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# Parabolic Weingarten surfaces in hyperbolic space

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**Abstract.** A surface in hyperbolic space $\mathbb{H}^3$ invariant by a group of parabolic isometries is called a parabolic surface. In this paper we investigate parabolic surfaces of $\mathbb{H}^3$ that satisfy a linear Weingarten relation of the form $a\kappa_1+b\kappa_2=c$ or $aH+bK=c$, where $a,b,c\in\mathbb{R}$ and, as usual, $\kappa_i$ are the principal curvatures, $H$ is the mean curvature and $K$ is de Gaussian curvature. We classify all parabolic linear Weingarten surfaces in hyperbolic space.

## 1. Introduction

A surface $S$ in 3-dimensional hyperbolic space $\mathbb{H}^3$ is called a *Weingarten surface* if there is some relation between its two principal curvatures $\kappa_1$ and $\kappa_2$, that is, if there is a smooth function $W$ of two variables such that $W(\kappa_1,\kappa_2)=0$. In particular, if $K$ and $H$ denote respectively the Gauss curvature and the mean curvature of $S$, the identity $W(\kappa_1,\kappa_2)=0$ implies a relation $U(K,H)=0$. In this paper we study Weingarten surfaces that satisfy the simplest case for $W$ and $U$, that is, of linear type:

$$a\ \kappa_1+b\ \kappa_2=c \tag{1}$$

and

$$a\ H+b\ K=c, \tag{2}$$

where $a,b,c\in\mathbb{R}$. We say in both cases that $S$ is a *linear Weingarten surface* and we abbreviate by $LW$-surface. In the set of $LW$-surfaces, it is worth mentioning

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three families of surfaces that correspond with trivial choices of $a$, $b$ and $c$:

(1) Umbilical surfaces, when $a = -b$ and $c = 0$ in (1).

(2) Surfaces with constant mean curvature: they appear if we choose $a = b$ in (1) or $b = 0$ in (2).

(3) Surfaces with constant Gaussian curvature, with the choice $a = 0$ in (2).

We call these three families of surfaces as *trivial* $LW$*-surfaces* and they have been studied in the literature. However, the classification of $LW$-surfaces in the general case is almost completely open today. Part of the present work is to provide new examples of these surfaces. Recently, there has been progress in the theory of Weingarten surfaces in both Euclidean and hyperbolic space, specially when the Weingarten relation is of type $H = f(H^2 - K)$ and $f$ elliptic. In such case, the surfaces satisfy a maximum principle that allows a best knowledge of the shape of such surfaces. These achievements can see, for example, in [1], [2], [4], [10]–[12].

A surface in $\mathbb{H}^3$ is called a *parabolic surface* if it is invariant by a group of parabolic isometries. Let us recall that a parabolic group of isometries of $\mathbb{H}^3$ is formed by isometries that leave fix one double point of the ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$. A such parabolic surface $S$ is determined by a generating curve $\alpha$ obtained by the intersection of $S$ with any geodesic plane orthogonal to the orbits of the group.

Our interest are $LW$-parabolic surfaces: in such setting, the relations (1) and (2) reduce to an ordinary differential equation that describes the shape of the profile curve that generates the surface. Parabolic surfaces in $\mathbb{H}^3$ were introduced by Do Carmo and Dajczer in [3] focusing in the study of surfaces with constant mean curvature (see also, [5]). With respect to the trivial $LW$-surfaces, we point out that umbilical surfaces in $\mathbb{H}^3$ are well known (see for example [14]); parabolic surfaces with constant mean curvature are given in the cited papers [3] and [5], and finally, parabolic surfaces with constant Gaussian curvature are described in [7], [8], [9].

We will ask about some facts for $LW$-parabolic surfaces. First, the question whether the surface can be extended to be complete, which it is given in terms of the generating curve. Second, if a complete parabolic Weingarten surface is embedded. For example, this occurs if the surface has constant Gaussian curvature [8], [9]. However, there exist constant mean curvature non-embedded surfaces that are complete [5]. Finally, the question about the behavior of the surface in relation with the ideal boundary $\mathbb{S}^2_\infty$. We know that the asymptotic boundary of surface contains the fixed point of the parabolic group of isometries.

This paper is organized as follows. In Section 2 we establish the differential equations that govern the parabolic $LW$-surfaces and some properties about their symmetries. In Sections 3 and 4 we study all parabolic surfaces in $\mathbb{H}^3$ that satisfy equations (1) and (2), respectively. We give a complete description of such surfaces, which depends on certain relations of the parameters $a$, $b$ and $c$. For the explicit classification, we refer the readers to Sections 3 and 4. However, we can announce some facts that are worth to point out.

*Any parabolic surface in $\mathbb{H}^3$ that satisfies the relation $a\kappa_1+b\kappa_2=c$ can be extended to be complete. The asymptotic boundary of any such surface is one point, one circle of two tangent circles. If it is one circle, the surface is umbilical. Moreover, there exist surfaces that are graphs on $\mathbb{S}^2_\infty$.*
*There exist complete parabolic surfaces in $\mathbb{H}^3$ that satisfy $aH+bK=c$. For these surfaces, the asymptotic boundary is the point $\infty$, one circle or two tangent circles at $\infty$. Some of the above surfaces are graphs on $\mathbb{S}^2_\infty$. There exist surfaces that can not extend to be complete*

## 2. Preliminaries and first properties

In this section we fix some notations and we give some properties about the symmetries of parabolic $LW$-surfaces. Let us consider the upper half-space model of the hyperbolic three-space $\mathbb{H}^3$, namely,

$$\mathbb{H}^3 =: \mathbb{R}^3_+ = \{(x,y,z)\in\mathbb{R}^3;\ z>0\}$$

equipped with the metric

$$\langle\,,\rangle = \frac{dx^2+dy^2+dz^2}{z^2}.$$

In what follows, we will use the words "vertical" or "horizontal" in the usual affine sense of $\mathbb{R}^3_+$. The ideal boundary $\mathbb{S}^2_\infty$ of $\mathbb{H}^3$ is identified with the one point compactification of the plane $\Pi\equiv\{z=0\}$, that is, $\mathbb{S}^2_\infty=\Pi\cup\{\infty\}$ and it corresponds with the asymptotic classes of geodesics rays of $\mathbb{H}^3$. The asymptotic boundary of a set $\Sigma\subset\mathbb{H}^3$ is defined as $\partial_\infty\Sigma=\overline{\Sigma}\cap\mathbb{S}^2_\infty$, where $\overline{\Sigma}$ is the closure of $\Sigma$ in $\{z\geq 0\}\cup\{\infty\}$. Let $L=\{(x,0,0),x\in\mathbb{R}\}$.

A parabolic group of isometries $G$ of $\mathbb{H}^3$ is a group of isometries that admits a fixed double point at $\mathbb{S}^2_\infty$. These isometries leave globally fixed each horocycle tangent to the fixed point. In our model, and without loss of generality, we take the point $\infty$ of $\mathbb{S}^2_\infty$ as the point that fixes $G$. Then the group $G$ is defined by the horizontal (Euclidean) translations in the direction of a horizontal vector $\xi$ with $\xi\in\Pi$: $G=\{T_a;a\in\mathbb{R},T_a(p)=p+a\xi\}$. The orbits are then horizontal straight lines parallel to $\xi$. We can also view this group as the set of reflections with respect to any geodesic plane orthogonal to $\xi$. Actually, the parabolic group $G$ is generated by all reflections with respect to the geodesic planes orthogonal to $\xi$. The space of orbits is then represented in any geodesic plane of this family. This will be done in our study.

Let $G$ be a group of parabolic isometries. Without loss of generality, we assume that the horizontal vector $\xi$ that defines the group of is the vector $\xi = (0,1,0)$. Let $P=\{(x,0,z);\ z>0\}$, which it is a vertical geodesic plane orthogonal to $\xi$. Then a surface $S$ invariant by $G$ intersects $P$ in a curve $\alpha$ called the *generating curve* of $S$. If $S$ is a parabolic $LW$-surface, we shall obtain an ordinary differential equation for the curve $\alpha$, equations (6) and (7) below. If we assume that $S$ is a complete surface, the possibilities about its asymptotic boundary $\partial_\infty S$ are: a circle ($\partial_\infty\alpha$ is a point of $L$ or one point of $L$ together $\infty$), two tangent circles ($\partial_\infty\alpha$ are two different points of $L$) or it is one point ($\partial_\infty\alpha=\emptyset$ or $\infty$).

