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Sharp Jordan-type inequalities for Bessel functions

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Abstract. In this paper our aim is to establish some sharp Jordan and Kober type inequalities for Bessel and modified Bessel functions of the first kind by using the monotone form of l'Hospital's rule. Moreover, by using the classical Cauchy mean value theorem inductively we deduce new series expansions for the Bessel and modified Bessel functions. These results extend and improve many known results in the literature.

1. Introduction

The following inequality is known in literature as Jordan's inequality [14, p. 33]

$$\frac{2}{\pi} \leq \frac{\sin x}{x} < 1, \quad 0 < x \leq \frac{\pi}{2},$$

while the inequality

$$1 - \frac{2}{\pi}x \leq \cos x \leq 1 - \frac{x^2}{\pi}, \quad 0 < x \leq \frac{\pi}{2},$$

is known as Kober's inequality [13]. Jordan's inequality plays an important role in many areas of mathematics and it has been studied recently by several mathematicians in order to sharpen this basic analytic inequality, see [7], [8], [11], [15]–[19], [21]–[34]. For more details about Jordan's inequality on refinements, generalizations, extensions and applications we refer to the reader the interesting

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recent survey paper [18] where more than 40 papers, which contains various improved versions of the above Jordan inequality, were cited. It is worth mentioning that an interesting approach to deduce sharp Jordan-type inequalities was given recently in the papers [32], [33], [34]. This paper is motivated by these papers, mentioned above, and our aim in Section 2 is to extend to Bessel functions all of the results from [32], [33], [34] regarding Jordan and Kober type inequalities and the new power series expansion for the sine function. We note that our approach is similar to those given in the above mentioned papers, however an important step in the proofs is simplified. Moreover, our approach in each cases gives us larger intervals of validity. In Section 3 we present the hyperbolic counterpart of the results of Section 2, namely the sharp Jordan and Kober type inequalities for the modified Bessel functions. In Section 4 we formulate an equivalent form of the main results of this paper, namely Theorems 1 and 2, in order to point out the connection between the results of [15] and [32], [33], [34]. We note also that the approach in [15] is somewhat different to that given in this paper, and our range of validity in some cases is much better than in [15].

To achieve our goal first let us recall some basic facts. Suppose that $\nu > -1$ and consider the normalized Bessel function of the first kind $\mathcal{J}_\nu : \mathbb{R} \to (-\infty, 1]$, defined by

$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} J_\nu(x) = \sum_{n\geq 0} \frac{(-1/4)^n}{(\nu+1)_n n!} x^{2n},$$

where, as usual, $(\nu+1)_n = \Gamma(\nu+n+1)/\Gamma(\nu+1)$ for each $n \geq 0$ is the so-called Pochhammer (or Appell) symbol, and J_ν, defined by [20, p. 40]

$$J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)},$$

stands for the Bessel function of the first kind of order ν. Note that the following differentiation formula

$$\mathcal{J}_\nu'(x) = -\frac{x}{2(\nu+1)} \mathcal{J}_{\nu+1}(x) \tag{1.1}$$

is valid for all $x \in \mathbb{R}$ and $\nu > -1$, which can be verified by using the series representation of the function $\mathcal{J}_\nu$. Moreover, it is worth mentioning that in particular we have

$$\mathcal{J}_{-1/2}(x) = \sqrt{\pi/2} \cdot x^{1/2} J_{-1/2}(x) = \cos x, \tag{1.2}$$

$$\mathcal{J}_{1/2}(x) = \sqrt{\pi/2} \cdot x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}, \tag{1.3}$$

$$\mathcal{J}_{3/2}(x) = 3\sqrt{\pi/2}\cdot x^{-3/2}J_{3/2}(x) = 3\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right), \tag{1.4}$$

respectively.

Now, for $\nu > -1$ let us consider the normalized modified Bessel function $\mathcal{I}_\nu : \mathbb{R} \to [1,\infty)$, defined by

$$\mathcal{I}_\nu(x) = 2^\nu\Gamma(\nu+1)x^{-\nu}I_\nu(x) = \sum_{n\geq 0}\frac{(1/4)^n}{(\nu+1)_n n!}x^{2n},$$

where I_ν is the modified Bessel function of the first kind, defined by [20, p. 77]

$$I_\nu(x) = \sum_{n\geq 0}\frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}.$$

Analogously with relations (1.2), (1.3) and (1.4) for the function $\mathcal{I}_p$ in particular we have

$$\mathcal{I}_{-1/2}(x) = \sqrt{\pi/2}\cdot x^{1/2}I_{-1/2}(x) = \cosh x, \tag{1.5}$$

$$\mathcal{I}_{1/2}(x) = \sqrt{\pi/2}\cdot x^{-1/2}I_{1/2}(x) = \frac{\sinh x}{x}, \tag{1.6}$$

$$\mathcal{I}_{3/2}(x) = 3\sqrt{\pi/2}\cdot x^{-3/2}I_{3/2}(x) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right), \tag{1.7}$$

respectively.

