Publ. Math. Debrecen
74/1-2 (2009), 45–58

A note on the diagonal mapping in spaces of analytic functions in the unit polydisc

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Abstract. We define two spaces $K^{p,q,\alpha,\beta}$ and $M^{p,\alpha}$ of analytic functions in the unit polydisc U^n of C^n, closely related to the mixed norm and the Bergman spaces on U^n, and for any holomorphic function F in $K^{p,q,\alpha,\beta}$ or in $M^{p,\alpha}$ we consider its restriction to the diagonal, i.e., the function in the unit disc U of C defined by $DF(z) = F(z,\dots,z)$, and prove that the diagonal mapping D maps $K^{p,q,\alpha,\beta}$ onto the mixed-norm space $H^{p,q,\beta+\frac{q}{p}(|\alpha|+2n-1)}(U)$ and the space $M^{p,\alpha}$ onto the Bergman space $A^{p,|\alpha|+2n-1}(U)$.

Introduction

Let U^n be the unit polydisc in C^n and T^n be its Shilov boundary (see [R]) ($U^1 = U$ and $T^1 = T$). Denote by dm_n the normalized volume measure in U^n, and by $d\sigma_n$ the normalized surface measure on T^n.

For any Lebesgue measurable function f in U^n, we define

$$M_p(r,f) = \left(\int_{T^n} |f(r\xi)|^p d\sigma_n(\xi) \right)^{1/p},$$

where $0 < p < \infty$ and $r\xi = (r_1\xi_1,\dots,r_n\xi_n)$.

If $0 < p < \infty$, $0 < q < \infty$, and $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_j > -1$, $j = 1,\dots,n$, let

$$\|f\|^q_{p,q,\alpha} = \int_{I^n} \left(\prod_{j=1}^n (1-r_j^2)^{\alpha_j} M_p(r,f)^q \right) dr,$$

Mathematics Subject Classification: 32A35, 32A37.
Key words and phrases: Bergman spaces, mixed norm spaces on the polydisc, diagonal mapping.
The research of the first two authors was supported by a grant from MNS ON144010, Serbia.

where $I^n=[0,1)^n$ and $dr=dr_1\dots dr_n$. The mixed norm space $H^{p,q,\alpha}=H^{p,q,\alpha}(U^n)$ is then defined to be the space of functions f holomorphic in U^n, ($f\in H(U^n)$) such that $\|f\|_{p,q,\alpha}<\infty$.

First we give a new characterization of the mixed norm spaces $H^{p,q,\alpha}(U)$.

Theorem 1. *Let $0<p,\,q<\infty$ and $-1<\beta,\,\gamma<\infty$. A function $f\in H(U)$ belongs to $H^{p,q,\beta+q/p(\gamma+1)}(U)$ if and only if*

$$\int_0^1\left(\int_{|z|<r}|f(z)|^p(1-|z|)^\gamma dm_1(z)\right)^{q/p}(1-r)^\beta dr<\infty.$$

Motivated by this characterization we define the spaces $K^{p,q,\alpha,\beta}(U^n)$ as follows.

The space $K^{p,q,\alpha,\beta}=K^{p,q,\alpha,\beta}(U^n)$ consists of all $f\in H(U^n)$ such that

$$\|f\|^q_{p,q,\alpha,\beta}=\int_0^1\left(\int_{|z_1|<r}\dots\int_{|z_n|<r}|f(z)|^p\prod_{j=1}^n(1-|z_j|^2)^{\alpha_j}dm_n(z)\right)^{q/p}(1-r)^\beta dr<\infty,$$

where, $0<p,\,q<\infty$, $-1<\beta<\infty$ and $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>-1$, $j=1,\dots,n$.

We note that the spaces $K^{p,q,\alpha,\beta}(U)$ were studied by [AJ] and [J].

To each $F\in H(U^n)$, we associate a function DF, defined on the unit disc U of C, by

$$DF(z)=F(z,\dots,z),\quad z\in U.$$

The problem of description of diagonal of subspaces of $H(U^n)$ was studied by many authors (see [DS], [MR], [HO], [RS], [Sh1], [Sh2], [S].)

Now we are ready to state the main result of this paper.

Theorem 2. *Let $0<p,\,q<\infty$, $-1<\beta<\infty$, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>-1$, $j=1,\dots,n$. Then*

$$DK^{p,q,\alpha,\beta}(U^n)=H^{p,q,\beta+q/p(|\alpha|+2n-1)}(U),$$

where $|\alpha|=\alpha_1+\dots+\alpha_n$.

