Publ. Math. Debrecen
74/1-2 (2009), 81–100

Banach–Stone theorems for vector-valued little Lipschitz functions

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Abstract. We give a complete description of linear biseparating maps between spaces of vector-valued little Lipschitz functions. We apply these results to study some automatic continuity properties of such maps and the onto linear isometries between such spaces.

1. Introduction

A linear map between spaces of Banach-valued functions is said to be *disjointness preserving* if it maps any pair of functions with disjoint cozero sets to functions with disjoint cozero sets. Given a nonempty set X and a Banach space E, let us recall that the *cozero set* of a function $f : X \to E$ is the set of all $x \in X$ for which $f(x) \neq 0$. Disjointness preserving maps have been investigated for many years under different names as *d-homomorphisms* [1], *Lamperti operators* [8] or *separating maps* [9]. We direct the reader to the memoir [2] of Abramovich and Kitover and the survey [17] of Narici and Beckenstein for a complete information on the matter in different contexts.

There exists an extensive literature about linear separating maps in spaces of vector-valued continuous functions. Hernández, Beckenstein and Narici [14] were the first to study the Banach–Stone type representation and the automatic

Mathematics Subject Classification: 46E40; 47B33, 47B38.
Key words and phrases: biseparating map, Banach–Stone theorem, disjointness preserving operator, vector-valued Lipschitz map, automatic continuity.
The first author was partially supported by Junta de Andalucía projects P06-FQM-1215 and P06-FQM-1438, and MEC project MTM-2006-04837.

continuity of linear separating maps between spaces $\mathcal{C}(X, E)$ of continuous functions from a Tihonov space X into a Banach space E, equipped with the compact-open topology. Furthermore, they applied the results obtained about separating maps to the study of surjective linear isometries between spaces $\mathcal{C}(X, E)$ for X compact with the supremum norm. Their method of proof is based essentially in the concept of support point of a separating map. It is worthwhile noting that Gau, Jeang and Wong [13] introduced a new point of view in the treatment of this subject.

The same problems were tackled later in several spaces of vector-valued continuous functions. For example, Araujo investigated into separating maps between some types of spaces $\mathcal{C}(X, E)$ ([3], [4], [5] and, with Jarosz, [7]), spaces of vector-valued uniformly continuous functions on complete metric spaces [3], and spaces of vector-valued differentiable functions on open subsets of $\mathbb{R}^n$ [6]. On the other hand, Dubarbie researched into linear separating maps between spaces of vector-valued absolutely continuous functions on compact subsets of the real line [11].

The aim of this paper is to study the Banach–Stone type representation and the automatic continuity of linear bijective maps that preserve disjointness in both directions (also called *biseparating*) between spaces of vector-valued little Lipschitz functions, as well as its application to research into surjective linear isometries on such spaces.

Given a metric space (X, d), a real number α in $(0, 1]$ and a nonzero Banach space E over the field $\mathbb{K}$ of real or complex numbers, we denote by $\mathrm{Lip}_\alpha(X, E)$ the Banach space of all functions $f : X \to E$ such that

$$p_\alpha(f) = \sup\{\|f(x) - f(y)\|/d(x, y)^\alpha : x, y \in X,\ x \neq y\} < +\infty$$

and

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in X\} < +\infty,$$

endowed with any of the natural norms:

$$\|f\| = \max\{p_\alpha(f), \|f\|_\infty\} \quad \text{or} \quad \|f\| = p_\alpha(f) + \|f\|_\infty.$$

The *little Lipschitz space* $\mathrm{lip}_\alpha(X, E)$ is then defined to be the closed subspace of $\mathrm{Lip}_\alpha(X, E)$ formed by all those functions f satisfying the property:

$$\forall \epsilon > 0,\ \exists \delta > 0 : x, y \in X,\ 0 < d(x, y) < \delta \Rightarrow \|f(x) - f(y)\|/d(x, y)^\alpha < \epsilon.$$

As far as we know, Johnson [16] was the first to consider the spaces $\mathrm{Lip}_\alpha(X, E)$ and $\mathrm{lip}_\alpha(X, E)$ for $E \neq \mathbb{K}$. Since then, these spaces have been the subject of considerable study (see, for example, [15] and its references).

We proceed to describe briefly the content of the paper. In Section 2, we introduce some notation and gather some basic material on Lipschitz functions and separating maps. Section 3 is devoted to the *support map* $h : Y \to X$ of a linear biseparating map $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ when X and Y are compact and $0 < \alpha < 1$. These assumptions are made to assure that $\mathrm{lip}_\alpha(X, E)$ separates closed sets of X. In general, this is no true for $\alpha = 1$. For example, if $X = [0, 1]$ with the usual distance, $\mathrm{lip}(X, \mathbb{K})$ contains only constant functions. We keep these assumptions and this notation in all what follows. This support map plays a key role in our reasonings which follow closely those of [4, 14].

Section 4 contains our main results. In Theorem 4.1, we prove that every linear biseparating map $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ is a weighted composition operator in the form

$$Tf(y) = \widehat{T}y(f(h(y))) \quad (f \in \mathrm{lip}_\alpha(X, E),\ y \in Y),$$

where h is a homeomorphism from Y onto X and $\widehat{T}y$ is a linear bijection from E onto F for each $y \in Y$. In Theorem 4.3, this representation is improved if T is in addition continuous. In this case, h becomes a Lipschitz homeomorphism and $\widehat{T}$ a continuous map from Y onto the set of all continuous linear bijections from E onto F with the *strong operator topology*. See Section 2 for terminology not explained. Previously, in Theorem 4.2, we characterize the continuity of T by means of the continuity of the functions $\widehat{T}y$.

Section 5 focuses on the automatic continuity of T. We state that a linear biseparating map $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ is continuous whenever E is finite-dimensional (Corollary 5.1) or X has no isolated points (Theorem 5.5).

In Section 6, we give an application to surjective linear isometries between spaces $\mathrm{lip}_\alpha(X, E)$. In the spirit of [14, Theorem 4.1], we prove that a surjective linear isometry $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ is biseparating if and only if there exist a Lipschitz homeomorphism $h : Y \to X$ and a continuous map $\widehat{T}$ from Y onto the set of all surjective linear isometries from E onto F with the strong operator topology such that $Tf(y) = \widehat{T}y(f(h(y)))$ for every $f \in \mathrm{lip}_\alpha(X, E)$ and all $y \in Y$.

The authors thank the referees for valuable comments to clarify and improve this manuscript.

2. Preliminaries and notation

We shall use d to denote the distance in any metric space, but this causes no confusion in the present paper. The following concepts are well-known.

Definition 2.1. Let X and Y be metric spaces. A map $f : X \to Y$ is said to be Lipschitz if there exists a constant $a \geq 0$ such that $d(f(x), f(y)) \leq a\, d(x, y)$ for all $x, y \in X$. If f is bijective and both f and f^{-1} are Lipschitz, then it is said that f is a Lipschitz homeomorphism.

The next lines are devoted to the Lipschitz function spaces which will be the subject of our study.

Let (X, d) be a metric space. For each $\alpha \in (0, 1]$, the map $d^\alpha : X \times X \to \mathbb{R}_0^+$ defined by $d^\alpha(x, y) = d(x, y)^\alpha$ is also a metric on X. As usual, $\mathbb{K}$ denotes the field of real or complex numbers. Let E be a nonzero Banach space over $\mathbb{K}$. We denote by $\mathrm{Lip}_\alpha(X, E)$ the Banach space of all functions $f : X \to E$ which are bounded and Lipschitz from (X, d^α) into $(E, \|\cdot\|)$, equipped with the supremum norm $\|f\| = \max\{p_\alpha(f), \|f\|_\infty\}$ or the sum norm $\|f\| = p_\alpha(f) + \|f\|_\infty$, where

$$p_\alpha(f) = \sup\{\|f(x) - f(y)\|/d^\alpha(x, y) : x, y \in X,\ x \neq y\}$$

and

$$\|f\|_\infty = \sup\{\|f(x)\| : x \in X\}.$$

The space $\mathrm{lip}_\alpha(X, E)$ is defined as the closed subspace of $\mathrm{Lip}_\alpha(X, E)$ consisting of those functions f with the property that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < d(x, y) < \delta$ implies $\|f(x) - f(y)\|/d(x, y)^\alpha < \epsilon$. To simplify the notation, we shall write $\mathrm{Lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(X)$ when $E = \mathbb{K}$, and $\mathrm{Lip}(X, E)$ and $\mathrm{lip}(X, E)$ in the case $\alpha = 1$.