Let $S$ be a parabolic (connected) surface in $\mathbb{H}^3$ and let $X(s,t)=(x(s),t,z(s))$ be a parametrization of $S$, where $t\in\mathbb{R}$ and the curve $\alpha$ will be assumed to be parametrized by the arc length with respect to the Euclidean metric, whose domain of definition $I$ is an open interval of real numbers including zero. The principal directions at each point are $\partial_s X$ and $\partial_t X$. Denote $\theta$ the angle that makes the velocity $\alpha'(s)$ with the $x$-axis, that is, $x'(s)=\cos\theta(s)$ and $z'(s)=\sin\theta(s)$ for a certain differentiable function $\theta$. The derivative $\theta'(s)$ of the function $\theta(s)$ is the Euclidean curvature of $\alpha$. From the hyperbolic viewpoint, the hyperbolic curvature of $\alpha$ at $s$ is exactly $z(s)\theta'(s)+\cos\theta(s)$.

Consider the Gauss map $N(s,t)$ induced by the immersion $X(s,t)$, that is, $N(s,t)=z(s)(-\sin\theta(s),0,\cos\theta(s))$. Then the principal curvatures $\kappa_i$ of $S$ are

$$\kappa_1(s,t)=z(s)\theta'(s)+\cos\theta(s),\quad \kappa_2(s,t)=\cos\theta(s) \tag{3}$$

and the mean curvature $H=\frac{\kappa_1+\kappa_2}{2}$ and Gaussian curvature $K=\kappa_1\kappa_2-1$ are

$$H(s,t)=\frac{z(s)}{2}\theta'(s)+\cos\theta(s),\quad K(s,t)=z(s)\cos\theta(s)\theta'(s)-\sin\theta(s)^2. \tag{4}$$

Thus, parabolic $LW$-surfaces in $\mathbb{H}^3$ are given by curves $\alpha$ whose coordinate functions satisfy

$$\begin{cases} x'(s)=\cos\theta(s)\\ z'(s)=\sin\theta(s)\end{cases} \tag{5}$$

together the equation

$$az(s)\theta'(s)+(a+b)\cos\theta(s)=c \tag{6}$$

or

$$\Big(\frac{a}{2}+b\cos\theta(s)\Big)\,z(s)\theta'(s)+a\cos\theta(s)-b\sin^2\theta(s)=c \tag{7}$$

depending if $S$ satisfies the Weingarten relation (1) or (2) respectively. We consider the initial conditions

$$x(0)=0,\quad z(0)=z_0>0,\quad \theta(0)=\theta_0. \tag{8}$$

In the upper half-space model of hyperbolic space, we say that a surface $S$ is a *vertical graph* if it writes as $S=(x,y,u(x,y));\ (x,y)\in\Omega\}$ for a certain positive $C^2$ function $u$ defined on a domain $\Omega$ of $\mathbb{R}^2=\{z=0\}$ (considered as asymptotic boundary of $\mathbb{H}^3$). In the case that $\Omega$ is $\mathbb{R}^2$, we say that $S$ is an *entire vertical graph*. For parabolic surfaces, $S$ is a vertical graph if the generating curve $\alpha$ writes as $\{(x,0,v(x));\ x\in I\}$ for a positive $C^2$ function $v$, where $I$ is an open interval of $\mathbb{R}$. In such case, $S$ is an entire graph if and only if $I=\mathbb{R}$. We also say that $\alpha$ is a vertical graph or an entire vertical graph.

We first prove two properties about the symmetries of the solutions of (6) and (7).

**Lemma 2.1.** *Let $\alpha$ be a solution of the initial value problem* (5)–(6) *or* (5)–(7)*. Suppose that $z'(s_0)=0$ for a real number $s_0$. Then $\alpha$ is symmetric with respect to the vertical line $x=x(s_0)$ of the $xz$-plane.*

Proof. We do the proof for a solution of (5)–(6) and the reasoning is analogous in the another case. Since $\sin\theta(s_0)=0$, then $\theta(s_0)=k\pi$ for some integer number $k$. The triplets of functions $\{x(s_0+s),z(s_0+s),\theta(s_0+s)\}$ and $\{2x(s_0)-x(s_0-s),z(s_0-s),-\theta(s_0-s)+2k\pi\}$ satisfy the same differential equations and the same initial conditions at $s=0$. The uniqueness of solutions concludes the result. $\square$

Given $a,b\in\mathbb{R}$ with $ab\neq 0$, examples of parabolic surfaces that satisfy the linear Weingarten relations (1) or (2) are the umbilical surfaces: it is suffices appropriates values (depending on $a$ and $b$) of the principal curvature to obtain the desired surfaces. As one example, given a relation $aH+bK=c$, there exists a horosphere that satisfies such Weingarten relation. Exactly, as $H=1$ and $K=0$, the horizontal plane $\{z=1\}$ satisfies $aH+bK=c$ by taking $a=b=c=1$. This means that in our results will appear this kind of surfaces. For this reason, we characterize them in terms of the above differential equations.

**Lemma 2.2.** *Let $\alpha$ be a solution of the initial value problem* (5)–(6) *or* (5)–(7)*. Suppose that $\theta'(s_0)=0$ for a real number $s_0$. Then $\alpha$ is a straight line* *and the corresponding surface is a totally geodesic plane, an equidistant surface or a horosphere.*

Proof. As in Lemma 2.1, we restrict to the case that $\alpha$ satisfies (5)–(6). If $\{x(s), z(s), \theta(s)\}$ is a such solution, then

$$\{\cos\theta(s_0)(s-s_0)+x(s_0), \sin\theta(s_0)(s-s_0)+z(s_0), \theta(s_0)\}$$

is a solution of (5)-(6) with the same initial conditions at $s = s_0$. Thus these three functions are the very solutions of the differential equations system. $\square$

Finally, and to end with this section, we consider the relation $a\kappa_1+b\kappa_2 = c$ in the case that $a$ or $b$ is zero. Then the surface has one constant principal curvature. Actually, each orbit $t \mapsto X(s,t)$ is a line of curvature, whose curvature, namely $\cos\theta(s)$, is constant along the line. On the other hand, the normal curvature of the curve $s \mapsto X(s,t)$ agrees with the (hyperbolic) curvature as a planar curve in $\mathbb{H}^3$. Thus, if it is constant, it is well known that the curve is a straight line or a Euclidean circle. We can see this as follows.

**Theorem 2.3.** *The only parabolic surfaces in $\mathbb{H}^3$ with one constant principal curvature are totally geodesic planes, equidistant surfaces, horospheres and Euclidean horizontal right-cylinders.*

Proof. We distinguish between the two principal curvatures $\kappa_1$ and $\kappa_2$. Assume that $\kappa_1 = c$, where $c$ is a constant. Then $\theta'(s)z(s) = c - \cos\theta(s)$. By differentiation of this expression and using (5) we obtain $\theta''(s) = 0$ for all $s$. Then $\theta'$ is constant and hence that from the Euclidean viewpoint, the curve is a piece of a straight line or a circle, which generates (pieces of) geodesic planes, equidistant surfaces, horospheres and horizontal right-cylinders.

Suppose now that $\kappa_2$ is constant, that is, $\cos\theta(s) = c$. This means that $\theta$ is constant, and so, $\alpha$ is a straight line. This gives totally geodesic planes (if $c = 0$), equidistant surfaces (if $0 < |c| < 1$) and horospheres (if $|c| = 1$). $\square$

After an isometry of the ambient space, the surfaces that are Euclidean horizontal right-cylinders are banana-shaped surfaces whose end points agree at one point of $\mathbb{S}^2_\infty$.