Note that in [RS] it was shown that

$$DH^{p,q,\alpha}(U^n)=H^{p,q,|\alpha|+(q/p+1)(n-1)}(U).$$

A special case of mixed norm spaces $H^{p,q,\alpha}(U^n)$ are the Bergman spaces $A^{p,\alpha}(U^n)$, $0<p<\infty$ and $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_j>-1$, $j=1,\dots,n$,

$$A^{p,\alpha}=A^{p,\alpha}(U^n)=H^{p,p,\alpha}(U^n).$$

J. Shapiro [S] and F. Shamoyan [Sh2] proved that

$$DA^{p,\alpha}(U^n) = A^{p,|\alpha|+2n-2}(U).$$

For example, if $\alpha' = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_n)$, then

$$DA^{p,\alpha'}(U^n) = A^{p,|\alpha|+2n-1}(U).$$

In this paper we define the space $M^{p,\alpha}(U^n)$, $0 < p < \infty$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$, that contains $A^{p,\alpha'}(U^n)$, for $\alpha' = (\alpha_1, \dots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \dots, \alpha_n)$ and show that $DM^{p,\alpha}(U^n) = A^{p,|\alpha|+2n-1}(U)$.

Let $0 < p < \infty$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$. Define $M^{p,\alpha} = M^{p,\alpha}(U^n)$ as the space of all $f \in H(U^n)$ such that

$$\|f\|_{p,\alpha}^p = \int_T \int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} |f(z)|^p \prod_{k=1}^n (1 - |z_k|^2)^{\alpha_k} dm_n(z) d\sigma_1(\xi) < \infty,$$

where $\Gamma_t(\xi) = \{z \in U : |z - \xi| < t(1 - |z|)\}$, $t > 1$, $\xi \in T$ is the Stolz angle with vertex ξ.

Theorem 3. *Let $0 < p < \infty$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\alpha_j > -1$, $j = 1, \dots, n$. Then*

$$DM^{p,\alpha}(U^n) = A^{p,|\alpha|+2n-1}(U),$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

1. Preliminaries

In this Section we gather several partially well-known lemmas that will be used in the proofs of our results.

Lemma 1. *Let $b > a > 0$, $c > 0$ and let $g : [0, 1] \to [0, \infty)$ be measurable. Assume either $0 < k < 1$ and g is increasing or $1 \le k < \infty$. Then*

$$\int_0^1 (1 - r)^{ka-1} \left(\int_0^1 \frac{g(\rho)(1 - \rho)^{c-1} d\rho}{(1 - r\rho)^b} \right)^k dr \le C \int_0^1 (1 - r)^{k(a+c-b)-1} g^k(r) dr.$$

See [RS, Proposition 3.1]. See also [AJ] and [M].

Lemma 2. *Let $0<p<\infty$ and $\beta>0$. Let (a_n) be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} 2^{-n\beta}a_n^p<\infty$. Then there is a constant $C>0$, depending only on p and β, such that*

$$\sum_{n=1}^{\infty} 2^{-n\beta}a_n^p \le C\bigg(a_0^p+\sum_{n=1}^{\infty} 2^{-n\beta}|a_n-a_{n-1}|^p\bigg).$$

See [MP, Lemma HL].

Lemma 3. *Let $0<p<1$, $\beta>-1$ and $\beta p+2p>1$. If $f\in H^{p,p,\beta p+2p-2}(U)$, then $f\in H^{1,1,\beta}(U)$. Moreover, there is a constant $C>0$, depending only on p and β, such that*

$$\|f\|_{1,1,\beta}\le C\|f\|_{p,p,\beta p+2p-2}.$$

Proof. By using the standard estimate

$$M_1(r^2,f)\le C(1-r)^{1-(1/p)}M_p(r,f),$$

(see [D]), we find that

$$\int_U |f(z)|(1-|z|)^{\beta}dm_1(z)\le C\int_0^1 M_p(r,f)(1-r)^{\beta-(1/p)+1}dr.$$

Let $r_n=1-2^{-n}$, $n=0,1,2,\dots$. Then we have

$$\begin{aligned}\|f\|_{1,1,\beta}^p &\le C\bigg(\sum_{n=1}^{\infty}\int_{r_{n-1}}^{r_n} M_p(r,f)(1-r)^{\beta-(1/p)+1}dr\bigg)^p\\ &\le C\sum_{n=1}^{\infty}2^{-np(\beta-(1/p)+2)}M_p^p(r_n,f)\\ &\le C\sum_{n=0}^{\infty}\int_{r_n}^{r_{n+1}} M_p^p(r,f)(1-r)^{p(\beta-(1/p)+2)-1}dr\\ &\le C\int_U |f(z)|^p(1-|z|)^{\beta p+2p-2}dm_1(z).\end{aligned}$$ □

An integration in polar coordinates shows that the following is true.