We next introduce the definition of separating map in the context of vector-valued little Lipschitz functions.

Definition 2.2. Given $f \in \mathrm{lip}_\alpha(X, E)$, we define the cozero set of f as

$$\mathrm{coz}(f) = \{x \in X : f(x) \neq 0\}.$$

Moreover, $\mathrm{supp}(f)$ denotes the closure of $\mathrm{coz}(f)$ in X.

Definition 2.3. A map $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ is said to be separating if $\mathrm{coz}(Tf) \cap \mathrm{coz}(Tg) = \emptyset$ whenever $f, g \in \mathrm{lip}_\alpha(X, E)$ satisfy $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$. Moreover, T is said to be biseparating if it is bijective and both T and T^{-1} are separating.

Equivalently, $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ is separating if $\|Tf(y)\|\, \|Tg(y)\| = 0$ for all $y \in Y$ whenever $f, g \in \mathrm{lip}_\alpha(X, E)$ satisfy $\|f(x)\|\|g(x)\| = 0$ for all $x \in X$.

Throughout this paper, for any $x \in X$ and $\delta > 0$, $B(x, \delta)$ will stand for the open ball $\{z \in X : d(z, x) < \delta\}$ and $B_c(x, \delta)$ for the corresponding closed ball.

Moreover, 1_X and I_X will denote the function constantly 1 on X and the identity function on X, respectively. As usual, $\mathrm{diam}(X)$ will stand for the diameter of X. For each $x \in X$, we shall denote by δ_x the evaluation map from $\mathrm{lip}_\alpha(X,E)$ to E defined by $\delta_x(f) = f(x)$.

We now present two families of functions in $\mathrm{Lip}(X)$, defined by means of the metric d, which will be very useful along the paper.

Proposition 2.1. *For each $x \in X$ and $\delta > 0$, $g_{x,\delta} : X \to [0,1]$ given by*

$$g_{x,\delta}(z) = \max\{1 - (1/\delta)d(z, B(x,\delta)), 0\}$$

belongs to $\mathrm{Lip}(X)$ *with $g_{x,\delta}(z) = 1$ for all $z \in B(x,\delta)$ and* $\mathrm{coz}(g_{x,\delta}) \subset B(x, 2\delta)$.

Proposition 2.2. *For any $x \in X$ and $\delta > 0$, $h_{x,\delta} : X \to [0,1]$ defined by*

$$h_{x,\delta}(z) = \max\{1 - (1/\delta)d(z,x), 0\},$$

is in $\mathrm{Lip}(X)$ *with $h_{x,\delta}(x) = 1$ and* $\mathrm{coz}(h_{x,\delta}) = B(x,\delta)$.

In order to prove our results, we shall need the following basic facts about vector-valued Lipschitz functions.

Remark 2.1. Let X be a compact metric space, E a nonzero Banach space and $\alpha \in (0,1)$.

(1) $\mathrm{Lip}(X,E)$ is contained in $\mathrm{lip}_\alpha(X,E)$, and both spaces are $\mathrm{Lip}(X)$-modules, that is, if $f \in \mathrm{Lip}(X)$ and $g \in \mathrm{Lip}(X,E)$ ($g \in \mathrm{lip}_\alpha(X,E)$), then $fg \in \mathrm{Lip}(X,E)$ (respectively, $fg \in \mathrm{lip}_\alpha(X,E)$).

(2) For any $f \in \mathrm{lip}_\alpha(X)$ and $e \in E\backslash\{0\}$, the map $f \otimes e : X \to E$ given by $(f \otimes e)(x) = f(x)e$ for all $x \in X$, belongs to $\mathrm{lip}_\alpha(X,E)$, $\|f \otimes e\| = \|f\|\|e\|$ and $\mathrm{coz}(f \otimes e) = \mathrm{coz}(f)$. In particular, for any $x \in X$, $\delta > 0$ and $e \in E\backslash\{0\}$, $h_{x,\delta} \otimes e \in \mathrm{lip}_\alpha(X,E)$, $(h_{x,\delta} \otimes e)(x) = e$ and $\mathrm{coz}(h_{x,\delta} \otimes e) = B(x,\delta)$.

Given nonzero Banach spaces E and F, we shall denote by $L(E,F)$, $B(E,F)$, $L^{-1}(E,F)$, $B^{-1}(E,F)$ and $I(E,F)$ the set of all linear maps (respectively, continuous linear maps, linear bijections, continuous linear bijections, surjective linear isometries) from E to F. We shall consider that $B(E,F)$, $B^{-1}(E,F)$ and $I(E,F)$ are endowed with the strong operator topology.

Definition 2.4. Let E and F be Banach spaces. The strong operator topology in $B(E,F)$ is the coarsest topology such that the mappings $T \hookrightarrow Te$ from $B(E,F)$ to F are continuous for every $e \in E$.

For comprehensive accounts on the strong operator topology and on Lipschitz functions, the reader is referred, for instance, to [12] and [18], respectively.

3. The support map of a linear biseparating map

From now on, unless otherwise stated, we shall suppose that $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ is a linear biseparating map, where X and Y are compact metric spaces, E and F are nonzero Banach spaces and $\alpha \in (0, 1)$.

Our main purpose in this section is to show that each linear biseparating map $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ gives rise to a homeomorphism $h : Y \to X$ (the support map of T) in such a way that $Tf(y) = 0$ provided that $f(h(y)) = 0$. From this fact we shall deduce easily a representation of Banach–Stone type for T in the following section.

The next concept has been a standard tool in the study of linear separating maps.

Definition 3.1. Let $y \in Y$. A point $x \in X$ is said to be a T-support point of y if for every $\delta > 0$, there exists $f \in \mathrm{lip}_\alpha(X, E)$ such that $\mathrm{coz}(f) \subset B(x, \delta)$ and $y \in \mathrm{coz}(Tf)$.

We begin by proving the uniqueness of the T-support point of each point $y \in Y$. To this end, we need the next lemma.

Lemma 3.1. *If $f, g \in \mathrm{lip}_\alpha(X, E)$ and $\mathrm{coz}(f) \subset \mathrm{coz}(g)$, then $\mathrm{coz}(Tf) \subset \mathrm{supp}(Tg)$.*

Proof. Let $y \in \mathrm{coz}(Tf)$ and suppose $y \notin \mathrm{supp}(Tg)$. Then there exists $\delta > 0$ such that $\mathrm{coz}(Tg) \subset Y \backslash B(y, \delta)$. By Remark 2.1, we can take a function $f_0 \in \mathrm{lip}_\alpha(Y, F)$ such that $\mathrm{coz}(f_0) = B(y, \delta)$. Since $\mathrm{coz}(f_0) \cap \mathrm{coz}(Tg) = \emptyset$ and T^{-1} is separating, we have $\mathrm{coz}(T^{-1}f_0) \cap \mathrm{coz}(g) = \emptyset$. This implies that $\mathrm{coz}(T^{-1}f_0) \cap \mathrm{coz}(f) = \emptyset$ and, since T is separating, it follows that $\mathrm{coz}(f_0) \cap \mathrm{coz}(Tf) = \emptyset$, but $y \in \mathrm{coz}(f_0) \cap \mathrm{coz}(Tf)$, a contradiction. $\square$

Lemma 3.2. *For each $y \in Y$, there exists a unique T-support point of y in X.*

Proof. Fix $y \in Y$ and define $F_y = \{f \in \mathrm{lip}_\alpha(Y, F) : y \in \mathrm{coz}(f)\}$. Clearly, $F_y \neq \emptyset$. Put $K_y = \bigcap_{f \in F_y} \mathrm{supp}(T^{-1}f)$. It is easy to see that every T-support point of y is in K_y.

We claim that K_y is nonempty. To show this, it is enough to see that the family $\{\mathrm{supp}(T^{-1}f) : f \in F_y\}$ has the finite intersection property, since all of its members are closed subsets of the compact space X. Pick $f_1, \ldots, f_n \in F_y$. Since $\bigcap_{i=1}^n \mathrm{coz}(f_i)$ is an open neighborhood of y in Y, we have $B(y, \delta) \subset \bigcap_{i=1}^n \mathrm{coz}(f_i)$ for some $\delta > 0$. By Remark 2.1, there exists $f \in \mathrm{lip}_\alpha(Y, F)$ such that $\mathrm{coz}(f) = B(y, \delta)$ and thus $\mathrm{coz}(f) \subset \bigcap_{i=1}^n \mathrm{coz}(f_i)$. Using Lemma 3.1 applied to T^{-1}, it follows that

$\mathrm{coz}(T^{-1}f) \subset \bigcap_{i=1}^n \mathrm{supp}(T^{-1}f_i)$. Notice that $\mathrm{coz}(T^{-1}f) \neq \emptyset$ since $f \neq 0$ and T^{-1} is an injective linear map. Hence $\bigcap_{i=1}^n \mathrm{supp}(T^{-1}f_i)$ is nonempty and this proves our claim.