## 3. Parabolic surfaces satisfying $\kappa_1 = m\kappa_2 + n$.

In this section we shall consider parabolic surfaces that satisfy the relation (1). The case that one of the principal curvatures $\kappa_i$ is constant has been completely studied in Theorem 2.3. Thus we deal with the case that both $a$ and $b$ are non-zero numbers. Then the relation (1) can written as

$$\kappa_1 = m\kappa_2 + n \tag{9}$$

where $m, n \in \mathbb{R}$, $m \neq 0$. By using (3), we have

$$\theta'(s) = \frac{(m-1)\cos\theta(s) + n}{z(s)}. \tag{10}$$

After a change of orientation on the surface, we suppose that $n \geq 0$. We discard the trivial $LW$-surfaces, that is, umbilical surfaces corresponding to $(m, n) = (1, 0)$ and the surfaces with constant mean curvature, that is, $m = -1$. We consider $\theta(0) = \theta_0 = 0$ in the initial condition. In particular and from Lemma 2.1, the generating curve $\alpha$ is symmetric with respect to the line $x = 0$ of the $xz$-plane $P$. Multiplying in (10) by $\sin\theta$ and integrating, we obtain

$$n + \cos\theta(s) = \frac{2-m}{z(s)} \int_0^s (\sin\theta(t)\cos\theta(t))\ dt + (n+1)\frac{z_0}{z(s)}. \tag{11}$$

Equation (10) yields at $s = 0$,

$$\theta'(0) = \frac{n+m-1}{z_0}.$$

By Lemma 2.2, if the function $\theta'(s)$ vanishes at some point $s$, then $\theta' = 0$ and $\alpha$ is a straight line. If $\theta'(0) \neq 0$, then $\theta(s)$ is a strictly monotonic function on $s$. Let $(-\bar{s}, \bar{s})$ be the maximal domain of solutions of (5)–(10) under the initial conditions (8). Denote $\theta_1 = \lim_{s\to\bar{s}} \theta(s)$. Depending on the sign of $\theta'(0)$, we consider three cases.

**3.1. Case $n + m - 1 > 0$.** Here $\theta'(0) > 0$ and so, $\theta$ is strictly increasing in its domain.

(1) Subcase $m < n + 1$. In particular, $n > 0$. We prove that $\theta$ attains the value $\pi/2$. Assume on the contrary, that is, $\theta_1 \leq \pi/2$ and we will arrive to a contradiction. As $z'(s) = \sin\theta(s) > 0$, $z(s)$ is strictly increasing in $(0, \bar{s})$. Then $z(s) \geq z_0$ and the derivatives of $\{x(s), z(s), \theta(s)\}$ in equations (5)-(10) are bounded. This means that $\bar{s} = \infty$. As $\lim_{s\to\infty} z'(s) = \sin\theta_1 > 0$, then $\lim_{s\to\infty} z(s) = \infty$. Let $s \to \infty$ in (11). If the integral that appears in the right-side is bounded, then $n + \cos\theta_1 = 0$, that is, $\cos\theta_1 = -n < 0$: contradiction. If the integral is not bounded, and using the L'Hôpital's rule, $n + \cos\theta_1 = (2-m)\cos\theta_1$, that is, $(m-1)\cos\theta_1 + n = 0$. Then $m - 1 \leq 0$ and the hypothesis $n + m - 1 > 0$ yields $\cos\theta_1 = n/(1-m) > 1$: contradiction.

Therefore, there exists a first value $s_1$ such that $\theta(s_1) = \pi/2$. We prove that $\theta(s)$ attains the value $\pi$. By contradiction, we assume $\theta_1 \leq \pi$ and $z(s)$ is strictly increasing again. We then have $\bar{s} = \infty$ again and since $\theta$ is bounded by above, $\theta'(s) \to 0$ as $s \to \infty$. If $z(s)$ is bounded, then (10) implies $(m-1)\cos\theta_1 + n = 0$. As $m - 1 = n = 0$ is impossible, then $m - 1 > 0$ since $\cos\theta_1 < 0$. But the hypothesis $m < n + 1$ implies that $\cos\theta_1 = -n/(m-1) < -1$, which it is a contradiction. Thus $z(s) \to \infty$ as $s \to \infty$. By using (11) again, and letting $s \to \infty$, we have $n + \cos\theta_1 = 0$. In particular, $0 < m < 2$. For the contradiction, we obtain a second integral from (10) multiplying by $\cos\theta(s)$:

$$\sin\theta(s) = \frac{s}{z(s)} + \frac{1}{z(s)} \int_0^s \left( n\cos\theta(t) + (m-2)\cos^2\theta(t) \right) \, dt.$$

If the integral is bounded, then $\sin^2\theta_1 = 1$: contradiction. Thus, the integral is not bounded and L'Hôpital rule implies $\sin^2\theta_1 = 1 + n\cos\theta_1 + (m-2)\cos^2\theta_1$. This equation, together with $n + \cos\theta_1 = 0$ yields $(m-2)\cos^2\theta_1 = 0$: contradiction.

As conclusion, there exists a first value $s_2$ such that $\theta(s_2) = \pi$. By Lemma 2.1, the curve $\alpha$ is symmetric with respect to the line $x = x(s_2)$. Moreover, and putting $T = 2s_2$, we have:

$$x(s+T) = x(s) + x(T), \quad z(s) = z(s+T), \quad \theta(s+T) = \theta(s) + 2\pi.$$

This means that $\alpha$ is invariant by a group of horizontal translations orthogonal to the orbits of the parabolic group.

(2) Subcase $m \geq n + 1$. With this hypothesis and as $\theta'(s) > 0$, the equation (10) implies that $\cos\theta(s) \neq -1$ for any $s$. Thus $-\pi < \theta(s) < \pi$. For $s > 0$, $z'(s) = \sin\theta(s) > 0$ and then $z(s)$ is increasing on $s$ and so, $\theta'(s)$ is a bounded function. This implies $\bar{s} = \infty$. We show below that either there exists $s_0 > 0$ such $\theta(s_0) = \pi/2$ or $\lim_{s \Rightarrow \infty} \theta(s) = \pi/2$. See Fig. 2.

As in the above subcase, and with the same notation, if $\theta(s) < \pi/2$ for any $s$, then $n + \cos\theta_1 = 0$ or $(m-1)\cos\theta_1 + n = 0$. As $\cos\theta_1 \geq 0$ and since $m - 1 \geq n$, it implies that this occurs if and only if $n = 0$ and $\theta_1 = \pi/2$. In such case, $z''(s) = \theta'(s)\cos\theta(s) > 0$, that is, $z(s)$ is a convex function. As conclusion, if $n > 0$, there exists a value $s_0$ such that $\theta(s_0) = \pi/2$, and there exists $\theta_1 \in (\pi/2, \pi]$ such that $\lim_{s\to\infty} \theta(s) = \theta_1$.

We summarize the above discussion in the following theorem.

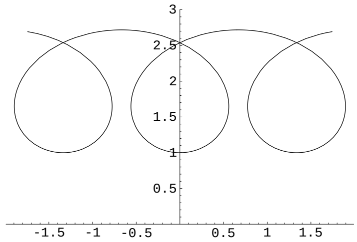



*Figure 1.* The generating curves of a parabolic surfaces with $\kappa_1 = m\kappa_2 + n$ with $n+m-1>0$ and subcase $m<n+1$. Here $m=1$ and $n=2$. The value $z_0$ is $z_0=1$.

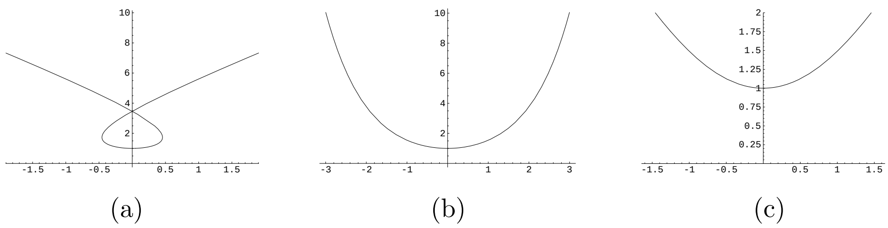



*Figure 2.* The generating curves of a parabolic surfaces with $\kappa_1 = m\kappa_2 + n$ with $n+m-1>0$ and subcase $m \geq n+1$. Here $z_0=1$. In (a), we have $m=3$ and $n=1$; in (b), $m=2$ and $n=0$; in (c), $m=3$, $n>0$ and $\theta(0)=\pi$.