Lemma 4. *Let $\beta>-1$ and $\gamma-\beta>2$. Then there is a constant C, depending only on β and γ, such that*

$$\int_{|z|<r}\frac{(1-|z|)^{\beta}}{|1-\bar wz|^{\gamma}}dm_1(z)\le\frac{C}{(1-r|w|)^{\gamma-\beta-2}},\quad |w|<1,\ 0<r\le 1.$$

Lemma 5. *Let $\beta > -1$, $\gamma > \beta + 2$ and $t > 1$. Then*

$$\int_{\Gamma_t(\xi)} \frac{(1-|z|)^\beta}{|1-\bar{z}w|^\gamma} dm_1(z) \leq \frac{C}{|1-\xi\bar{w}|^{\gamma-\beta-2}}, \quad \xi \in T, \ |w| < 1,$$

where C is a constant depending only on β, γ and t.

Lemma 6. *Let $\beta > 1$. Then*

$$\int_T \frac{d\sigma_1(\xi)}{|1-\xi z|^\beta} \leq \frac{C}{(1-|z|)^{\beta-1}}, \quad |z| < 1.$$

See [D].

2. On characterization of the mixed norm spaces in the unit polydisc U

In this section we prove Theorem 1.

Proof of Theorem 1. First, assume that $\|f\|_{p,q,\beta+q/p(\gamma+1)} < \infty$. For $0 < t < 1$ define $f_t(z) = f(tz)$, $z \in U$. Let $r_n = 1 - 2^{-n}$, $n = 0, 1, \ldots$. Then

$$\int_0^1 (1-r)^\beta \left(\int_{|z|<r} |f_t(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} dr$$

$$= \sum_{n=1}^\infty \int_{r_{n-1}}^{r_n} (1-r)^\beta \left(\int_{|z|<r} |f_t(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} dr \leq C \sum_{n=1}^\infty 2^{-n(\beta+1)} A_n^{q/p},$$

where

$$A_n = \int_{|z|<r_n} |f_t(z)|^p (1-|z|)^\gamma dm_1(z).$$

Now by using Lemma 2 we find that

$$\begin{aligned}
&\int_0^1 (1-r)^\beta \left(\int_{|z|<r} |f_t(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} dr \\
&\leq C \sum_{n=1}^\infty 2^{-n(\beta+1)} \left(\int_{r_{n-1} \leq |z| < r_n} |f_t(z)|^p (1-|z|)^\gamma dm_1(z) \right)^{q/p} \\
&\leq C \sum_{n=1}^\infty 2^{-n(\beta+1)} M_p(r_n, f_t)^q 2^{-n(\gamma+1)q/p}
\end{aligned}$$

$$\le C\sum_{n=1}^{\infty}\int_{r_n}^{r_{n+1}} M_p(r,f_t)^q(1-r)^{\beta+q/p(\gamma+1)}dr$$
$$\le C\int_0^1(1-r)^{\beta+q/p(\gamma+1)}M_p(r,f_t)^q dr.$$

Letting $t\to 1$, we get

$$\int_0^1(1-r)^{\beta}\bigg(\int_{|z|<r}|f(z)|^p(1-|z|)^{\gamma}dm_1(z)\bigg)^{q/p}dr\le C\|f\|^q_{p,q,\beta+q/p(\gamma+1)}.$$

Conversely,

$$\|f\|^q_{p,q,\beta+q/p(\gamma+1)}=\int_0^1(1-r)^{\beta+q/p(\gamma+1)}M_p(r,f)^q dr$$
$$=\sum_{n=1}^{\infty}\int_{r_{n-1}}^{r_n}(1-r)^{\beta+q/p(\gamma+1)}M_p(r,f)^q dr$$
$$\le C\sum_{n=1}^{\infty}2^{-n(\beta+q/p(\gamma+1)+1)}M_p(r_n,f)^q$$
$$\le C\sum_{n=1}^{\infty}\bigg(\int_{r_n<|z|<r_{n+1}}|f(z)|^p(1-|z|)^{\gamma}dm_1(z)\bigg)^{q/p}2^{-n(\beta+1)}$$
$$\le C\sum_{n=1}^{\infty}\bigg(\int_{r_n<|z|<r_{n+1}}|f(z)|^p(1-|z|)^{\gamma}dm_1(z)\bigg)^{q/p}\int_{r_{n+1}}^{r_{n+2}}(1-r)^{\beta}dr$$
$$\le C\sum_{n=1}^{\infty}\int_{r_{n+1}}^{r_{n+2}}(1-r)^{\beta}\bigg(\int_{|z|<r}|f(z)|^p(1-|z|)^{\gamma}dm_1(z)\bigg)^{q/p}dr$$
$$\le C\int_0^1(1-r)^{\beta}\bigg(\int_{|z|<r}|f(z)|^p(1-|z|)^{\gamma}dm_1(z)\bigg)^{q/p}dr.$$

3. On the diagonal of $K^{p,q,\alpha,\beta}(U^n)$

In this section we prove Theorem 2.