We show that K_y is a singleton. Suppose that x and w are distinct points of K_y and let $\delta = d(x, w) > 0$. Take $f = h_{y,\delta} \otimes u \in \mathrm{lip}_\alpha(Y, F)$ for some vector $u \in F \backslash \{0\}$. By Remark 2.1, $\mathrm{coz}(f) = B(y, \delta)$. Since $f \in F_y$, we see that $x, w \in \mathrm{supp}(T^{-1}f)$. Consider $g_{x,\delta/4} \in \mathrm{Lip}(X)$ (see Proposition 2.1). A trivial verification shows that $\mathrm{coz}(g_{x,\delta/4}) \subset X\backslash B(w, \delta/4)$ and $\mathrm{coz}(1_X - g_{x,\delta/4}) \subset X\backslash B(x, \delta/4)$. Since $y \in \mathrm{coz}(f)$ and $f = T(g_{x,\delta/4}T^{-1}f) + T((1_X - g_{x,\delta/4})T^{-1}f)$, we deduce that $y \in \mathrm{coz}(T(g_{x,\delta/4}T^{-1}f))$ or $y \in \mathrm{coz}(T((1_X - g_{x,\delta/4})T^{-1}f))$. Observe that $1_X T^{-1}f,\ g_{x,\delta/4}T^{-1}f \in \mathrm{lip}_\alpha(X, E)$ by Remark 2.1.

If $y \in \mathrm{coz}(T(g_{x,\delta/4}T^{-1}f))$, then $T(g_{x,\delta/4}T^{-1}f) \in F_y$, and since $w \in K_y$, we deduce that $w \in \mathrm{supp}(g_{x,\delta/4}T^{-1}f)$, but $\mathrm{coz}(g_{x,\delta/4}) \subset X\backslash B(w, \delta/4)$, which gives a contradiction.

Reasoning similarly, if $y \in \mathrm{coz}(T((1_X - g_{x,\delta/4})T^{-1}f))$, then $x \in \mathrm{supp}((1_X - g_{x,\delta/4})T^{-1}f)$ since $T((1_X - g_{x,\delta/4})T^{-1}f) \in F_y$ and $x \in K_y$, but this contradicts $\mathrm{coz}(1_X - g_{x,\delta/4}) \subset X\backslash B(x, \delta/4)$. This proves that K_y has a unique point x.

Finally, we check that x is a T-support point of y. Let $\delta > 0$. We must find a function $f \in \mathrm{lip}_\alpha(X, E)$ such that $\mathrm{coz}(f) \subset B(x, \delta)$ and $y \in \mathrm{coz}(Tf)$. Take $g = h_{y,\delta} \otimes u \in \mathrm{Lip}(Y, F)$ for some $u \in F\backslash\{0\}$, and $g_{x,\delta/2} \in \mathrm{Lip}(X)$. Recall that $\mathrm{coz}(1_X - g_{x,\delta/2}) \subset X\backslash B(x, \delta/2)$ and $\mathrm{coz}(g_{x,\delta/2}) \subset B(x, \delta)$. It is clear that $T^{-1}g = g_{x,\delta/2}T^{-1}g + (1_X - g_{x,\delta/2})T^{-1}g$ and therefore $g = T(g_{x,\delta/2}T^{-1}g) + T((1_X - g_{x,\delta/2})T^{-1}g)$. Since $y \in \mathrm{coz}(g)$, we deduce that $y \in \mathrm{coz}(T(g_{x,\delta/2}T^{-1}g))$ or $y \in \mathrm{coz}(T((1_X - g_{x,\delta/2})T^{-1}g))$. If $y \in \mathrm{coz}(T((1_X - g_{x,\delta/2})T^{-1}g))$, as $x \in K_y$ we have $x \in \mathrm{supp}((1_X - g_{x,\delta/2})T^{-1}g)$, but $\mathrm{coz}(1_X - g_{x,\delta/2}) \subset X\backslash B(x, \delta/2)$. This contradiction forces that $y \in \mathrm{coz}(T(g_{x,\delta/2}T^{-1}g))$. The function $f = g_{x,\delta/2}T^{-1}g$ satisfies then the required conditions. □

The preceding lemma allows us to consider the following map.

Definition 3.2. For each $y \in Y$, $h(y)$ is the unique T-support point of y in X. Following the literature, we shall say that $h : Y \to X$ is the support map of T.

We now prove that if f vanishes in a neighborhood of $h(y)$, then $Tf(y) = 0$.

Lemma 3.3. *If $y \in Y$, $f \in \mathrm{lip}_\alpha(X, E)$ and $h(y) \notin \mathrm{supp}(f)$, then $y \notin \mathrm{supp}(Tf)$ and, in particular, $Tf(y) = 0$.*

Proof. Take $\delta > 0$ such that $\mathrm{coz}(f) \subset X\backslash B(h(y), \delta)$. Since $h(y)$ is the T-support point of y, we can take a function $g \in \mathrm{lip}_\alpha(X, E)$ such that $\mathrm{coz}(g) \subset B(h(y), \delta)$ and $y \in \mathrm{coz}(Tg)$. It follows that $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$, hence $\mathrm{coz}(Tf) \cap$

$\mathrm{coz}(Tg) = \emptyset$ since T is separating. Moreover, $\mathrm{coz}(Tg)$ is an open neighborhood of y and therefore $B(y, \epsilon) \subset \mathrm{coz}(Tg)$ for some $\epsilon > 0$. Hence $\mathrm{coz}(Tf) \cap B(y, \epsilon) = \emptyset$ and so $y \notin \mathrm{supp}(Tf)$. □

Among the properties of h, we point out the following:

Lemma 3.4. *The support map $h : Y \to X$ of T is a homeomorphism, and its inverse h^{-1} is the support map of T^{-1}.*

Proof. If we prove that $h : Y \to X$ is a continuous bijection, then the first assertion follows since Y is compact and X is Hausdorff. To prove the continuity of h, let $y \in Y$ and let $\{y_n\}$ be a sequence in Y converging to y. By the compactness of X, we can suppose, taking a subsequence if necessary, that $\{h(y_n)\}$ converges to a point $x \in X$. Assume $x \neq h(y)$. Put $\delta = d(x, h(y)) > 0$ and since $h(y)$ is the T-support of y, we can take a function $f \in \mathrm{lip}_\alpha(X, E)$ such that $\mathrm{coz}(f) \subset B(h(y), \delta/3)$ and $y \in \mathrm{coz}(Tf)$. On the other hand, since $\{h(y_n)\}$ converges to x, there exists $m \in \mathbb{N}$ such that $h(y_n) \in B(x, \delta/3)$ for all $n \geq m$. Fix $n \geq m$. It is easy to see that $B(h(y_n), \delta/3) \subset X \backslash B_c(h(y), \delta/3)$. Then $\mathrm{coz}(f) \subset X \backslash B(h(y_n), \delta/3)$, which yields $h(y_n) \notin \mathrm{supp}(f)$, and therefore $Tf(y_n) = 0$ by Lemma 3.3. Hence $Tf(y_n) = 0$ for all $n \geq m$. By the continuity of Tf, it follows that $Tf(y) = 0$ which is a contradiction. Hence h is continuous.

Next, we see that $h : Y \to X$ is surjective. Suppose there exists a point $x \in X \backslash h(Y)$. Then $d(x, h(Y)) > 0$ since $h(Y)$ is closed in X. Put $\delta = d(x, h(Y))$ and consider $f = h_{x,\delta/2} \otimes e$ in $\mathrm{Lip}(X, E)$ for some $e \in E \backslash \{0\}$. Since $\mathrm{coz}(f) = B(x, \delta/2)$, it is clear that $\mathrm{supp}(f) \subset B_c(x, \delta/2)$ and therefore $h(y) \notin \mathrm{supp}(f)$ for all $y \in Y$. By Lemma 3.3, it follows that Tf is the zero map on Y. Hence f is the zero map on X, but as $f(x) = e$, we arrive at a contradiction.