**Theorem 3.1.** *Let $\alpha(s)=(x(s),0,z(s))$ be the generating curve of a LW-parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ whose principal curvatures satisfy the relation $\kappa_1 = m\kappa_2+n$. Consider $n\geq 0$ and that $\theta(0)=0$ in the initial condition* (8)*. Assume $n+m-1>0$.*

(1) *If $m<n+1$, then $\alpha$ is invariant by a group of translations in the $x$-direction. Moreover, $\alpha$ has self-intersections and it presents one maximum and one minimum in each period, with vertical points between maximum and minimum. The velocity $\alpha'$ twirls around the origin. See Figure 1.*

(2) *Assume $m\geq n+1$. Then the curve is complete. If $n>0$, then $\alpha$ has a minimum with self-intersections, it is the union of two horizonal graphs (see Figure 2, case* (a)*). If $n=0$, then $\alpha$ is a convex entire vertical graph with a minimum (see Figure 2, case* (b)*).*

In the above theorem we have considered the initial condition $\theta(0)=0$. Now we change this value by $\theta(0)=\theta_0$, $\theta_0\neq 0$. By using the uniqueness of solutions for an O.D.E. we have only to study those values $\theta_0$ that do not belong to the range of the function $\theta(s)$.

**Theorem 3.2.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a LW-parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ whose principal curvatures satisfy the relation $\kappa_1 = m\kappa_2 + n$. Consider $n \geq 0$ and that $\theta(0) = \theta_0 \neq 0$ in the initial condition* (8)*. Assume $n + m - 1 > 0$. Besides the examples obtained in Theorem 3.1, $\alpha$ is a geodesic or a entire vertical graph.*

Proof. In the subcase $m < n + 1$, the function $\theta(s)$ takes all the possible values of the interval $[0, 2\pi]$. Then, let us consider $m \geq n + 1 > 0$. First, we assume $n = 0$. Let $\theta_0 = \pi/2$. Then the solution of (5)-(8) is $\alpha(s) = (0, 0, s + z_0)$, that is, $S$ is a geodesic plane. On the other hand, if $\theta_0 = \pi$ and $(x(s), z(s), \theta(s))$ is the corresponding solution, then $(x(-s), z(-s), \theta(-s) - \pi)$ is the solution for $\theta_0 = 0$, and this case has been already studied.

Let $n > 0$. From Theorem 3.1, we only have to consider $\theta_0 = \pi$. This case is similar to the considered one when $\theta_0 = 0$. As $\theta'(0) < 0$, $\theta(s)$ is a decreasing function and $\cos\theta(s) < -n/(m-1)$. In particular, $x'(s) \neq 0$ and $z''(s) > 0$. This implies that $\alpha$ is a convex entire graph where $\lim_{s=\infty}\theta(s) = \theta_1 \in (\pi/2, \pi)$ with $\cos\theta_1 = n/(m-1)$. See Fig. 2, (c). □

**3.2. Case $n + m - 1 = 0$.** In the case $n + m - 1 = 0$, $\theta'(0) = 0$ and thus, $\theta'(s) = 0$ for any $s$. As $\theta(0) = 0$, then $\theta(s) = 0$ for any $s$. This implies that $\alpha(s)$ is a horocycle.

**Theorem 3.3.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$. Assume that the principal curvatures of $S$ satisfy the relation $\kappa_1 = m\kappa_2 + n$ with $n + m - 1 = 0$ and $n \geq 0$. If $\theta(0) = 0$ in the initial condition* (8)*, then $S$ is a horosphere (see Figure 3, case* (a)*).*

As in the previous section, we analyze the case $\theta_0 \neq 0$.

**Theorem 3.4.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$. Assume that the principal curvatures of $S$ satisfy the relation $\kappa_1 = m\kappa_2 + n$ with $n + m - 1 = 0$. If $\theta(0) \in (0, 2\pi)$ in the initial condition* (8)*, then (see Figure* (3)*, case* (b)*)*

(1) *$\alpha$ has one maximum with self-intersections.*

(2) *$\alpha$ is complete and asymptotic to the line $L$ at infinity.*

Proof. As $n \geq 0$, then $m - 1 < 0$ (recall that we discard the umbilical case $m = 1, n = 0$). Assume $\theta_0 = \pi$. Then by (10), $\theta'(s) > 0$ and $\cos\theta(s) \neq 1$. This means that $0 < \theta(s) < 2\pi$. For $s > 0$, $z'(s) < 0$ and $z(s) \leq z_0$. Thus $\theta'(s)$ is a bounded function and $\bar{s} = \infty$. Then $\theta'(s) \to 0$ as $s \to \infty$. From (10), this yields that

$$\lim_{s\to\infty} \cos\theta(s) = 0, \qquad \lim_{s\to\infty} z(s) = 0$$

and then the function $\theta(s)$ takes all the values of $(0, 2\pi)$. $\square$

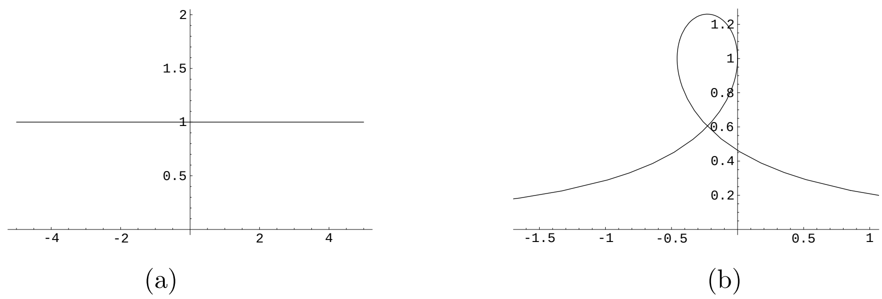



*Figure 3.* The generating curve of a parabolic surface with $\kappa_1 = m\kappa_2 + n$ with $n + m - 1 = 0$. Here $z_0 = 1$ and $m = -2$ and $n = 3$. In the case (a), $\theta(0) = 0$ and in the case (b), $\theta(0) = \pi/2$ (see Theorem 3.4).

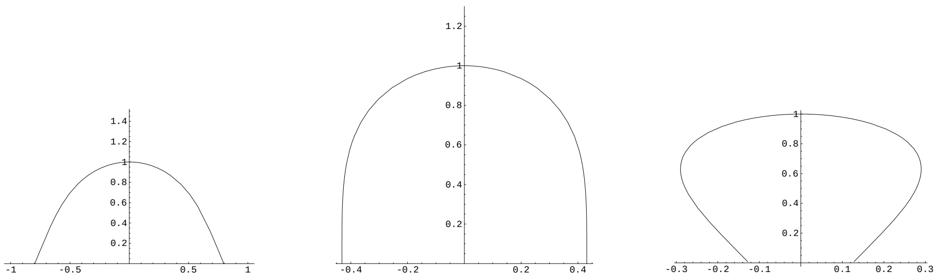
**3.3. Case** $n + m - 1 < 0$. With this assumption, $\theta(s)$ is a strictly decreasing function. As $n \geq 0$ and from (10), $\cos\theta(s) \neq 0$. This implies that $\theta(s)$ is a bounded function with $-\pi/2 < \theta(s) < \pi/2$. If $\bar{s} = \infty$ and as $z(s) > 0$, then both functions $\theta'(s)$ and $z'(s)$ go to 0 as $s \to \infty$. By (10) and (8), we have $(m-1)\cos\theta_1 + n = 0$ and $\sin\theta_1 = 0$: contradiction. This proves that $\bar{s} < \infty$.