Proof of Theorem 2. For $k=0,1,\dots$, let

$$U_k=\{U_{k,j}:j=0,1,\dots,2^{k+1}-1\},$$

where

$$U_{k,j}=\bigg\{z\in U:1-\frac{1}{2^k}\le|z|<1-\frac{1}{2^{k+1}},\frac{2\pi j}{2^{k+1}}\le\arg z<\frac{2\pi(j+1)}{2^{k+1}}\bigg\},$$

and let

$$V_k = \{V_{k,j} : j = 0, 1, \dots, 2^{k+1} - 1\},$$

where

$$V_{k,j} = \left\{ z \in U : 1 - \frac{1}{2^{k-1}} \le |z| < 1 - \frac{1}{2^{k+2}},\ \frac{2\pi(j - 1/2)}{2^{k+1}} \le \arg z < \frac{2\pi(j + 3/2)}{2^{k+1}} \right\}.$$

To prove Theorem 2, we will use the dyadic decomposition of the polydisc (see [DjS]):

$$U_{k,l_1,\dots,l_n} = U_{k,l_1} \times \cdots \times U_{k,l_n},$$

where $k = 0, 1, \dots$ and $U_{k,l_j} \in U_k$, $j = 1, \dots, n$.

Now, let $r_m = 1 - \frac{1}{2^m}$, $m = 0, 1, \dots$. Then by using Theorem 1 we find that

$$\begin{aligned}
&\|Df\|^q_{p,q,\beta+q/p(|\alpha|+2n-1)} \\
&\le C \int_0^1 (1-r)^\beta \left(\int_{|z|<r} |Df(z)|^p (1-|z|^2)^{|\alpha|+2n-2} dm_1(z) \right)^{q/p} dr \\
&= C \sum_{m=0}^{\infty} \int_{r_m}^{r_{m+1}} (1-r)^\beta \left(\int_{|z|<r} |Df(z)|^p (1-|z|^2)^{|\alpha|+2n-2} dm_1(z) \right)^{q/p} dr \\
&\le C \sum_{m=0}^{\infty} 2^{-m(\beta+1)} \left(\int_{|z|<r_{m+1}} |Df(z)|^p (1-|z|^2)^{|\alpha|+2n-2} dm_1(z) \right)^{q/p} \\
&\le C \sum_{m=0}^{\infty} 2^{-m(\beta+1)} \left(\sum_{k\le m} \sum_{l_1,\dots,l_n} \left(\max_{z\in U_{k,l_1,\dots,l_n}} |f(z)|^p \right) 2^{-k(|\alpha|+2n-2)} 2^{-2k} \right)^{q/p}.
\end{aligned} \tag{1}$$

From the n-subharmonicity of $|f(z)|^p$, $0 < p < \infty$, it follows that

$$\max_{z\in U_{k,l_1,\dots,l_n}} |f(z)|^p \le C 2^{2kn} \int_{V_{k,l_1,\dots,l_n}} |f(w_1,\dots,w_n)|^p dm_n(w). \tag{2}$$

See [DjS].

The family of enlarged sets $V_{k,j}$, $k = 0, 1, \dots,$ $j = 0, 1, \dots, 2^{k+1} - 1$, is a finite covering of U^n. Thus,

$$\begin{aligned}
&\sum_{k\le m} \sum_{l_1,\dots,l_n} \int_{V_{k,l_1,\dots,l_n}} |f(w_1,\dots,w_n)|^p \prod_{j=1}^n (1-|w_j|^2)^{\alpha_j} dm_n(w) \\
&\quad \le C \int_{|w_1|<r_{m+2}} \cdots \int_{|w_n|<r_{m+2}} |f(w_1,\dots,w_n)|^p \prod_{j=1}^n (1-|w_j|)^{\alpha_j} dm_n(w).
\end{aligned} \tag{3}$$

Combining (1), (2) and (3) we find that

$$\|Df\|^q_{\beta+q/p(\alpha|+2n-1)}$$
$$\leq C\sum_{m=0}^{\infty}2^{-m(\beta+1)}\int_{|w_1|<r_{m+2}}\cdots\int_{|w_n|<r_{m+2}}|f(w_1,\dots,w_n)|^p$$
$$\times\prod_{j=1}^{n}(1-|w_j|^2)^{\alpha_j}dm_n(w)$$
$$\leq C\sum_{m=0}^{\infty}\int_{r_{m+2}}^{r_{m+3}}(1-r)^{\beta}\Big(\int_{|w_1|<r}\cdots\int_{|w_n|<r}|f(w_1,\dots,w_n|^p$$
$$\times\prod_{j=1}^{n}(1-|w_j|^2)^{\alpha_j}dm_n(w)\Big)^{q/p}dr$$
$$\leq C\int_0^1(1-r)^{\beta}\Big(\int_{|w_1|<r}\cdots\int_{|w_n|<r}|f(w)|^p\prod_{j=1}^{n}(1-|w_j|^2)^{\alpha_j}dm_n(w)\Big)^{q/p}dr.$$

It remains to show that

$$D:K^{p,q,\alpha,\beta}(U^n)\to H^{p,q,\beta+q/p(|\alpha|+2n-1)}(U)$$

is onto.