Finally, we show that $h : Y \to X$ is injective. Since $T^{-1} : \mathrm{lip}_\alpha(Y, F) \to \mathrm{lip}_\alpha(X, E)$ is a linear biseparating map, we can consider the support map $k : X \to Y$ of T^{-1}. We claim that $k \circ h = I_Y$. Suppose, contrary to our claim, that there exists some $y_0 \in Y$ such that $(k \circ h)(y_0) = y_1 \neq y_0$. Let $\delta = d(y_1, y_0)/2 > 0$. Since y_1 is the T^{-1}-support point of $h(y_0)$, there exists g in $\mathrm{lip}_\alpha(Y, F)$ such that $\mathrm{coz}(g) \subset B(y_1, \delta)$ and $h(y_0) \in \mathrm{coz}(T^{-1}g)$. Consider $f_0 = h_{y_0,\delta} \otimes u$ in $\mathrm{Lip}(Y, F)$ for some $u \in F \backslash \{0\}$. Since $\mathrm{coz}(f_0) = B(y_0, \delta)$, it is easily seen that $\mathrm{coz}(f_0) \cap \mathrm{coz}(g) = \emptyset$ and then $\mathrm{coz}(T^{-1}f_0) \cap \mathrm{coz}(T^{-1}g) = \emptyset$ since T^{-1} is separating. Moreover, $\mathrm{coz}(T^{-1}g)$ is an open neighborhood of $h(y_0)$, and taking into account that $h(y_0)$ is the T-support point of y_0, there exists $f \in \mathrm{lip}_\alpha(X, E)$ such that $\mathrm{coz}(f) \subset \mathrm{coz}(T^{-1}g)$ and $y_0 \in \mathrm{coz}(Tf)$. By Lemma 3.1, $\mathrm{coz}(Tf) \subset \mathrm{supp}(g)$. Since $\mathrm{coz}(f_0) \cap \mathrm{coz}(g) = \emptyset$, we have $\mathrm{coz}(f_0) \cap \mathrm{coz}(Tf) = \emptyset$, but $y_0 \in \mathrm{coz}(f_0) \cap \mathrm{coz}(Tf)$, a contradiction. This proves our claim and therefore h is injective. Since h is also surjective, we conclude that h is bijective and $h^{-1} = k$. $\square$

We have seen in Lemma 3.3 that if f vanishes in a neighborhood of $h(y)$, then $Tf(y) = 0$. Our principal objective in this section is to show that it holds also if $f(h(y)) = 0$. We shall use the next result in the proof of this fact.

Lemma 3.5. *Let $x \in X$ and let $\{x_n\}$ be a sequence of distinct points in $X\backslash\{x\}$ converging to x. Then, for each $f \in \mathrm{lip}_\alpha(X, E)$ with $f(x) = 0$, there exist a subsequence $\{x_{n_k}\}$, a sequence $\{s_k\}$ in $(0, 1)$ with $s_{k+1} < (1/2)s_k$ for all $k \in \mathbb{N}$, and a sequence $\{h_k\}$ in $\mathrm{lip}_\alpha(X, E)$ such that $h_k = f$ on $B(x_{n_k}, s_k)$, $\mathrm{coz}(h_k) \subset B(x_{n_k}, 2s_k)$ and $\|h_k\| \leq 10/k^2$ for all $k \in \mathbb{N}$. Moreover, $\{B(x_{n_k}, 2s_k)\}$ is a sequence of pairwise disjoint open balls of X.*

Proof. Fix $f \in \mathrm{lip}_\alpha(X, E)$ with $f(x) = 0$. Then there exists $\delta_1 \in (0, 1)$ such that $d(z, w) \leq \delta_1$ implies $\|f(z) - f(w)\| \leq d(z, w)^\alpha$. Since $\{x_n\}$ converges to x, there exists $n_1 \in \mathbb{N}$ such that $d(x, x_{n_1}) < (3/4)\delta_1$. Let $s_1 = (1/6)d(x, x_{n_1})$, consider $g_1 = g_{x_{n_1}, s_1}$ and $h_1 = fg_1$. Notice that g_1 belongs to $\mathrm{Lip}(X)$, $0 \leq g_1 \leq 1$, $\mathrm{coz}(g_1) \subset B(x_{n_1}, 2s_1)$ and $p_1(g_1) \leq 1/s_1$. We have $h_1 \in \mathrm{lip}_\alpha(X, E)$, $h_1 = f$ on $B(x_{n_1}, s_1)$ and $\mathrm{coz}(h_1) \subset B(x_{n_1}, 2s_1)$.

Next, we prove that $\|h_1(w)\| < (8s_1)^\alpha$ for all $w \in X$, and since $8s_1 < \delta_1 < 1$, we have $\|h_1\|_\infty \leq 1$. Observe that $h_1 = 0$ on $X\backslash B(x_{n_1}, 2s_1)$, and if $w \in B(x_{n_1}, 2s_1)$, then $d(x, w) \leq d(x, x_{n_1}) + d(x_{n_1}, w) < 8s_1 < \delta_1$, hence

$$\|f(w)\| = \|f(x) - f(w)\| \leq d(x, w)^\alpha < (8s_1)^\alpha,$$

and thus $\|h_1(w)\| < (8s_1)^\alpha$.

We use the estimation of f above to obtain $p_\alpha(h_1) \leq 1 + 4 \cdot 2^\alpha$. Let $z, w \in X$. An easy calculation yields

$$\|h_1(z) - h_1(w)\| \leq g_1(z)\|f(z) - f(w)\| + \|f(w)\||g_1(z) - g_1(w)|.$$

For any $z, w \in B(x_{n_1}, 2s_1)$, we have $d(z, w) < 4s_1 < \delta_1$, hence

$$\begin{aligned}\|h_1(z) - h_1(w)\| &\leq d(z, w)^\alpha + (8s_1)^\alpha(1/s_1)d(z, w)\\ &\leq d(z, w)^\alpha + (8s_1)^\alpha(1/s_1)(4s_1)^{1-\alpha}d(z, w)^\alpha = (1 + 4 \cdot 2^\alpha)d(z, w)^\alpha.\end{aligned}$$

On the other hand, for $z \in X\backslash B(x_{n_1}, 2s_1)$ and $w \in B(x_{n_1}, 2s_1)$, we have

$$\|h_1(z) - h_1(w)\| \leq \|f(w)\||g_1(z) - g_1(w)|.$$

If $d(z,w) \leq 4s_1$, it follows that

$$\|h_1(z) - h_1(w)\| \leq (8s_1)^\alpha (1/s_1) d(z,w) \leq 4 \cdot 2^\alpha d(z,w)^\alpha,$$

and if $d(z,w) > 4s_1$,

$$\|h_1(z) - h_1(w)\| \leq \|f(w)\| \leq (8s_1)^\alpha \leq 2^\alpha d(z,w)^\alpha.$$

Finally, $h_1(z) = 0 = h_1(w)$ if $z, w \in X \backslash B(x_{n_1}, 2s_1)$. We have so proved that

$$\|h_1(z) - h_1(w)\| \leq (1 + 4 \cdot 2^\alpha) d(z,w)^\alpha, \quad \forall z, w \in X.$$

From the above we deduce that

$$\|h_1\| \leq \|h_1\|_\infty + p_\alpha(h_1) \leq 1 + (1 + 4 \cdot 2^\alpha) \leq 10.$$

Similarly, choose $\delta_2 \in (0, 4s_1)$ such that $d(z,w) \leq \delta_2$ implies $\|f(z) - f(w)\| \leq (1/2^2) d(z,w)^\alpha$. Then take $n_2 \in \mathbb{N}$ with $n_2 > n_1$ such that $d(x, x_{n_2}) < (3/4)\delta_2$. Let $s_2 = (1/6) d(x, x_{n_2})$, put $g_2 = g_{x_{n_2}, s_2}$ and $h_2 = f g_2$. Clearly, $h_2 \in \mathrm{lip}_\alpha(X, E)$ with $h_2 = f$ on $B(x_{n_2}, s_2)$ and $\mathrm{coz}(h_2) \subset B(x_{n_2}, 2s_2)$.

