(0, 0, d(w,z),\ d(w,z),\ d(w,z))\\ &\le \psi(d(w,z)) < d(w,z),\end{aligned}$$

which gives that $w = z$. Similarly, one can show that z is a unique common fixed point of G and J. This completes the proof.

The following theorem is immediate.

Theorem 3.2. *Theorem* 3.1 *holds good if we replace the condition* (3.1.1) *by*

$$\delta(Fx, Gy) \le \phi(\delta(Ix, Fy),\ \delta(Jy, Gy),\ D(Ix, Gy),\ D(Jy, Fx),\ d(Ix, Jy)).$$

Remark 1. A careful observation of the proof reveals that the condition required on the constants a and b in Theorem 3.2 is merely $a + b \le 2$.

Remark 2. By setting $I = J$ in Theorem 3.2, we get an improved version of Theorem 3 of Khan-Kubiaczyk [6] as we require the continuity of any one of the mappings instead of all three. Also, the commutativity condition is replaced by slight commutativity.

Remark 3. By setting $I = J =$ Identity mappings, we get Theorem 1 of Khan-Kubiaczyk [6].

As has already been remarked in [4], if the slight commutativity is replaced by weak commutativity in Theorem 3.1, then the continuity of anyone of the singlevalued mappings I or J is necessary. Thus we have the following result:

Theorem 3.3. *Theorem* 3.1 *holds good if we replace the slight commutativity with weak commutativity and the continuity of any one of the four mappings with the continuity of any one of the two single valued mappings.*

Proof. As proved in Theorem 3.1, $\{V_n\}$ is a decreasing sequence and $V_n \to 0$ as $n \to \infty$. Thus for $\varepsilon > 0$ there exists a positive integer p such that for $m, n > p$ we have

$$\delta(Fx_{2m},\ Fx_{2n}) < \varepsilon,\ \delta(Fx_{2m},\ Gx_{2n+1}) < \varepsilon,\ \delta(Gx_{2m+1}, Gx_{2n+1}) < \varepsilon.$$

To show that $\{y_{2n}\}$ is a Cauchy sequence, choose z_{2n} arbitrary in Fx_{2n} for $n = 0, 1, 2, \dots$. Then

$$d(z_{2m}, z_{2n}) \leq \delta(z_{2m},\ Fx_{2n}) \leq \delta(Fx_{2m},\ Fx_{2n}) < \varepsilon \tag{3.3.1}$$

for $m, n > p$. Thus $\{z_{2n}\}$ is a Cauchy sequence hence it converges to a point z in X.

We now assume I to be continuous, then depending on ε, one can find $\sigma > 0$ such that $d(Iz_{2m},\ Iz_{2n}) < \varepsilon$ whenever $d(z_{2m},\ z_{2n}) < \sigma$. Hence there exists an integer q with $m, n > q$ such that $d(z_{2n}\ z_{2m}) < \sigma$. For $m, n > q$ we have

$$d(Iz_{2m},\ Iz_{2n}) < \varepsilon. \tag{3.3.2}$$

Since the inequality (3.3.2) holds for arbitrary $z_{2n} \varepsilon Fx_{2n}$ we have

$$d(Iz_{2m},\ IFx_{2n}) < \varepsilon. \tag{3.3.3}$$

We now set $z_{2n} = y_{2n+1} = Jx_{2n+1} \varepsilon Fx_{2n}$. It folows that the sequence $\{y_{2n+1}\} = \{Jx_{2n+1}\}$ converges to z. Similarly one can also show that the sequence $\{y_{2n}\} = \{Ix_{2n}\}$ converges to z. So from (3.3.1) and (3.3.3), for $m, n > \max\{p, q\}$, we have

$$\delta(y_{2m+1},\ Fx_{2n}) < \varepsilon,\ \delta(Iy_{2m+1}, IFx_{2n}) < \varepsilon.$$

Similarly it can be argued that for $m, n > \max\{p, q\}$

$$\delta(y_{2m+2}, Gx_{2n+1}) < \varepsilon,\ \delta(Iy_{2m+2}, IG_{2n+1}) < \varepsilon.$$

Now, using inequality (3.1.1), for $n > \max\{p,q\}$ we obtain

$$
\begin{aligned}
&d(Iy_{2n+1}, y_{2n+2}) \le \delta(IFx_{2n}, Gx_{2n+1})\\
&\le \delta(IFx_{2n},\ FIx_{2n}) + \delta(FIx_{2n}, Gx_{2n+1})\\
&\le \max\{\delta(y_{2n}, Fx_{2n}), \mathrm{diam} IF_{2n}\} + \delta(FIx_{2n}, Gx_{2n+1})\\
&\le \max\{d(y_{2n}, y_{2n+1}) + \varepsilon, 2\delta(y_{2n+1}, IFx_{2n})\} + \phi(\delta(Iy_{2n}, FIx_{2n}),\\
&\quad \delta(y_{2n+1}, Gx_{2n+1}), \delta(Iy_{2n}, Gx_{2n+1}), \delta(y_{2n+1}, FIx_{2n}),\ \delta(Iy_{2n}, y_{2n+1}))\\
&\le \max\{d(y_{2n},\ y_{2n+1}) + \varepsilon,\ 2\varepsilon\} + \phi(d(Iy_{2n}, Iy_{2n+1})\\
&\quad + \varepsilon + \max\{d(y_{2n}, y_{2n+1}) + \varepsilon, 2\varepsilon\},\ \{d(y_{2n}, y_{2n+1}) + \varepsilon\},\\
&\quad \{d(Iy_{2n},\ y_{2n+2}) + \varepsilon\}, \{d(y_{2n+1},\ Iy_{2n+1}) + \varepsilon\\
&\quad + \max\{d(y_{2n}, y_{2n+1}) + \varepsilon, 2\varepsilon\}, d(Iy_{2n}, y_{2n+1})).
\end{aligned}
$$

Making $n \to \infty$, we obtain

$$d(Iz, z) \le 2\varepsilon + \phi(3\varepsilon, \varepsilon, \{d(Iz, z) + \varepsilon\}, d(Iz, z) + 3\varepsilon,\ d(Iz, z))$$

which for $\varepsilon \to 0+$ reduces to

$$
\begin{aligned}
d(Iz, z) &\le \phi(0, 0, d(Iz, z),\ d(Iz, z),\ d(Iz, z))\\
&\le \psi(d(Iz, z)) < d(Iz, z),
\end{aligned}
$$

giving thereby $Iz = z$.

The remaining part of the proof is similar to that of Theorem 3.1 hence it is omitted.

As has already been noted in [4], if the slight commutativity is replaced by quasi-commutativity in Theorem 3.1, then the continuity of any one of the two set-valued mappings is necessary. Thus, we get the following result:

Theorem 3.4. *Theorem* 3.1 *holds good if we replace the slight commutativity with quasi commutativity, and the continuity of any one of the four mappings with the continuity of any one of the two set valued mappings.*

Proof. The proof is similar to that of Theorem 3.1 except for some minor changes, hence it is omitted.

Remark 4. Results analogous to Theorem 3.3 and Theorem 3.4 and similar to Theorem 3.2, can be stated which also include Theorem 1 and Theorem 3 of Khan-Kubiaczyk [6].

Remark 5. By suitably restricting the four mappings one can derive a multitude of fixed point theorems which were proved while generalizing the results of Singh–Meade [9].

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