2. Jordan-type inequalities for Bessel functions and a new power series representation of Bessel functions

2.1. The first main result and its proof. Our main result of this section provides an extension of the main results of [32], [33], [34] concerning Jordan type inequalities and the new power series expansion of the sine function.

Theorem 1. *Let $\nu > -1$ and let $j_{\nu,1}$ be the first positive zero of the Bessel function of the first kind J_ν. Then for all $0 < x \leq r \leq j_{\nu+1,1}$ the following sharp Jordan-type inequalities hold*

$$S_{\nu,n}(x) + \alpha_\nu(r)(r^2-x^2)^{n+1} \leq \mathcal{J}_\nu(x) \leq S_{\nu,n}(x) + \beta_\nu(r)(r^2-x^2)^{n+1}, \tag{2.1}$$

where

$$S_{\nu,n}(x) = \sum_{k=0}^{n} a_{\nu,k}(r)(r^2 - x^2)^k,$$

n is a natural number and the coefficients $a_{\nu,k}(r)$ are defined explicitly by

$$a_{\nu,k}(r) = \frac{\mathcal{J}_{\nu+k}(r)}{4^k k!(\nu+1)_k} \tag{2.2}$$

for all $k \in \{0, 1, \dots, n+1\}$ or recursively by

$$a_{\nu,0}(r) = \mathcal{J}_\nu(r), \quad a_{\nu,1}(r) = \frac{1}{4(\nu+1)}\mathcal{J}_{\nu+1}(r),$$

$$a_{\nu,k+1}(r) = \frac{\nu+k}{(k+1)r^2} a_{\nu,k}(r) - \frac{1}{4k(k+1)r^2} a_{\nu,k-1}(r) \tag{2.3}$$

for all $k \in \{1, 2, \dots, n\}$. Moreover, the constants

$$\alpha_\nu(r) = a_{\nu,n+1}(r) \quad \text{and} \quad \beta_\nu(r) = \frac{1 - \sum_{k=0}^{n} a_{\nu,k}(r) r^{2k}}{r^{2(n+1)}}$$

are the best possible. In addition, there exist $\zeta \in (x, r)$ depending on n such that

$$\mathcal{J}_\nu(x) = \sum_{k=0}^{n} a_{\nu,k}(r)(r^2 - x^2)^k + \frac{\mathcal{J}_{\nu+n+1}(\zeta)}{4^{n+1}(n+1)!(\nu+1)_{n+1}}(r^2 - x^2)^{n+1}, \tag{2.4}$$

which leads to the power series expansion

$$\mathcal{J}_\nu(x) = \sum_{n \geq 0} a_{\nu,n}(r)(r^2 - x^2)^n. \tag{2.5}$$

Proof. Let $m \in \{1, 2, \dots, n+1\}$ and consider the functions $f_m, g_m\colon [0, r] \to \mathbb{R}$, defined by

$$f_1(x) = \mathcal{J}_\nu(x) - \sum_{k=0}^{n} a_{\nu,k}(r)(r^2 - x^2)^k,$$

$$f_m(x) = \frac{\mathcal{J}_{\nu+m-1}(x)}{4^{m-1}(\nu+1)_{m-1}} - \sum_{k=m-1}^{n} k(k-1)\dots(k-m+2) a_{\nu,k}(r)(r^2 - x^2)^{k-m+1}$$

for all $m \geq 2$ and

$$g_1(x) = (r^2 - x^2)^{n+1},$$

$$g_m(x) = (n+1)n(n-1)\dots(n-m+3)(r^2 - x^2)^{n-m+2}, \quad m \geq 2.$$

Now consider the function $Q_1 : (0, r) \to \mathbb{R}$, defined by $Q_1(x) = f_1(x)/g_1(x)$. Observe that the inequality (2.1) is equivalent to $\alpha_\nu(r) \leq Q_1(x) \leq \beta_\nu(r)$. Thus, to prove (2.1), in what follows we show that Q_1 is decreasing. In view of (1.1) it is easy to verify that for all $s \in \{0,1,2,\dots,n\}$ we have $f_{s+1}(r) = g_{s+1}(r) = 0$ and consequently for each $x \in (0,r)$ and $s \in \{1,2,\dots,n\}$ one has

$$\frac{f_s'(x)}{g_s'(x)} = \frac{f_{s+1}(x)}{g_{s+1}(x)} = \frac{f_{s+1}(x) - f_{s+1}(r)}{g_{s+1}(x) - g_{s+1}(r)}.$$