We next estimate $\|h_2\|_\infty$ and $p_\alpha(h_2)$. For any $w \in B(x_{n_2}, 2s_2)$, it is clear that $d(x,w) < 8s_2 < \delta_2$, hence

$$\|f(w)\| = \|f(x) - f(w)\| \leq (1/2^2) d(x,w)^\alpha < (1/2^2)(8s_2)^\alpha < 1/2^2,$$

and thus $\|h_2(w)\| < 1/2^2$. Since $h_2 = 0$ on $X \backslash B(x_{n_2}, 2s_2)$, we deduce that $\|h_2\|_\infty \leq 1/2^2$. We now claim that

$$\|h_2(z) - h_2(w)\| \leq ((1 + 4 \cdot 2^\alpha)/2^2) d(z,w)^\alpha, \quad \forall z, w \in X.$$

Indeed, we have

$$\|h_2(z) - h_2(w)\| \leq g_2(z) \|f(z) - f(w)\| + \|f(w)\| |g_2(z) - g_2(w)|.$$

Given $z, w \in B(x_{n_2}, 2s_2)$, we have $d(z,w) < 4s_2 < \delta_2$. It follows that

$$\|h_2(z) - h_2(w)\| \leq (1/2^2) d(z,w)^\alpha + (1/2^2)(8s_2)^\alpha (1/s_2) d(z,w)$$
$$\leq (1/2^2) d(z,w)^\alpha + (1/2^2)(8s_2)^\alpha (1/s_2)(4s_2)^{1-\alpha} d(z,w)^\alpha = ((1 + 4 \cdot 2^\alpha)/2^2) d(z,w)^\alpha.$$

For $z \in X \backslash B(x_{n_2}, 2s_2)$ and $w \in B(x_{n_2}, 2s_2)$, we have $g_2(z) = 0$, hence

$$\|h_2(z) - h_2(w)\| \leq \|f(w)\| |g_2(z) - g_2(w)|.$$

If $d(z,w) \le 4s_2$, we obtain that

$$\|h_2(z) - h_2(w)\| \le (1/2^2)(8s_2)^\alpha(1/s_2)d(z,w) \le 2^\alpha d(z,w)^\alpha,$$

and if $d(z,w) > 4s_2$,

$$\|h_2(z) - h_2(w)\| \le \|f(w)\| \le (1/2^2)(8s_2)^\alpha \le (1/2^2)2^\alpha d(z,w)^\alpha.$$

Moreover, $h_2(z) = 0 = h_2(w)$ for any $z, w \in X \backslash B(x_{n_2}, 2s_2)$. We have thus

$$\|h_2(z) - h_2(w)\| \le ((1 + 4 \cdot 2^\alpha)/2^2)d(z,w)^\alpha, \quad \forall z, w \in X.$$

From what has been proved, we see that

$$\|h_2\| \le (1/2^2) + (1 + 4 \cdot 2^\alpha)/2^2 \le 10/2^2.$$

Following the process, we can inductively construct a subsequence $\{x_{n_k}\}$, a sequence $\{s_k\}$ in $(0,1)$ defined by $s_k = (1/6)d(x, x_{n_k})$ for all $k \in \mathbb{N}$, a sequence $\{B(x_{n_k}, 2s_k)\}$ of open balls of X, and a sequence $\{h_k\}$ in $\mathrm{lip}_\alpha(X,E)$, defined by $h_k = fg_k$ with $g_k = g_{x_{n_k}, s_k}$, satisfying that $h_k = f$ on $B(x_{n_k}, s_k)$, $\mathrm{coz}(h_k) \subset B(x_{n_k}, 2s_k)$, and $\|h_k\| \le 10/k^2$ since $\|h_k\|_\infty \le 1/k^2$ and $p_\alpha(h_k) \le (1 + 4 \cdot 2^\alpha)/k^2$ for all $k \in \mathbb{N}$.

Finally, we prove that $B(x_{n_k}, 2s_k) \cap B(x_{n_l}, 2s_l) = \emptyset$ if $k \ne l$. Take $k < l$ and assume there is $z \in X$ such that $d(z, x_{n_k}) < (1/3)d(x, x_{n_k})$ and $d(z, x_{n_l}) < (1/3)d(x, x_{n_l})$. Observe that, for any $i \in \mathbb{N}$, we have

$$d(x, x_{n_{i+1}}) < (3/4)\delta_{i+1} < 3s_i = (1/2)d(x, x_{n_i})$$

(hence $s_{i+1} < (1/2)s_i$). It follows that $d(x, x_{n_l}) < (1/2^{l-k})d(x, x_{n_k})$. Then

$$\begin{aligned} d(x_{n_k}, x_{n_l}) &\ge d(x, x_{n_k}) - d(x, x_{n_l}) \\ &> d(x, x_{n_k}) - (1/2^{l-k})d(x, x_{n_k}) = (1 - 1/2^{l-k})d(x, x_{n_k}), \end{aligned}$$

but, on the other hand,

$$\begin{aligned} d(x_{n_k}, x_{n_l}) &\le d(x_{n_k}, z) + d(z, x_{n_l}) \\ &< (1/3)d(x, x_{n_k}) + (1/3)(1/2^{l-k})d(x, x_{n_k}) \le (1 - 1/2^{l-k})d(x, x_{n_k}). \end{aligned}$$

Thus, we arrive at a contradiction. □

We have now gathered all the ingredients for the proof of the following result.

Lemma 3.6. *If* $y \in Y$, $f \in \mathrm{lip}_\alpha(X, E)$ *and* $f(h(y)) = 0$, *then* $Tf(y) = 0$.

Proof. Let us suppose that $f(h(y)) = 0$, but $Tf(y) \neq 0$. If $h(y)$ is an isolated point of X, since $f(h(y)) = 0$, then $h(y) \notin \mathrm{supp}(f)$, hence $Tf(y) = 0$ by Lemma 3.3, which is a contradiction. Assume now that $h(y)$ is not isolated in X. Put $h(y) = x$ and let $\{x_n\}$ be a sequence of distinct points in $X \backslash \{x\}$ converging to x. Since h is surjective, for each $n \in \mathbb{N}$ there exists $y_n \in Y$ such that $h(y_n) = x_n$. By the continuity of h^{-1}, $\{y_n\}$ converges to y, and we may suppose, taking a subsequence if necessary, that $\|Tf(y_n)\| > \|Tf(y)\|/2$ for all $n \in \mathbb{N}$.

By Lemma 3.5, there exist a subsequence $\{x_{n_k}\}$, a sequence $\{s_k\}$ in $(0, 1/2^3)$, a sequence $\{B(x_{n_k}, 2s_k)\}$ of pairwise disjoint open balls in X, and a sequence $\{h_k\}$ in $\mathrm{lip}_\alpha(X, E)$ such that $h_k = f$ on $B(x_{n_k}, s_k)$, $\mathrm{coz}(h_k) \subset B(x_{n_k}, 2s_k)$ and $\|h_k\| \leq 10/k^2$ for all $k \in \mathbb{N}$. As $\|k^{1/2}h_k\| \leq 10/k^{3/2}$ for all $k \in \mathbb{N}$, we can define a function g in $\mathrm{lip}_\alpha(X, E)$ by $g = \sum_{k=1}^{+\infty} k^{1/2}h_k$. For each $k \in \mathbb{N}$, it is immediate that $g = k^{1/2}f$ on $B(x_{n_k}, s_k)$ since the sets $\mathrm{coz}(h_k)$ are pairwise disjoint. Hence $Tg(y_{n_k}) = k^{1/2}Tf(y_{n_k})$ by Lemma 3.3, and thus $\|Tg(y_{n_k})\| = k^{1/2}\|Tf(y_{n_k})\| > k^{1/2}\|Tf(y)\|/2$. As a consequence, Tg is unbounded. Since this is not possible, we conclude that lemma holds. □

4. Banach–Stone type representation

We are now in position to establish a first representation of Banach–Stone type for linear biseparating maps between spaces $\mathrm{lip}_\alpha(X, E)$. First of all, we introduce the notion of associate map to a linear map.

Definition 4.1. Let $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ be a linear map. We call associate map of T to the map $\widehat{T} : Y \to L(E, F)$ defined by $\widehat{T}y(e) = T(1_X \otimes e)(y)$ for all $y \in Y$ and $e \in E$.

Theorem 4.1. *Let* $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ *be a linear biseparating map. Then there exist a map* $\widehat{T} : Y \to L^{-1}(E, F)$ *and a homeomorphism* $h : Y \to X$ *such that* $Tf(y) = \widehat{T}y(f(h(y)))$ *for all* $f \in \mathrm{lip}_\alpha(X, E)$ *and* $y \in Y$.