As consequence, $z(s) \to 0$ since on the contrary, $\theta'(s)$ would be bounded and $\bar{s} = \infty$. We now use (11). Letting $s \to \bar{s}$ and by L'Hôpital rule again, we obtain $(m-1)\cos\theta_1 + n = 0$, that is, $\cos\theta_1 \geq -n/(m-1)$. Finally, $z''(s) = \theta'(s)\cos\theta(s) < 0$, that is, $\alpha$ is concave. As conclusion of this reasoning:

**Theorem 3.5.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a LW-parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ whose principal curvatures satisfy the relation $\kappa_1 = m\kappa_2 + n$. Consider $n \geq 0$ and that $\theta(0) = 0$ in the initial condition* (8)*. Assume $n + m - 1 < 0$. Then $\alpha$ is a concave (non-entire) vertical graph with one maximum and it intersects $L$ with a contact angle $\theta_1$, $\cos\theta_1 = -n/(m-1)$ (see Figure 4, case* (a)*). In the particular case that $n = 0$, then $\alpha$ meets orthogonally $L$ (see Figure 4, case* (b)*).*

*In the case $\theta(0) \neq 0$, we obtain that $\alpha$ is a geodesic or it is not a vertical graph intersecting the ideal boundary $L$ at two points. See Figure 4, case* (c)*.*

Proof. The only case to study is when $\theta(0) \neq 0$. As when $\theta_0 = 0$, the range of $\theta$ is $(-\theta_1, \theta_1)$, we consider $\theta(0) = \pi$. Then $\theta'(0) > 0$ and so, $\theta(s)$ is strictly increasing, with $-1 \leq \cos\theta(s) < -n/(m-1)$. As above, one can show that $\theta(s)$ takes all values of the interval $(\theta_1, 2\pi - \theta_1)$ and that intersects the ideal boundary at $L$ at two points. This means that the assumption $\theta_0 = \pi$ covers all possibilities on the initial angle. Moreover, the function $\theta(s)$ reaches the values $\pi/2$ and $3\pi/2$, that is, $\alpha$ is not a vertical graph. See Figure 4, (c). In the case that $n = 0$, we point out that if $\theta_0 = \pi/2$, then $\alpha$ is a vertical line, that is, it is a geodesic. $\square$



(a) (b) (c)

*Figure 4.* The generating curve of a parabolic surface with $\kappa_1 = m\kappa_2 + n$ with $n + m - 1 < 0$. Here $m = -2$, $z_0 = 1$ and: (a) $n = 1$; (b), $n = 0$; (c), $n = 1$ and $\theta(0) = \pi$.

As conclusion of this section, we have:

**Corollary 3.6.** *Let $S$ be a parabolic surface in $\mathbb{H}^3$ that satisfies $\kappa_1 = m\kappa_2 + n$, for some constants $m$ and $n$. Consider $\alpha(s) = (x(s), 0, z(s))$ the generating curve. If $\theta(0) = 0$ in the initial condition* (8)*, then:*

(1) *The asymptotic boundary $\partial_\infty S$ of $S$ is $\{\infty\}$, two tangent circles or one circle. In the latter case, the surface must be umbilical.*
(2) *The surface $S$ is complete.*
(3) *If $S$ is embedded, then it is an entire vertical graph.*

## 4. Parabolic surfaces satisfying $aH + bK = c$.

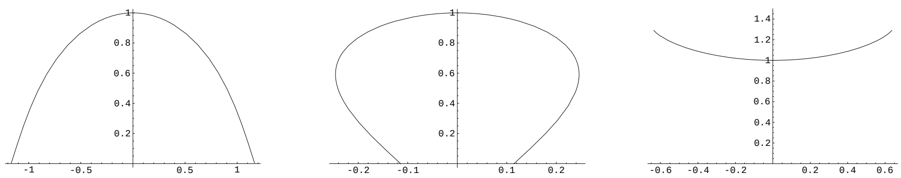
In this section we consider parabolic $LW$-surfaces that satisfy the relation (2). As in Section 3, we shall consider generating curves $\alpha$ with some horizontal tangent line. Recall that this means that $\theta_0 = 0$ on the initial condition (8). We also discard the trivial $LW$-surfaces that satisfy equation (2), that is, the cases that $a$ or $b$ are zero. Depending on the constant $c$, we have two possibilities: (i) $c = 0$. Without loss of generality, we assume that $a = 2$; (ii) $c \neq 0$. Then we take $c = 1$.

### 4.1. Case $2H + bK = 0$.

**Theorem 4.1.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ that satisfies* (5)–7. *Assume that $\theta(0) = 0$. If $2H + bK = 0$ with $b \neq 0$, then it holds the following:*

(1) *If $b > -1$, $\alpha$ is complete with one maximum and intersects $L$ at an angle $\theta_1$ such that $2\cos\theta_1 - b\sin^2\theta_1 = 0$. Moreover, if $b \geq 0$, $\alpha$ is a concave (non-entire) vertical graph, whereas if $-1 < b < 0$, $\alpha$ is not a vertical graph. See Figure 5* (a) *and* (b) *respectively. The asymptotic boundary of $\alpha$ is two points and the one of the surface that generates $\alpha$ is two tangent circles at infinity.*

(2) *If $b < -1$, then $\alpha$ is a convex (non-entire) vertical graph with a minimum. Moreover, the contact angle between $\alpha$ and $L$ is $\theta_1$ with $1 + b\cos\theta_1 = 0$. The curve (and the corresponding surface) is not complete. See Figure 5,* (c).



(a) (b) (c)

*Figure 5.* The generating curve of a parabolic surface with $2H{+}bK{=}0$. Here $z_0 = 1$ and $\theta(0) = 0$. Case(a), $b = 1$; case (b), $b = -0,7$; case (c), $b = -3$.

Proof. Equation (7) writes as

$$\big(1 + b\cos\theta(s)\big)z(s)\theta'(s) + 2\cos\theta(s) - b\sin^2\theta(s) = 0. \tag{12}$$

Let us denote $(-\bar{s}, \bar{s})$ the maximal domain of the solutions of (5)–(7) with initial conditions (8). At $s = 0$, we have $(1{+}b)z_0\theta'(0){+}2 = 0$. We know from Lemma 2.2 that $\theta'(s) \neq 0$ unless that $\alpha$ is a straight line. On the other hand, for any solution of (12), the function $1 + b\cos\theta(s)$ can not vanish, since on the contrary, we would have $\cos\theta(s) = -1/b$ and (12) gives $(-b^2 - 1)/b = 0$: contradiction.

As $\theta'(0) = -2/(z_0(1+b))$, we have the subcases $b+1>0$ and $b+1<0$ depending on the sign of $\theta'(0)$. In both settings, let $\theta_1 = \lim_{s\to\bar{s}}\theta(s)$.

(1) Subcase $b>-1$. Then $\theta(s)$ is a decreasing function. If $\cos\theta(s)=0$, then (12) yields $z(s)\theta'(s)-b=0$. Thus, if $b>0$ then $\cos\theta(s)\neq 0$ and $-\pi/2 < \theta(s) < \pi/2$.

  (a) Let $b>0$. For $s>0$, $z'(s)=\sin\theta(s)<0$, that is, $z(s)$ is decreasing. If $\bar{s}=\infty$, then $\lim_{s\to\infty} z'(s)=0$. But (5) yields $\theta_1=0$: contradiction. Therefore $\bar{s}<\infty$. If $z(s)\to z(\bar{s})>0$, then $\theta'(s)\to-\infty$ as $s\to\bar{s}$. Using (12), $\lim_{s\to\bar{s}} 1+b\cos\theta(s)=0$, which it is a contradiction because $1+b\cos\theta(s)\geq 1$. As conclusion, $z(\bar{s})=0$. From (12), we obtain that in the contact point between $\alpha$ and the line $L$, both curves make an angle $\theta_1$ such that $2\cos\theta_1 - b\sin^2\theta_1=0$. On the other hand, $x'(s)=\cos\theta(s)\neq 0$ and so $\alpha$ is a vertical graph on some bounded interval of $L$. In the particular case that $b=0$, that is, $S$ is a minimal surface, then $\alpha$ is a curve is a vertical graph that meets $L$ at right angle: this was done in [5].