Let $f\in H^{p,q,\beta+q/p(|\alpha|+2n-1)}(U)$. Define a function F on U^n by

$$F(z_1,\cdots,z_n)=s\int_U\frac{f(w)(1-|w|^2)^{s-1}}{\prod_{j=1}^{n}(1-z_j\bar{w})^{(s+1)/n}}dm_1(w),$$

where s is a positive real number. Then F is holomorphic in U^n and it is easy to see that

$$DF(z)=F(z,\cdots,z)=f(z),\quad z\in U.$$

Let $0<p\leq 1$. We may assume that $p(s+1)-n\max_{1\leq j\leq n}\alpha_j-2n>(\beta+1)p/q$. Then using Lemma 3 and Lemma 4 we find that

$$\|F\|^q_{p,q,\alpha,\beta}=\int_0^1\Big(\int_{|z_1|<r}\cdots\int_{|z_n|<r}|F(z)|^p\prod_{j=1}^{n}(1-|z_j|^2)^{\alpha_j}dm_n(z)\Big)^{q/p}(1-r)^{\beta}dr$$
$$\leq C\int_0^1\Big(\int_{|z_1|<r}\cdots\int_{|z_n|<r}\Big(\int_U\frac{|f(w)|^p(1-|w|)^{p(s-1)+2p-2}dm_1(w)}{\prod_{j=1}^{n}|1-z_j\bar{w}|^{(s+1)p/n}}\Big)$$
$$\times\prod_{j=1}^{n}(1-|z_j|)^{\alpha_j}dm_n(z)\Big)^{q/p}(1-r)^{\beta}dr$$

$$\leq C\int_0^1\left(\int_0^1 M_p^p(\rho,f)\frac{(1-\rho)^{p(s-1)+2p-2}}{(1-r\rho)^{p(s+1)-|\alpha|-2n}}d\rho\right)^{q/p}(1-r)^\beta dr.$$

An application of Lemma 1 gives

$$\|F\|^q_{p,q,\alpha,\beta}\leq C\int_0^1(1-r)^{\beta+q/p(|\alpha|+2n-1)}M_p(r,f)^q dr$$
$$=C\|f\|^q_{p,q,\beta+q/p(|\alpha|+2n-1)}.$$

Thus, $F\in K^{p,q,\alpha,\beta}(U^n)$ and $DF=f$.

Assume now that $1<p<\infty$ and $(s+1)>n\max_{1\leq k\leq n}(\alpha_k+2)$. Let γ_1 and γ_2 be positive real numbers such that $\gamma_1+\gamma_2=(s+1)/n$, and

$$\frac{s+1}{pn}-\frac{\min_{1\leq k\leq n}\alpha_k+1}{p}<\gamma_1<\frac{s+1}{pn}.$$

Let q denote the conjugate of p, i.e. $1/p+1/q=1$. Then we have

$$\|F\|^q_{p,q,\alpha,\beta}\leq C\int_0^1\left(\int_{|z_1|<r}\cdots\int_{|z_n|<r}\left(\int_U\frac{|f(w)|(1-|w|^2)^{s-1}dm_1(w)}{\prod_{j=1}^n|1-z_j\bar{w}|^{(s+1)/n}}\right)^p\right.$$
$$\left.\times\prod_{j=1}^n(1-|z_j|^2)^{\alpha_j}dm_n(z)\right)^{q/p}(1-r)^\beta dr. \tag{4}$$

By using Hölder's inequality we find that

$$\left(\int_U\frac{|f(w)|(1-|w|^2)^{s-1}dm_1(w)}{\prod_{j=1}^n|1-z_j\bar{w}|^{(s+1)/n}}\right)^p\leq\int_U\frac{|f(w)|^p(1-|w|^2)^{s-1}dm_1(w)}{\prod_{j=1}^n|1-z_j\bar{w}|^{\gamma_1 p}}$$
$$\times\left(\int_U\frac{(1-|w|^2)^{s-1}dm_1(w)}{\prod_{j=1}^n|1-z_j\bar{w}|^{\gamma_2 q}}\right)^{p/q}. \tag{5}$$

Using the Hölder's inequality and Lemma 4 we see that

$$\left(\int_U\frac{(1-|w|^2)^{s-1}dm_1(w)}{\prod_{j=1}^n|1-z_j\bar{w}|^{\gamma_2 q}}\right)^{p/q}\leq\prod_{j=1}^n\left(\int_U\frac{(1-|w|^2)^{s-1}dm_1(w)}{|1-z_j\bar{w}|^{n\gamma_2 q}}\right)^{p/(qn)}$$
$$\leq\prod_{j=1}^n\frac{1}{(1-|z_j|)^{(s+1)/n-p\gamma_1}}. \tag{6}$$