Proof. Let $\widehat{T}$ and h be the associate map of T and the support map of T, respectively. The map h is a homeomorphism from Y onto X by Lemma 3.4. To obtain the representation of T, take $f \in \mathrm{lip}_\alpha(X, E)$ and $y \in Y$. Define $g = f - (1_X \otimes f(h(y)))$. It is clear that $g \in \mathrm{lip}_\alpha(X, E)$ and $g(h(y)) = 0$. From Lemma 3.6 it follows that $Tg(y) = 0$, and so

$$Tf(y) = T(1_X \otimes f(h(y)))(y) = \widehat{T}y(f(h(y))).$$

The preceding representation can be applied to T^{-1}, since T^{-1} is also linear biseparating. Then we can write:

$$T^{-1}g(x) = \widehat{T^{-1}}x(g(k(x)), \quad \forall g \in \mathrm{lip}_\alpha(Y,F),\ \forall x \in X,$$

where the associate map $\widehat{T^{-1}} : X \to L(F,E)$ of T^{-1} comes given by

$$\widehat{T^{-1}}x(u) = T^{-1}(1_Y \otimes u)(x), \quad \forall x \in X,\ \forall u \in F,$$

and the support map $k : X \to Y$ of T^{-1} is a homeomorphism.

Next, we prove that $\widehat{T}y$ is a bijection from E onto F for each $y \in Y$. Fix $y \in Y$. To show that $\widehat{T}y \in L(E,F)$ is injective, let $e \in E$ and suppose $\widehat{T}y(e) = 0$, that is $T(1_X \otimes e)(y) = 0$. By the surjectivity of k, we have $y = k(x)$ for some $x \in X$. Then

$$e = (1_X \otimes e)(x) = T^{-1}(T(1_X \otimes e))(x) = \widehat{T^{-1}}x(T(1_X \otimes e)(y)) = \widehat{T^{-1}}x(0) = 0,$$

and, consequently, $\widehat{T}y$ is injective. To prove the surjectivity of $\widehat{T}y$, take $u \in F$. Since T is surjective, there exists $f \in \mathrm{lip}_\alpha(X,E)$ such that $Tf = 1_Y \otimes u$. Let $e = f(h(y)) \in E$. We get $\widehat{T}y(e) = \widehat{T}y(f(h(y))) = Tf(y) = u$, which is the desired conclusion. □

We next characterize the continuity of linear biseparating maps.

Theorem 4.2. *Let $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ be a linear biseparating map. The following statements are equivalent:*

(1) *T is continuous.*

(2) *$\widehat{T}y$ is continuous for all $y \in Y$.*

(3) *$\delta_y \circ T$ is continuous for all $y \in Y$.*

Proof. (1) $\Longrightarrow$ (2): Fix $y \in Y$. Since $\widehat{T}y$ is linear and

$$\|\widehat{T}y(e)\| \leq \|T(1_X \otimes e)\|_\infty \leq \|T(1_X \otimes e)\| \leq \|T\|\|1_X \otimes e\| = \|T\|\|e\|$$

for all $e \in E$, we see that $\widehat{T}y$ is continuous.

(2) $\Longrightarrow$ (3): Let $y \in Y$. Clearly, $\delta_y \circ T$ is linear and

$$\|(\delta_y \circ T)(f)\| = \|Tf(y)\| = \|\widehat{T}y(f(h(y)))\| \leq \|\widehat{T}y\|\|f\|_\infty \leq \|\widehat{T}y\|\|f\|$$

for all $f \in \mathrm{lip}_\alpha(X,E)$ by using Theorem 4.1. Then $\delta_y \circ T$ is continuous.

(3) $\Longrightarrow$ (1): We claim that T has closed graph. Indeed, let us suppose that

the sequences $\{f_n\}$ in $\mathrm{lip}_\alpha(X,E)$ and $\{Tf_n\}$ converge to $f \in \mathrm{lip}_\alpha(X,E)$ and $g \in \mathrm{lip}_\alpha(Y,F)$, respectively. Fix $y \in Y$. We have

$$\|Tf(y) - Tf_n(y)\| = \|(\delta_y \circ T)(f) - (\delta_y \circ T)(f_n)\| \to 0$$

by the continuity of $\delta_y \circ T$, and

$$\|Tf_n(y) - g(y)\| \leq \|Tf_n - g\|_\infty \leq \|Tf_n - g\| \to 0.$$

It follows that $Tf(y) = g(y)$. Hence $Tf = g$, and this proves our claim. Since T is a linear map between Banach spaces, the Closed Graph Theorem shows that T is continuous. $\square$

We can improve the representation of T if it is in addition continuous.

Theorem 4.3. *Let $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ be a continuous linear biseparating map. Then there exist a continuous map $\widehat{T} : Y \to B^{-1}(E,F)$ and a Lipschitz homeomorphism $h : Y \to X$ such that $Tf(y) = \widehat{T}y(f(h(y)))$ for all $f \in \mathrm{lip}_\alpha(X,E)$ and $y \in Y$.*

Proof. Let $\widehat{T}$ and h be as in Theorem 4.1. For each $y \in Y$, $\widehat{T}y \in L^{-1}(E,F)$ by Theorem 4.1, and since T is continuous, that $\widehat{T}y \in B^{-1}(E,F)$ follows from Theorem 4.2.

Taking into account that $B^{-1}(E,F)$ is endowed with the strong operator topology, to prove that $\widehat{T}$ is continuous it is enough to show that for every $e \in E$, the map $y \mapsto \widehat{T}y(e)$ from Y into F is continuous. Let $\{y_n\}$ be a sequence in Y which converges to $y \in Y$. Then the sequence $\{\widehat{T}y_n(e)\}$ converges to $\widehat{T}y(e)$. This is deduced immediately from the inequality:

$$\|\widehat{T}y(e) - \widehat{T}z(e)\| \leq p_\alpha(T(1_X \otimes e))d(y,z)^\alpha \leq \|T\|\|e\|d(y,z)^\alpha, \quad \forall z \in Y.$$

Clearly, it follows that the map $y \mapsto \|\widehat{T}y(e)\|$ from Y into $\mathbb{R}$ is continuous. Moreover, if $e \in E \backslash \{0\}$, we have $\widehat{T}y(e) \neq 0$ for all $y \in Y$ since $\widehat{T}y \in L^{-1}(E,F)$, and, in consequence, $\min\{\|\widehat{T}y(e)\| : y \in Y\} > 0$.

We next prove that h is Lipschitz. For any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, define $f_{y_1,y_2} : X \to \mathbb{R}$ by

$$f_{y_1,y_2}(z) = \max\{2d(h(y_1),z)^\alpha - d(h(y_1),h(y_2))^\alpha, 0\}.$$

It is clear that $f_{y_1,y_2} \in \mathrm{Lip}_\alpha(X)$ and $\|f_{y_1,y_2}\| \leq 2(1 + \mathrm{diam}(X)^\alpha)$. We claim that $f_{y_1,y_2} \in \mathrm{Lip}(X)$. To prove this, define $g_{y_1} \in \mathrm{Lip}(X)$ by $g_{y_1}(z) = d(h(y_1),z)$ and $h_{y_1,y_2} : [0, \mathrm{diam}(X)] \to \mathbb{R}$ by

$$h_{y_1,y_2}(t) = \max\{2t^\alpha - d(h(y_1),h(y_2))^\alpha, 0\}.$$

It is easily seen that h_{y_1,y_2} is differentiable at $[0, \mathrm{diam}(X)]\backslash\{t_0\}$ with bounded derivative by $2\alpha t_0^{\alpha-1}$, where $t_0 = (1/2)^{1/\alpha} d(h(y_1), h(y_2))$. Let $t, s \in [0, \mathrm{diam}(X)]$, $t \neq s$. If $t, s \in [0, t_0]$, we have $h_{y_1,y_2}(t) = h_{y_1,y_2}(s) = 0$. If $t, s \in (t_0, \mathrm{diam}(X)]$,

$$|h_{y_1,y_2}(t) - h_{y_1,y_2}(s)| = |h'_{y_1,y_2}(s_0)|\,|t-s| \leq 2\alpha t_0^{\alpha-1}|t-s|$$

where s_0 is between t and s by using the Mean Value Theorem. If $t \leq t_0 < s$ (or $s \leq t_0 < t$), it is easy to check that

$$|h_{y_1,y_2}(t) - h_{y_1,y_2}(s)| = |h_{y_1,y_2}(t_0) - h_{y_1,y_2}(s)| \leq 2\alpha t_0^{\alpha-1}|t_0 - s| \leq 2\alpha t_0^{\alpha-1}|t-s|.$$

It follows that $h_{y_1,y_2} \in \mathrm{Lip}([0, \mathrm{diam}(X)])$. Since $f_{y_1,y_2} = h_{y_1,y_2} \circ g_{y_1}$, our claim is proved.