  (b) Consider $-1<b<0$. If $\cos\theta(s)\neq 0$, then $\bar{s}<\infty$ as above. We follow the same reasoning. If $z(\bar{s})>0$ then $\theta'(s)\to-\infty$. As $2\cos\theta(s) - b\sin^2\theta(s)\to -(1+b^2)/b>0$, then $\lim_{s\to\bar{s}}\big(1+b\cos\theta(s)\big)=0$. We use the L'Hôpital rule,

$$\lim_{s\to\bar{s}}(1+b\cos\theta(s))\theta'(s) = \frac{\lim_{s\to\bar{s}} b\theta'(s)\sin\theta(s)}{\lim_{s\to\bar{s}}\frac{\theta''(s)}{\theta'(s)^2}} > 0,$$

  since $\theta''(s)<0$ near $\bar{s}$ in contradiction with (12). Thus $z(\bar{s})=0$ and using (12) again, we obtain $2\cos\theta_1 - b\sin^2\theta_1=0$: contradiction, since $b<0$.

  As conclusion, the function $\theta(s)$ reaches the value $-\pi/2$ at some point. However, $\theta(s)>-\pi$ using (12) again. In the case that $\bar{s}=\infty$, then $z'(s)\to 0$, that is, $\theta_1=-\pi$. But equation (12) and the fact that $\theta'(s)\to 0$ gives a contradiction. Therefore, $\bar{s}<\infty$. We prove that $z(\bar{s})=0$. On the contrary, that is, $z(\bar{s})>0$, then $\theta'(\bar{s})=-\infty$ and for $\theta(s)<-\pi/2$, we would have

$$\theta'(s) \geq \frac{-2\cos\theta(s)+b\sin^2\theta(s)}{z(s)} \geq \frac{-2\cos\theta(s)+b\sin^2\theta(s)}{z_0},$$

  and $\theta'(s)$ would be bounded. This contradiction proves the claim on $z(\bar{s})$. The angle $\theta_1$ which $\alpha$ intersects $L$ satisfies $2\cos\theta_1 - b\sin^2\theta_1 = 0$

by using (12) again. As $x'(s)$ vanishes at some point, then $\alpha$ is not a vertical graph.

(2) Subcase $b < -1$. Now $\theta(s)$ is a strictly increasing function in its domain. From (12) and as $b < 0$, the function $\cos\theta(s)$ can not vanish. Thus, $\theta(s)$ is bounded by $-\pi/2 < \theta(s) < \pi/2$. For $s > 0$, $z(s)$ is an increasing function. Assuming that $\bar{s} = \infty$, we will arrive to a contradiction. Since $\theta$ is bounded and $\bar{s} = \infty$, then

$$\lim_{s\to\infty} \theta'(s) = 0$$

and $\theta''(s)$ is negative near $s = \infty$. A differentiation of (12) leads to

$$(1 + b\cos\theta(s))\big[z(s)\theta''(s) - \sin\theta(s)\theta'(s)\big] - b\sin\theta(s)z(s)\theta'(s)^2 = 0. \tag{13}$$

It follows from (13) that $\theta''(s)$ is positive near to $s = 0$. Then $\theta''(s)$ must vanish at some number $s$. However, if $\theta''(s) = 0$ it follows from (13) and the fact that $1 + b\cos\theta(s) < 0$ that for this number $s$, we have

$$0 = -\sin\theta(s)\theta'(s)\big[(1 + b\cos\theta(s)) + bz(s)\theta'(s)\big] > 0.$$

This contradiction proves that $\bar{s} < \infty$. This means that the surface $S$ is not complete. Moreover $\lim_{s\to\bar{s}} \theta'(s) = \infty$ and from (5), the function $z(s)$ is bounded. Letting $s \to \bar{s}$ in (12) we obtain that $1 + b\cos\theta_1 = 0$. On the other hand, $x'(s) \neq 0$ and so, $\alpha$ is a vertical graph over a bounded interval of $L$, and as $z''(s) = \theta'(s)\cos\theta(s) > 0$, then $\alpha$ is a convex vertical graph. $\square$

For the subcase $-1 < b < 0$, we assert the following:

**Corollary 4.2.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ that satisfies* (5) *and $2H + bK = 0$ with $-1 < b < 0$. Assume that $\theta(0) = 0$. Then there exists a value $b_0 \in (-1, 0)$ such that (See Figure 6):*

(1) *If $b_0 < b < 0$, $\alpha$ is an embedded curve whose asymptotic boundary is two points. The corresponding surface is embedded and the asymptotic boundary is two tangent circles at infinity.*

(2) *If $b = b_0$, $\alpha$ is embedded with only one point as asymptotic boundary. The surface that generates is embedded with a circle as asymptotic boundary.*

(3) *If $-1 < b < b_0$, $\alpha$ is not embedded with two points as asymptotic boundary. The surface that generates is not embedded and it has two tangent circles at infinity as asymptotic boundary.*

Proof. The reasoning is similar as in the case that $H$ is constant [5, chap. 3] and we omit the proof. The contact point between $\alpha$ and $L$ is $\alpha(\bar{s}) = \lim_{s\to\bar{s}} \alpha(s)$. Because the symmetry of $\alpha$ (Lemma 2.1), the curve $\alpha$ is or not embedded depending on $x(\bar{s})$ is $> 0$ or $< 0$, respectively. The contact angle $\theta_1$ is given by $\cos\theta_1 = \cos\theta_1(b) = (-1+\sqrt{1+b^2}\,)/b$, which is increasing on $b$, with $\cos\theta_1(-1) = 1-\sqrt{2}$ and $\cos\theta_1(0) = 0$. $\square$

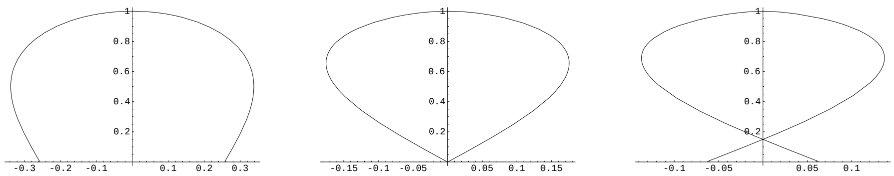



*Figure 6.* Parabolic surfaces that satisfy $2H + bK = 0$, subcase $-1 < b < 0$. Here $z_0 = 1$ and $\theta(0) = 0$. Case(a), $b \in (b_0, 0)$; case (b), $b = b_0$; case (c), $b \in (-1, b_0)$.

**4.2. Case $aH + bK = 1$.** In this subsection we consider parabolic $LW$-surfaces that satisfy equation (7). After a change of variables, we can suppose that $c = 1$. We discard the trivial $LW$-surfaces, that is, that $a$ or $b$ are 0. We also exclude that situation that for some $s$, $\theta'(s) = 0$, and then $\alpha$ would be a straight line by Lemma 2.2. Due to the variety of case that appear and in order to show the arguments, we only study the case that $a$ is positive (if $a < 0$, we could reverse the orientation of the surface but the Weingarten relation is now $(-a)H + bK = -1$, which is different that the one that we consider here).

With these assumptions, Equation (7) writes

$$\Big(\frac{a}{2} + b\cos\theta(s)\Big)\, z(s)\theta'(s) + a\cos\theta(s) - b\sin^2\theta(s) = 1 \tag{14}$$

or

$$\theta'(s) = 2\,\frac{1 - a\cos\theta(s) + b\sin^2\theta(s)}{z(s)(a + 2b\cos\theta(s))}. \tag{15}$$

A first case occurs when $a^2 + 4b^2 + 4b = 0$, where we obtain explicit solutions of (14).

**Theorem 4.3.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ that satisfies (5)–(7) with $\theta(0) = 0$. Assume $aH + bK = 1$ with $b \neq 0$, and that $a^2 + 4b^2 + 4b = 0$. Then $\alpha$ describes an open of a Euclidean circle in the $xz$-plane $P$.*

PROOF. Equation (14) reduces into

$$-2bz(s)\theta'(s) = a + 2b\cos\theta(s).$$

By differentiation with respect to $s$, we obtain $z(s)\theta''(s) = 0$, that is, $\theta'(s)$ is a constant function. Since $\theta'(s)$ describes the Euclidean curvature of $\alpha$, we conclude that $\alpha$ parametrizes a Euclidean circle in the $xz$-plane $P$. This circle may not to be completely included in the halfspace $\mathbb{R}^3_+$. □

We point out that if $\partial_\infty\alpha = \emptyset$, that is, $\alpha$ does not intersect $L$, then the resulting surfaces is one of the type obtained in Theorem 2.3.