Thus, from (4), (5) and (6) it follows that

$$\|F\|^q_{p,q,\alpha,\beta}\leq C\int_0^1\left(\int_{|z_1|<r}\cdots\int_{|z_n|<r}\int_U\frac{|f(w)|^p(1-|w|^2)^{s-1}dm_1(w)}{\prod_{j=1}^n|1-z_j\bar{w}|^{\gamma_1 p}}\right.$$

$$\times \prod_{j=1}^{n}(1-|z_j|^2)^{\alpha_j-(s+1)/n+\gamma_1 p}dm_n(z)\Big)^{q/p}(1-r)^{\beta}dr$$

$$\leq C\int_0^1\bigg(\int_U |f(w)|^p(1-|w|^2)^{s-1}$$

$$\times\bigg(\prod_{j=1}^{n}\int_{|z_j|<r}\frac{(1-|z_j|)^{\alpha_j-(s+1)/n+\gamma_1 p}dm_1(z_j)}{|1-z_j\bar{w}|^{\gamma_1 p}}\bigg)dm_1(w)\bigg)^{q/p}(1-r)^{\beta}dr.$$

Now by using Lemma 4 and Lemma 1 we get

$$\begin{aligned}\|F\|^q_{p,q,\alpha,\beta} &\leq C\int_0^1\bigg(\int_U\frac{|f(w)|^p(1-|w|^2)^{s-1}dm_1(w)}{(1-r|w|)^{s+1-|\alpha|-2n}}\bigg)^{q/p}(1-r)^{\beta}dr\\ &\leq C\int_0^1 M_p(r,f)^q(1-r)^{\beta+q/p(|\alpha|+2n-1)}dr\\ &= C\|f\|^q_{p,q,\beta+q/p(|\alpha|+2n-1)}.\end{aligned}$$

This finishes the proof of Theorem 2.

4. On the diagonal of $M^{p,\alpha}(U^n)$

PROOF OF THEOREM 3. Let $f\in H^{p,\alpha}(U^n)$. To show that

$$\|f\|^p_{p,\alpha}\geq C\int_U |Df(z)|^p(1-|z|^2)^{|\alpha|+2n-1}dm_1(z) \tag{7}$$

we will use the following inequality

$$\begin{aligned}&\int_U |Df(z)|^p(1-|z|^2)^{|\alpha|+2n-1}dm_1(z)\\ &=\sum_{k,j}\int_{U_{k,j}}|Df(z)|^p(1-|z|^2)^{|\alpha|+2n-1}dm_1(z)\\ &\leq C\sum_{k,j}2^{-k(|\alpha|+2n)}\int_{2\pi j/2^{k+1}}^{2\pi(j+1)/2^{k+1}}\sup_{1-2^{-k}\leq\rho<1-2^{-(k+1)}}|f(\rho e^{i\theta},\dots,\rho e^{i\theta})|^p d\theta\\ &\leq C\sum_{k}2^{-k(|\alpha|+2n)}\int_0^{2\pi}\sup_{1-2^{-k}\leq\rho<1-2^{-(k+1)}}|f(\rho e^{i\theta},\dots,\rho e^{i\theta})|^p d\theta\\ &\leq C\sum_{k_1,\dots,k_n}\int_{1-2^{-(k_1+1)}}^{1-2^{-(k_1+2)}}(1-\rho_1^2)^{\alpha_1+1}d\rho_1\cdots\int_{1-2^{-(k_n+1)}}^{1-2^{-(k_n+2)}}(1-\rho_n^2)^{\alpha_n+1}d\rho_n\end{aligned}$$

$$\times \int_0^{2\pi} |f(\rho_1 e^{i\theta}, \cdots, \rho_n e^{i\theta})|^p d\theta$$

$$\leq C \int_T \Big(\int_0^1 \cdots \int_0^1 |f(\rho_1 \xi, \ldots, \rho_n \xi)|^p \prod_{j=1}^n (1-\rho_j^2)^{\alpha_j+1} d\rho_1 \cdots, d\rho_n \Big) d\sigma_1(\xi). \quad (8)$$

If u is a function defined on U, and $0 < \alpha,\ p < \infty$, than as a special case of the following known asymptotic relation (see [C])

$$\int_U |g(z)|^p d\mu(z) \cong \int_T \left(\int_{\Gamma_t(\xi)} \frac{|g(z)|^p}{1-|z|} d\mu(z) \right) d\sigma_1(\xi),$$

where $d\mu$ is a positive Borel measure on U, we have

$$\int_0^1 |u(re^{i\theta})|^p (1-r)^\alpha dr \leq C \int_{\Gamma_t(e^{i\theta})} |u(z)|^p (1-|z|^2)^{\alpha-1} dm_1(z).$$