Now, fix $e \in E$, $\|e\| = 1$. Clearly, $f_{y_1,y_2} \otimes e \in \mathrm{lip}_\alpha(X, E)$ and $\|f_{y_1,y_2} \otimes e\| \leq 2(1 + \mathrm{diam}(X)^\alpha)$ for distinct $y_1, y_2 \in Y$. Since T is linear and continuous, it follows that there exists a constant $c > 0$ such that

$$p_\alpha(T(f_{y_1,y_2} \otimes e)) \leq \|T(f_{y_1,y_2} \otimes e)\| \leq c, \quad \forall\, y_1, y_2 \in Y,\ y_1 \neq y_2.$$

As a consequence, for any $y_1, y_2 \in Y$ with $y_1 \neq y_2$, we have

$$\|T(f_{y_1,y_2} \otimes e)(y_1) - T(f_{y_1,y_2} \otimes e)(y_2)\| \leq c\, d(y_1, y_2)^\alpha.$$

Using Theorem 4.1 yields

$$\begin{aligned} T(f_{y_1,y_2} \otimes e)(y_1) &= \widehat{T}y_1((f_{y_1,y_2} \otimes e)(h(y_1))) = \widehat{T}y_1(0) = 0, \\ T(f_{y_1,y_2} \otimes e)(y_2) &= \widehat{T}y_2((f_{y_1,y_2} \otimes e)(h(y_2))) = d(h(y_1), h(y_2))^\alpha \widehat{T}y_2(e). \end{aligned}$$

Then we deduce that

$$d(h(y_1), h(y_2))^\alpha \|\widehat{T}y_2(e)\| \leq c\, d(y_1, y_2)^\alpha.$$

Putting $b = \min\{\|\widehat{T}y(e)\| : y \in Y\} > 0$ gives $d(h(y_1), h(y_2)) \leq (c/b)^{1/\alpha}\, d(y_1, y_2)$, and so h is Lipschitz. Furthermore, by Lemma 3.4, h^{-1} is the support map of the linear biseparating map T^{-1} and so h^{-1} is also Lipschitz by proved above. Hence h is a Lipschitz homeomorphism. □

5. Automatic continuity

In this section we shall study some automatic continuity properties of linear biseparating maps between spaces $\mathrm{lip}_\alpha(X, E)$. From Theorems 4.1 and 4.2 we deduce immediately the following result in this line.

Corollary 5.1. *Let $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ be a linear biseparating map and suppose that E is finite-dimensional. Then F is finite-dimensional with $\dim(E) = \dim(F)$ and T (and T^{-1}) is continuous.*

Next, we infer some results for scalar-valued little Lipschitz functions. Note that every linear biseparating map between spaces $\mathrm{lip}_\alpha(X)$ is continuous. We first derive from Theorem 4.3 the following:

Corollary 5.2. *A map $T : \mathrm{lip}_\alpha(X) \to \mathrm{lip}_\alpha(Y)$ is linear biseparating if and only if there exist a nonvanishing function $a \in \mathrm{lip}_\alpha(Y)$ and a Lipschitz homeomorphism $h : Y \to X$ such that $Tf = a \cdot (f \circ h)$ for every $f \in \mathrm{lip}_\alpha(X)$.*

Linear biseparating maps between spaces $\mathrm{lip}_\alpha(X)$ turn out to natural generalizations for algebra isomorphisms. We give the general form of such maps:

Corollary 5.3. *A map $T : \mathrm{lip}_\alpha(X) \to \mathrm{lip}_\alpha(Y)$ is an algebra isomorphism if and only if there exists a Lipschitz homeomorphism $h : Y \to X$ such that $Tf = f \circ h$ for every $f \in \mathrm{lip}_\alpha(X)$.*

We finish the applications in the scalar-valued case with the next result of Banach–Stone type.

Corollary 5.4. *Let X, Y be compact metric spaces and $\alpha \in (0,1)$. The following statements are equivalent:*

(1) *X and Y are Lipschitz homeomorphic.*

(2) *$\mathrm{lip}_\alpha(X)$ and $\mathrm{lip}_\alpha(Y)$ are algebra isomorphic.*

(3) *There is a linear biseparating map from $\mathrm{lip}_\alpha(X)$ onto $\mathrm{lip}_\alpha(Y)$.*

The proof of our next theorem requires a little more work.

Theorem 5.5. *Let $T : \mathrm{lip}_\alpha(X,E) \to \mathrm{lip}_\alpha(Y,F)$ be a linear biseparating map and suppose that X has no isolated points. Then T is continuous.*

Proof. We use the Closed Graph Theorem. Let $\{f_n\}$ be a sequence in $\mathrm{lip}_\alpha(X,E)$ such that $\|f_n\| \to 0$, and $\|Tf_n - k\| \to 0$ for some $k \in \mathrm{lip}_\alpha(Y,F)$. We claim that $k = 0$. Assume, contrary to what we claim, that $k(y) \neq 0$ for some $y \in Y$. Then there exists $\epsilon > 0$ such that $\|k(z)\| > (1/2)\|k(y)\|$ for all $z \in B(y,\epsilon)$. Moreover, since $\|Tf_n - k\| \to 0$, we can suppose, taking a subsequence if necessary, that $\|Tf_n(z) - k(z)\| < (1/4)\|k(y)\|$ for all $z \in B(y,\epsilon)$ and $n \in \mathbb{N}$. Hence $\|Tf_n(z)\| > (1/4)\|k(y)\|$ for all $z \in B(y,\epsilon)$ and $n \in \mathbb{N}$.

Let $h : Y \to X$ be the support map of T. By the continuity of h^{-1}, there exists $\delta > 0$ such that $h^{-1}(x) \in B(y,\epsilon)$ for all $x \in B(h(y),\delta)$. Since X has no isolated points, we can inductively construct a sequence $\{x_n\}$ of distinct points of $B(h(y),\delta)$ such that $0 < d(h(y),x_{n+1}) < (1/3)d(h(y),x_n)$ for all $n \in \mathbb{N}$. For each n, take $r_n = (1/4)d(x_n,h(y))$ and $g_n = g_{x_n,r_n} \in \mathrm{Lip}(X)$.

Notice that $\|g_n f_m\| \neq 0$ for all $n,m \in \mathbb{N}$. In contrary case, we would have $f_m(x) = 0$ for all $x \in B(x_n,r_n)$, hence $Tf_m(h^{-1}(x_n)) = 0$ by Lemma 3.3, but this is impossible since $\|Tf_m(h^{-1}(x_n))\| > (1/4)\|k(y)\| > 0$.

Since $\|f_n\| \to 0$, we can construct a subsequence $\{f_{\sigma(n)}\}$ such that

$$p_\alpha(f_{\sigma(n+1)}) + (1+p_\alpha(g_{n+1}))\|f_{\sigma(n+1)}\|_\infty \leq (1/(n+1)^3)\|g_n f_{\sigma(n)}\|$$

for all $n \in \mathbb{N}$, by induction. Indeed, let $\sigma(1) = 1$. There exists $\sigma(2) \in \mathbb{N}$ with $\sigma(2) > 1$ such that

$$p_\alpha(f_{\sigma(2)}) + (1+p_\alpha(g_2))\|f_{\sigma(2)}\|_\infty \leq (1/8)\|g_1 f_1\|,$$

since $\lim_{m\to+\infty}(p_\alpha(f_m) + (1+p_\alpha(g_2))\|f_m\|_\infty) = 0$. Suppose that for n, there are $\sigma(1),\dots,\sigma(n+1) \in \mathbb{N}$ with $\sigma(1) < \sigma(2) < \cdots < \sigma(n+1)$, such that

$$p_\alpha(f_{\sigma(j+1)}) + (1+p_\alpha(g_{j+1}))\|f_{\sigma(j+1)}\|_\infty \leq (1/(j+1)^3)\|g_j f_{\sigma(j)}\|$$

for $j = 1,\dots,n$. As $\lim_{m\to+\infty}(p_\alpha(f_m) + (1+p_\alpha(g_{n+2}))\|f_m\|_\infty) = 0$, there exists $\sigma(n+2) \in \mathbb{N}$ with $\sigma(n+2) > \sigma(n+1)$ such that

$$p_\alpha(f_{\sigma(n+2)}) + (1+p_\alpha(g_{n+2}))\|f_{\sigma(n+2)}\|_\infty \leq (1/(n+2)^3)\|g_{n+1} f_{\sigma(n+1)}\|.$$

On the other hand, we have $B(x_{n+m},2r_{n+m}) \subset B(h(y),2r_n)$ for all $n,m \in \mathbb{N}$, and from this it is deduced that the balls $B(x_n,2r_n)$ with n varying on $\mathbb{N}$ are pairwise disjoint.