From now, and along this section, we assume $a^2+4b^2+4b \neq 0$. Let us denote $(-\bar{s}, \bar{s})$ the maximal domain of the solutions. At $s = 0$, we have

$$\theta'(0) = \frac{2}{z_0}\frac{1-a}{a+2b}.$$

We study the different settings that appear depending on the sign of $\theta'(0)$. Moreover, and as $\theta'(s) \neq 0$, the numerator can not vanish, that is,

$$a + 2b\cos\theta(s) \neq \pm\sqrt{a^2+4b^2+4b} \tag{16}$$

and it has the same sign that the sign at $s = 0$, that is, the sign of $1 - a$. By the monotonicity of $\theta(s)$, let $\theta_1 = \lim_{s\to\bar{s}}\theta(s)$. We distinguish three cases, namely, $0 < a < 1$, $a = 1$ and $a > 1$. This will be done in the next three Theorems 4.4, 4.5 and 4.7 respectively.

**Theorem 4.4** (Case $0 < a < 1$). *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ that satisfies* (5)–(7) *with $\theta(0) = 0$. Assume $aH + bK = 1$ with $b \neq 0$, and that $0 < a < 1$.*

(1) *If $a + 2b < 0$, $\alpha$ has one maximum and $\alpha$ is a concave (non-entire) vertical graph. If $b < -(1+\sqrt{1-a^2})/2$, $\alpha$ is complete and intersects $L$ at an angle $\theta_1$ such that $2\cos\theta_1 - b\sin^2\theta_1 = 0$. The asymptotic boundary of $\alpha$ is two points. If $-(1+\sqrt{1-a^2})/2 < b < -a/2$, then $\alpha$ is not complete. See Figure 7 cases* (a) *and* (b) *respectively.*

(2) *Assume $a+2b > 0$. If $a-2b > 0$, then $\alpha$ is complete, it is invariant by a group of translations in the $x$-direction, $\alpha$ has self-intersections and it presents one maximum and one minimum in each period. The velocity $\alpha'$ turns around the origin. If $a - 2b \leq 0$, then $\alpha$ is not complete. Moreover it is not a vertical graph with a minimum. See Figure 8, cases* (a) *and* (b) *respectively.*

Proof. The second derivative of $\theta''(s)$ is

$$-\theta'(s)\sin\theta(s)\left[b\theta'(s)+\left(\frac{a}{2}+b\cos\theta(s)\right)\right]+\left(\frac{a}{2}+b\cos\theta(s)\right)z(s)\theta''(s)=0. \quad (17)$$

(1) Case $a+2b<0$. Then $\theta'(0)<0$ and $\theta(s)$ is strictly decreasing. If $\cos\theta(s)=0$ at some point $s$, then (14) gives $(a/2)z(s)\theta'(s)-b-1=0$. Thus, if $b\geq -1$, $\cos\theta(s)\neq 0$ and $-\pi/2<\theta(s)<\pi/2$. In the case that $b<-1$ and as $a+2b\cos\theta(s)<0$, it follows from (14) that $a\cos\theta(s)-b\sin^2\theta(s)-1<0$ for any value of $s$. In particular, we have $\cos\theta(s)\neq 0$ for any $s$ again. This proves that $x'(s)=\cos\theta(s)\neq 0$ and so, $\alpha$ is a vertical graph on $L$. This graph is concave since $z''(s)=\theta'(s)\cos\theta(s)<0$. Moreover, this implies that $\bar{s}<\infty$ since on the contrary, and as $z(s)$ is decreasing with $z(s)>0$, we would have $z'(s)\to 0$, that is, $\theta(s)\to 0$: contradiction.

For $s>0$, $z'(s)=\sin\theta(s)<0$ and $z(s)$ is strictly decreasing. Set $z(s)\to z(\bar{s})\geq 0$. The two roots of $4b^2+4b+a^2=0$ on $b$ are $b=-\frac{1}{2}(1\pm\sqrt{1-a^2}\,)$. Moreover, and from $a+2b<0$, we have

$$-\frac{1}{2}(1+\sqrt{1-a^2}\,)<\frac{-a}{2}<-\frac{1}{2}(1-\sqrt{1-a^2}\,).$$

(a) Subcase $b<-(1+\sqrt{1-a^2}\,)/2$. With this assumption, $a^2+4b^2+4b>0$. From (16) and the fact that $a<1$, we obtain

$$a+2b\cos\theta(s)<-\sqrt{a^2+4b^2+4b}. \quad (18)$$

If $z(\bar{s})>0$, then $\lim_{s\to\bar{s}}\theta'(s)=-\infty$. In particular and from (15), $a+2b\cos\theta(\bar{s})=0$: contradiction with (18). Thus, $z(\bar{s})=0$ and $\alpha$ intersects $L$ with an angle $\theta_1$ satisfying $a\cos\theta_1-b\sin^2\theta_1-1=0$.

(b) Subcase $-(1+\sqrt{1-a^2}\,)/2<b<-a/2$. Now $a^2+4b^2+4b<0$. The function $1-a\cos\theta(s)+b\sin^2\theta(s)$ is strictly decreasing and its value at $\bar{s}$ satisfies $\cos\theta(s)>-a/2b$. Thus

$$1-a\cos\theta(s)+b\sin^2\theta(s)\geq\frac{a^2+4b^2+4b}{4b}>0. \quad (19)$$

Assume $z(\bar{s})=0$. Then (19) and (15) imply that $\theta'(\bar{s})=-\infty$. On the other hand, using (17) and (15), we have

$$\frac{\theta''(s)}{\theta'(s)^2}=\frac{b\sin\theta(s)}{z(s)\left(\frac{a}{2}+b\cos\theta(s)\right)}+\frac{\sin\theta(s)\left(\frac{a}{2}+b\cos\theta(s)\right)}{1-a\cos\theta(s)+b\sin^2\theta(s)}.$$

From this equation and as $\sin\theta(\bar{s}) \neq 0$, we conclude

$$\lim_{s\to\bar{s}} \frac{\theta''(s)}{\theta'(s)^2} = -\infty.$$

On the other hand, using L'Hôpital rule, we have

$$\lim_{s\to\bar{s}} z(s)\theta'(s) = \lim_{s\to\bar{s}} -\frac{\sin\theta(s)}{\frac{\theta''(s)}{\theta'(s)^2}} = 0.$$

As the numerator in (15) is bounded from below for a positive number, see (19), we obtain a contradiction by letting $s \to \bar{s}$. Thus, $z(\bar{s}) > 0$. This means that $\lim_{s\to\bar{s}} \theta'(s) = -\infty$ and from (15), that

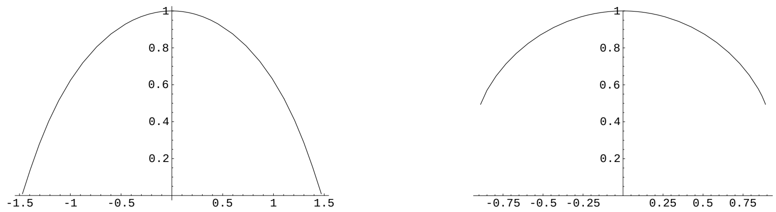
$$\lim_{s\to\bar{s}} \left(\frac{a}{2} + b\cos\theta(s)\right) = 0.$$

(2) Case $a+2b > 0$. Then $\theta'(0) > 0$ and $\theta(s)$ is strictly increasing. We distinguish two possibilities:

(a) Subcase $a - 2b > 0$. We prove that $\theta(s)$ reaches the value $\pi$. On the contrary, $\theta(s) < \pi$ and $z(s)$ is an increasing function. The hypothesis $a - 2b > 0$ together $a + 2b > 0$ implies that $a + 2b\cos\theta(s) \geq \delta > 0$ for some number $\delta$. From (15), $\theta'(s)$ is bounded and then $\bar{s} = \infty$. In particular, $\lim_{s\to\infty} \theta'(s) = 0$. As both $a - 2b$ and $a + 2b$ are positive numbers, the function $b\theta'(s) + (a + 2b\cos\theta(s))$ is positive near $\bar{s} = \infty$. Then using (17), $\theta''(s)$ is positive for a certain value of $s$ big enough, which it is impossible. As conclusion, $\theta(s)$ reaches the value $\pi$ at some $s = s_0$. By Lemma 2.1, $\alpha$ is symmetric with respect to the line $x = x(s_0)$ and the velocity vector of $\alpha$ rotates until to the initial position. This means that $\alpha$ is invariant by a group of horizontal translation.