Using this inequality, by each variable, we get the following

$$\int_0^1 \cdots \int_0^1 |f(\rho_1 e^{i\theta}, \ldots, \rho_n e^{i\theta})|^p \prod_{j=1}^n (1-\rho_j^2)^{\alpha_j+1} d\rho_1 \cdots d\rho_n$$

$$\leq C \int_{\Gamma_t(e^{i\theta})} \cdots \int_{\Gamma_t(e^{i\theta})} |f(z)|^p \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dm_n(z).$$

Integrating bothe sides by T and using (8) we get (7). The argument given here is taken from [LS].

The inequality (7) shows that the diagonal mapping D maps $M^{p,\alpha}(U^n)$ into $A^{p,|\alpha|+2n-1}(U)$. That D is onto follows from the following two facts:

$$A^{p,\alpha'}(U^n) \subset M^{p,\alpha}(U^n) \quad \text{and} \quad DA^{p,\alpha'}(U^n) = A^{p,|\alpha|+2n-1}(U),$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\alpha' = (\alpha_1, \cdots, \alpha_{j-1}, \alpha_j + 1, \alpha_{j+1}, \cdots, \alpha_n)$.

For a reader convenience, we show that an extension operator, the right inverse of the diagonal operator, is bounded from $A^{p,|\alpha|+2n-1}(U)$ into $M^{p,\alpha}$ and consequently $D : M^{p,\alpha}(U^n) \to A^{p,|\alpha|+2n-1}(U)$ is onto.

Let $f \in A^{p,|\alpha|+2n-1}(U)$. Define a function F on U^n by

$$F(z_1, \ldots, z_n) = s \int_U \frac{f(w)(1-|w|^2)^{s-1}}{\prod_{j=1}^n (1-z_j \bar{w})^{(s+1)/n}} dm_1(w),$$

where s is a positive real number. Obviously, $DF = f$, by the known Bergman representation formulas.

First, assume that $0 < p \leq 1$. We may assume that $p(s+1) > n \max_{1\leq j\leq n} \alpha_j + 2n+1$. By using Lemma 3, Lemma 5 and Lemma 6 we obtain

$$\begin{aligned}
\|F\|_{p,\alpha}^p &\leq C \int_T \left(\int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} \int_U \frac{|f(w)|^p (1-|w|^2)^{p(s-1)+2p-2} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{(s+1)p/n}} \right. \\
&\quad \left. \times \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dm_1(z_j) \right) d\sigma_1(\xi) \\
&\leq C \int_T \left(\int_U |f(w)|^p (1-|w|^2)^{p(s-1)+2p-2} \right. \\
&\quad \left. \times \prod_{j=1}^n \int_{\Gamma_t(\xi)} \frac{(1-|z_j|^2)^{\alpha_j} dm_1(z_j)}{|1 - z_j \bar{w}|^{(s+1)p/n}} \right) dm_1(w) \Bigg) d\sigma_1(\xi) \\
&\leq C \int_T \left(\int_U \frac{|f(w)|^p (1-|w|^2)^{p(s-1)+2p-2}}{|1 - \xi \bar{w}|^{p(s+1)-|\alpha|-2n}} dm_1(w) \right) d\sigma_1(\xi) \\
&\leq C \int_U |f(w)|^p (1-|w|^2)^{|\alpha|+2n-1} dm_1(w).
\end{aligned}$$

Case $1 < p < \infty$. We may assume that $s > n \max_{1\leq j\leq n} \alpha_j + 2n$.

Let q be the conjugate of p and let γ_1 and γ_2 be positive numbers such that $\gamma_1 + \gamma_2 = (s+1)/n$. and

$$\frac{s+1}{pn} - \frac{\alpha_j + 1}{p} < \gamma_1 < \frac{s+1}{pn}, \quad \text{for } j = 1, \cdots, n.$$

By using Hölder's inequality we find that

$$\begin{aligned}
\|F\|_{p,\alpha}^p \leq C \int_T \left(\int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} \left(\int_U \frac{|f(w)|^p (1-|w|^2)^{s-1}}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_1 p}} dm_1(w) \right. \right. \\
\times \left. \left(\int_U \frac{(1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_2 q}} \right)^{p/q} \right) \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j} dm_1(z_j) \Bigg) d\sigma_1(\xi). \quad (9)
\end{aligned}$$

An application of Hölders inequality and Lemma 4 gives

$$\begin{aligned}
&\left(\int_U \frac{(1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_2 q}} \right)^{p/q} \\
&\quad \leq \prod_{j=1}^n \left(\int_U \frac{(1-|w|^2)^{s-1} dm_1(w)}{|1 - z_j \bar{w}|^{n\gamma_2 q}} \right)^{p/(qn)} \leq C \prod_{j=1}^n (1-|z_j|)^{\gamma_1 p - (s+1)/n}. \quad (10)
\end{aligned}$$