For each n, consider $ng_n f_{\sigma(n)}$. It is clear that $ng_n f_{\sigma(n)} \in \mathrm{lip}_\alpha(X,E)$ with $\mathrm{coz}(ng_n f_{\sigma(n)} - nf_{\sigma(n)}) \subset X\backslash B(x_n,r_n)$. Therefore $x_n \notin \mathrm{supp}(ng_n f_{\sigma(n)} - nf_{\sigma(n)})$, and according to Lemma 3.3, we have $T(ng_n f_{\sigma(n)} - nf_{\sigma(n)})(h^{-1}(x_n)) = 0$. Then $T(ng_n f_{\sigma(n)})(h^{-1}(x_n)) = nTf_{\sigma(n)}(h^{-1}(x_n))$, and thus

$$\|T(ng_n f_{\sigma(n)})(h^{-1}(x_n))\| > (n/4)\|k(y)\|.$$

Since the balls $B(x_n,2r_n)$ with $n \in \mathbb{N}$ are pairwise disjoint and $\mathrm{coz}(ng_n f_{\sigma(n)}) \subset \mathrm{coz}(g_n) \subset B(x_n,2r_n)$, the sets $\mathrm{coz}(ng_n f_{\sigma(n)})$ are pairwise disjoint. Hence we can consider the function $g : X \to E$ defined by

$$g = \sum_{n=1}^{+\infty} ng_n f_{\sigma(n)}.$$

We next prove that $g \in \mathrm{lip}_\alpha(X,E)$. First, an easy proof by induction shows that $\|g_n f_{\sigma(n)}\| \leq \|g_1 f_1\|$ for all $n \in \mathbb{N}$. Then, for any $n \geq 2$, we have

$$\begin{aligned} np_\alpha(g_n f_{\sigma(n)}) &\leq n[\|g_n\|_\infty p(f_{\sigma(n)}) + p_\alpha(g_n)\|f_{\sigma(n)}\|_\infty] \\ &\leq n[p_\alpha(f_{\sigma(n)}) + (1 + p_\alpha(g_n))\|f_{\sigma(n)}\|_\infty] \\ &\leq n(1/n^3)\|g_{n-1} f_{\sigma(n-1)}\| \leq (1/n^2)\|g_1 f_1\| \end{aligned}$$

and

$$\begin{aligned} n\|g_n f_{\sigma(n)}\|_\infty \leq n\|f_{\sigma(n)}\|_\infty &\leq n[p_\alpha(f_{\sigma(n)}) + (1 + p_\alpha(g_n))\|f_{\sigma(n)}\|_\infty] \\ &\leq n(1/n^3)\|g_{n-1} f_{\sigma(n-1)}\| \leq (1/n^2)\|g_1 f_1\|. \end{aligned}$$

It follows that, for all $n \in \mathbb{N}$,

$$n\|g_n f_{\sigma(n)}\| \leq np_\alpha(g_n f_{\sigma(n)}) + n\|g_n f_{\sigma(n)}\|_\infty \leq (2/n^2)\|g_1 f_1\|,$$

and since $\mathrm{lip}_\alpha(X, E)$ is complete, we conclude that $g \in \mathrm{lip}_\alpha(X, E)$.

On the other hand, given $m \in \mathbb{N}$, we have $ng_n(x)f_{\sigma(n)}(x) = 0$ for all $x \in B(x_m, 2r_m)$ and $n \neq m$, hence $g(x) - mg_m(x)f_{\sigma(m)}(x) = 0$ for all $x \in B(x_m, 2r_m)$, and thus $x_m \notin \mathrm{supp}(g - mg_m f_{\sigma(m)})$. Hence $T(g - mg_m f_{\sigma(m)})(h^{-1}(x_m)) = 0$ by Lemma 3.3. Then $\|Tg(h^{-1}(x_m))\| = \|T(mg_m f_{\sigma(m)})(h^{-1}(x_m))\| > (m/4)\|k(y)\|$. Hence Tg is not bounded, but this is not possible. This finishes the proof. $\square$

6. An application to the study of isometries

We shall now apply the previous results to characterize biseparating linear isometries between spaces $\mathrm{lip}_\alpha(X, E)$. We adapt [10, Definition 8.2 (ii)] as follows.

Definition 6.1. A surjective linear isometry $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ is said to be a strong Banach-Stone map if there exist a continuous map $\widehat{T} : Y \to I(E, F)$ and a homeomorphism $h : Y \to X$ such that for every $f \in \mathrm{lip}_\alpha(X, E)$ and $y \in Y$, $Tf(y) = \widehat{T}y(f(h(y)))$. A strong Banach-Stone map $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ is said to be Lipschitz if h is a Lipschitz homeomorphism.

Theorem 6.1. *Let $T : \mathrm{lip}_\alpha(X, E) \to \mathrm{lip}_\alpha(Y, F)$ be a surjective linear isometry. Then T is a Lipschitz strong Banach-Stone map if and only if T is biseparating.*

Proof. Suppose that $Tf(y) = \widehat{T}y(f(h(y)))$ for all $f \in \mathrm{lip}_\alpha(X, E)$ and $y \in Y$, where $\widehat{T} : Y \to I(E, F)$ is continuous and $h : Y \to X$ is a Lipschitz homeomorphism.

Let $f, g \in \mathrm{lip}_\alpha(X, E)$ with $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$ and suppose that $y \in \mathrm{coz}(Tf) \cap \mathrm{coz}(Tg)$. Then $\widehat{T}y(f(h(y))) \neq 0 \neq \widehat{T}y(g(h(y)))$. Since $\widehat{T}y$ is linear, we have $f(h(y)) \neq 0 \neq g(h(y))$, a contradiction. Hence T is separating.

Assume that T^{-1} is not separating. Then there exist $f, g \in \mathrm{lip}_\alpha(Y, F)$ with $\mathrm{coz}(f) \cap \mathrm{coz}(g) = \emptyset$ such that $\mathrm{coz}(T^{-1}f) \cap \mathrm{coz}(T^{-1}g) \neq \emptyset$. Take x in $\mathrm{coz}(T^{-1}f) \cap \mathrm{coz}(T^{-1}g)$ and let y be in Y such that $h(y) = x$. We see at once that

$$f(y) = T(T^{-1}f)(y) = \widehat{T}y(T^{-1}f(h(y))) = \widehat{T}y(T^{-1}f(x)) \neq 0,$$

since $\widehat{T}y$ is linear injective. A similar argument yields $g(y) \neq 0$. Hence y belongs to $\mathrm{coz}(f) \cap \mathrm{coz}(g)$, a contradiction.

Conversely, assume that T is biseparating. Theorem 4.3 will then give us that T is a Lipschitz strong Banach–Stone map if we show that $\widehat{T}y : E \to F$ is an isometry for all $y \in Y$. To prove this, we first see that T sends nonvanishing functions of $\mathrm{lip}_\alpha(X, E)$ into nonvanishing functions of $\mathrm{lip}_\alpha(Y, F)$. Suppose there exists $f \in \mathrm{lip}_\alpha(X, E)$ such that $f(x) \neq 0$ for all $x \in X$, but $Tf(y) = 0$ for some $y \in Y$. Let $k : X \to Y$ be the support map of T^{-1}. By the surjectivity of k, we have $k(x) = y$ for some $x \in X$. Since $Tf(y) = 0$, applying Lemma 3.6 to k and T^{-1} we deduce that $f(x) = T^{-1}(Tf)(x) = 0$, which is impossible. Hence T maps nonvanishing functions into nonvanishing functions.

Let us suppose now that $\widehat{T}y$ is not an isometry for some $y \in Y$. Then there exists $e \in E$ with $\|e\| = 1$ for which $\|\widehat{T}y(e)\| < 1$. By the surjectivity of T, there is a function $f \in \mathrm{lip}_\alpha(X, E)$ such that $Tf = 1_Y \otimes T(1_X \otimes e)(y)$. Thus

$$\|f\|_\infty \leq \|f\| = \|Tf\| = \|T(1_X \otimes e)(y)\| = \|\widehat{T}y(e)\| < 1$$

and therefore $(1_X \otimes e) - f$ never vanishes on X. However, $T(1_X \otimes e)(y) = Tf(y)$, and thus $T((1_X \otimes e) - f)$ vanishes on Y. This contradiction completes the proof. $\square$

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(Received March 3, 2008; revised October 1, 2008)