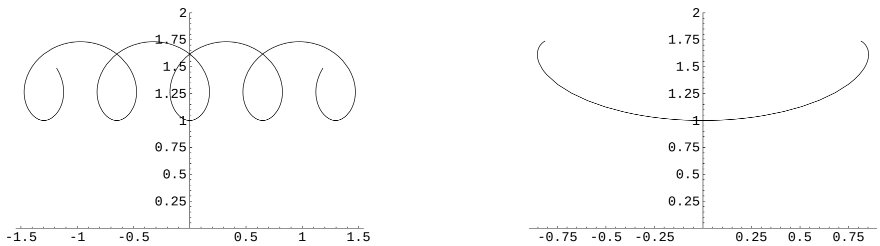
(b) Subcase $a-2b \leq 0$. As $\theta'(s) > 0$, Equation (15) says that $\cos\theta(s) \neq -1$, and so, $\theta(s)$ is bounded by $-\pi < \theta(s) < \pi$. As in the above subcase, if $\bar{s} = \infty$, then $\theta'(s) \to 0$, and this is a contradiction. Then $\bar{s} < \infty$ and $\lim_{s\to\bar{s}} \theta'(s) = \infty$. In particular, $\cos\theta(\bar{s}) = -a/(2b)$ and $\theta(s)$ reaches the value $\pi/2$. □

**Theorem 4.5** (Case $a = 1$)**.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ that satisfies* (5)–(7) *with $\theta(0) = 0$. Assume $H + bK = 1$ with $b \neq 0$. Then $\alpha$ is a horocycle and $S$ is a horosphere.*




(a) (b)

*Figure 7.* The generating curve of a parabolic surface with $aH{+}bK{=}1$, with $0 < a < 1$ and $a + 2b < 0$. Here $z_0 = 1$, $\theta(0) = 0$ and $a = 0.5$. In the case (a), $b = -1$ and in the case (b), $b = -0.8$.




(a) (b)

*Figure 8.* The generating curve of a parabolic surface with $aH{+}bK{=}1$, with $0 < a < 1$ and $a + 2b > 0$. Here $z_0 = 1$, $\theta(0) = 0$ and $a = 0.5$. In the case (a), $b = -0.2$ and in the case (b), $b = 0.3$.

Proof. If $a = 1$ and since $\theta'(0) = 0$, $\alpha$ is a horizontal straight-line. $\square$

*Remark 4.6.* As we pointed out at the beginning of this section, the value of $a$ in the Weingarten relation (7) is assumed positive. For example, related with Theorem 4.5, and in the case that $a = -1$, there are other parabolic surfaces satisfying $-H{+}bK = 1$ that are not horospheres. Exactly, non-embedded surfaces appear and whose generating curve is asymptotic to $\partial_\infty \mathbb{H}^3$, in a similar way than in the case of constant mean curvature ($b = 0$): see [5], [13].

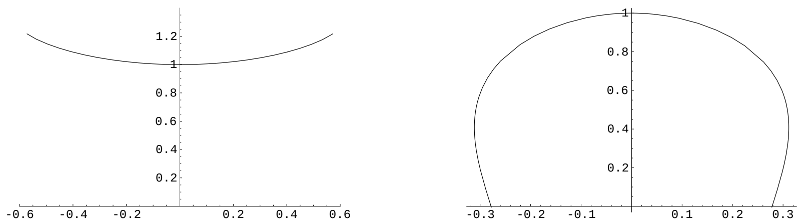
**Theorem 4.7** (Case $a > 1$)**.** *Let $\alpha(s) = (x(s), 0, z(s))$ be the generating curve of a parabolic surface $S$ in hyperbolic space $\mathbb{H}^3$ that satisfies* (5)–(7) *with $\theta(0) = 0$. Assume $aH + bK = 1$ with $a > 1$ and that $b \neq 0$.*

(1) *If $a{+}2b < 0$, $\alpha$ is not complete. Moreover, $\alpha$ is a convex (non-entire) vertical with a minimum. See Figure 9 case* (a)*. The surface is not complete.*

(2) *If* $a + 2b > 0$, *then* $\alpha$ *is complete with a maximum. Moreover, it intersects* $L$ *with an angle* $\theta_1$ *such that* $1 - a\cos\theta_1 + b\sin^2\theta_1 = 0$. *The asymptotic boundary of* $\alpha$ *is two points. See Figure 9, case* (b).

Proof. (1) Case $a + 2b < 0$. Now, the function $a + 2b\cos\theta(s) < 0$, in particular, $\cos\theta(s) \neq 0$. This means $-\pi/2 < \theta(s) < \pi/2$ and $\alpha$ is a vertical graph. Moreover, $\cos\theta(s) > -a/2b$, that is, $\theta_1 < \pi/2$. As $\theta'(0) > 0$, then $\theta(s)$ is a strictly increasing function and the same occurs for $z(s)$ for $s > 0$. We claim that $\bar{s} < \infty$. Assuming the contrary and as $\theta(s) < \theta_1$, we have that $\lim_{s\to\infty}\theta'(s) = 0$. From (17), $\theta''(s) > 0$ in a neighborhood of $\infty$, which it is impossible. As conclusion, $\bar{s} < \infty$. Then $\lim_{s\to\bar{s}}\theta'(s) = \infty$. Since $z(s)$ is defined in a bounded interval and its derivative is bounded, then $a/2 + b\cos\theta(s) \to 0$ as $s \to \bar{s}$.

(2) Case $a + 2b > 0$. Now $\theta'(s) < 0$, $\theta(s)$ is a decreasing function and $a + 2b\cos\theta(s) > 0$. From (14), if $a - 2b > 0$, then $\cos\theta(s) \neq -1$ for any $s$ and if $a - 2b \leq 0$, then $\cos\theta(s) \neq 0$. As conclusion, $\theta(s)$ is a bounded function. We prove that $\bar{s} < \infty$. On the contrary and since $z(s) > 0$, both functions $z'(s)$ and $\theta'(s)$ go to 0 as $s \to \infty$. This means that $\theta_1 = -\pi$ and from (14) and letting $s \to \infty$ we conclude that $a = 1$. This contradiction proves that $\bar{s} < \infty$. We have two possibilities: either $\theta'(\bar{s}) = -\infty$ or $z(\bar{s}) = 0$. The first case is impossible because $\phi(s) := a + 2b\cos\theta(s) \geq \delta > 0$, for some number $\delta > 0$: if $a - 2b > 0$, then $\phi(s) \geq a - 2b$ and if $a - 2b \leq 0$, then $\cos\theta(s) > 0$ and so, $\phi(s) \geq a$. As conclusion, $z(\bar{s}) = 0$, that is, $\alpha$ intersects the line $L$. $\square$




(a) (b)

*Figure 9.* The generating curve of a parabolic surface with $aH+bK=1$, with $a > 1$. Here $z_0 = 1$ and $\theta(0) = 0$. In the case (a), $a = 2$ and $b = -2$ and in the case (b), $a = 4$ and $b = -1.5$.

*Remark 4.8.* We compare the results obtained in this Section with the ones in Corollary 3.6. Here, we have showed the existence of parabolic surfaces in $\mathbb{H}^3$

that satisfy the relation $aH + bK = c$ with the property that either i) are not complete or ii) are embedded, complete but not vertical graphs.

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