By using (9), (10) and Lemma 6 we see that

$$\begin{aligned}
\|F\|_{p,\alpha}^p &\le C \int_T \bigg(\int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} \int_U \frac{|f(w)|^p (1-|w|^2)^{s-1} dm_1(w)}{\prod_{j=1}^n |1 - z_j \bar{w}|^{\gamma_1 p}} \\
&\quad \times \prod_{j=1}^n (1-|z_j|^2)^{\alpha_j + \gamma_1 p - (s+1)/n} dm_1(z_j) \bigg) d\sigma_1(\xi) \\
&= C \int_T \bigg(\int_U |f(w)|^p (1-|w|^2)^{s-1} dm_1(w) \\
&\quad \times \prod_{j=1}^n \int_{\Gamma_t(\xi)} \frac{(1-|z_j|^2)^{\alpha_j + \gamma_1 p - (s+1)/n}}{|1 - z_j \bar{w}|^{\gamma_1 p}} dm_1(z_j) \bigg) d\sigma_1(\xi) \\
&\le C \int_T \bigg(\int_U \frac{|f(w)|^p (1-|w|^2)^{s-1}}{|1-\xi\bar{w}|^{s+1-|\alpha|-2n}} dm_1(w) \bigg) d\sigma_1(\xi) \\
&= C \int_U |f(w)|^p (1-|w|^2)^{s-1} \bigg(\int_T \frac{d\sigma_1(\xi)}{|1-\xi\bar{w}|^{s+1-|\alpha|-2n}} \bigg) dm_1(w) \\
&\le C \int_U |f(w)|^p (1-|w|^2)^{|\alpha|+2n-1} dm_1(w).
\end{aligned}$$

References

[AJ] P. Ahern, M. Jevtić, Duality and multipliers for mixed norm spaces, *Michigan Math. J.* **30** (1983), 53–64.

[C] W. Cohn, Generalized area operators on Hardy spaces, *J. Math. Anal. Appl.* **216** (1997), 112–121.

[DjS] A. E. Djrbashian and F. A. Shamoyan, Topics in the Theory of A^p_α Spaces, *Leipzig, Teubner*, 1988.

[D] P. L. Duren, Theory of H^p Spaces, *Academic Press*, *New York*, 1970, reprinted by Dover, Mineola, NY 2000.

[DS] P. L. Duren and A. L. Shields, Restrictions of H^p functions to the diagonal of the polydisc, *Duke Math. J.* **42** (1975), 751–753.

[HO] C. Horowitz and D. Oberlin, Restriction of H^p functions to the diagonal of U^n, *Indiana Univ. Math. J.* **24** (1975), 767–772.

[J] M. Jevtić, Sur la derivee de la fonction atomique, *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), 201–203 (in *French. English summary*).

[LS] S. Li and R. Shamoyan, On a differential operator connected with finite sum of functions analytic in the polydisk, (*preprint*).

[MP] M. Mateljević and M. Pavlović, Multipliers of H^p and $BMOA$, *Pacific Journal of Mathematics* **146** (1990), 71–84.

[M] V. Mazya, Sobolev Spaces, *Moscow*, 1985.

[MR] B. Moulin and J. P.Rosay, Sur la restriction des fonctions pluri-sous-harmoniques a la diagonale du polydisque, *Indiana Univ. Math. J.* **26** (1977), 869–873 (in *French*).

[RS] G. Ren and J. Shi, The diagonal mapping in mixed norm spaces, *Studia Mathematica* **163**(2) (2004), 103–117.

[R] W. Rudin, Function Theory in Polydiscs, *W. A. Benjamin, Inc.*, *New York – Amsterdam*, 1969.

[Sh1] F. A. Shamoyan, Diagonal mapping and problems of representation in anisotropic spaces of functions that are holomorphic in a polydisc, *Siberian Math. J.* **31** (1990), 350–365.

[Sh2] F. A. Shamoyan, Applications of Dzhrbashyan integral representations to some problems of analysis, *Dokl. Akad. Nauk SSSR* **261**, no. 3 (1981), 557–561 (in *Russian)*).

[RSh] R. F. Shamoyan, The generalized Hardy transformation and Toeplitz operators in BMOA-type spaces, *Ukrain. Mat. Zh.* **53**, no. 9 (2001), 1260–1271 (in *Russian*).

[S] J. Shapiro, Mackey topologies, reproducing kernels and diagonal maps on the Hardy and Bergman spaces, *Duke Mathematical J.* **43** (1976), 187–202.

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(Received February 21, 2008; revised September 21, 2008)