Taking into account the above relation and using the monotone form of l'Hospital's rule [2, Lemma 2.2] $n+1$ times it is clear that to prove that Q_1 is decreasing it is enough to show that the function

$$x \mapsto \frac{f_{n+1}'(x)}{g_{n+1}'(x)} = a_{\nu,n+1}(r)\frac{\mathcal{J}_{\nu+n+1}(x)}{\mathcal{J}_{\nu+n+1}(r)}$$

is decreasing on $(0,r)$. But, it is known [9, Theorem 3] that for all $\nu > -1$ the function $x \mapsto \mathcal{J}_\nu(x)$ is decreasing on $(0, j_{\nu,1})$. Changing ν with $\nu+n+1$ we obtain that the function $x \mapsto \mathcal{J}_{\nu+n+1}(x)$ is decreasing on $(0, j_{\nu+2,1}) \subset (0, j_{\nu+n+1,1})$ and with this the monotonicity of Q_1 is proved. Here we used the well-known fact that the function $\nu \mapsto j_{\nu,1}$ is increasing on $(-1,\infty)$. To complete the proof of (2.1) all that remains is to prove that the constants $\alpha_\nu(r)$ and $\beta_\nu(r)$ in (2.1) are the best possible. For this observe that

$$\lim_{x \searrow 0} Q_1(x) = \frac{1 - \sum_{k=0}^{n} a_{\nu,k}(r) r^{2k}}{r^{2(n+1)}} = \beta_\nu(r)$$

and by using the l'Hospital's rule $n+1$ times

$$\lim_{x \nearrow r} Q_1(x) = \lim_{x \nearrow r} \frac{f_1'(x)}{g_1'(x)} = \lim_{x \nearrow r} \frac{f_2'(x)}{g_2'(x)} = \dots = \lim_{x \nearrow r} \frac{f_{n+1}'(x)}{g_{n+1}'(x)} = a_{\nu,n+1}(r) = \alpha_\nu(r).$$

Finally, to prove the recursive relation for the coefficients $a_{\nu,k}(r)$, recall the recurrence relation [20, p. 45]

$$J_{\nu-1}(x) + J_{\nu+1}(x) = (2\nu/x) J_\nu(x),$$

from which we easily obtain

$$4\nu(\nu+1)\mathcal{J}_{\nu-1}(x) + x^2 \mathcal{J}_{\nu+1}(x) = 4\nu(\nu+1)\mathcal{J}_\nu(x).$$

Changing in the above relation ν with $\nu+k$, and using (2.2), we obtain (2.3).

Now, let us focus on (2.4). By the well-known Cauchy mean value theorem there exist a constant $\xi_1 \in (x,r)$ such that

$$Q_1(x) = \frac{f_1(x)}{g_1(x)} = \frac{f_1(x) - f_1(r)}{g_1(x) - g_1(r)} = \frac{f_1'(\xi_1)}{g_1'(\xi_1)},$$

but as we have mentioned above we have

$$\frac{f_1'(\xi_1)}{g_1'(\xi_1)} = \frac{f_2(\xi_1)}{g_2(\xi_1)} = \frac{f_2(\xi_1) - f_2(r)}{g_2(\xi_1) - g_2(r)}.$$

Thus, using again the Cauchy mean value theorem n times we obtain that there exist $\xi_{n+1} \in (\xi_n, r)$, where $\xi_i \in (\xi_{i-1}, r)$ for all $i \in \{2, 3, \ldots, n\}$, such that

$$Q_1(x) = \frac{f_1'(\xi_1)}{g_1'(\xi_1)} = \ldots = \frac{f_n'(\xi_n)}{g_n'(\xi_n)} = \frac{f_{n+1}'(\xi_{n+1})}{g_{n+1}'(\xi_{n+1})} = a_{\nu,n+1}(r)\frac{\mathcal{J}_{\nu+n+1}(\xi_{n+1})}{\mathcal{J}_{\nu+n+1}(r)}.$$

Now denoting ξ_{n+1} with ζ and by using (2.2) we obtain (2.4). On the other hand when n tends to infinity ξ_{n+1} tends to r, and thus $\mathcal{J}_{\nu+n+1}(\xi_{n+1})/\mathcal{J}_{\nu+n+1}(r)$ tends to 1. Moreover, it is easy to show that $a_{\nu,n+1}(r)$ as well as $a_{\nu,n+1}(r)(r^2 - x^2)^{n+1}$ tends to zero as n tends to infinity. With other words, the reminder term in (2.4) tends to zero as n tends to infinity, which leads to the series expansion (2.5). □

2.2. Particular cases and remarks. 1. First let us focus on inequality (2.1) and suppose that $\nu = -1/2$. Then by using (1.2), (1.3) and the notation

$$S_{\nu,n}(x) = \sum_{k=0}^{n} a_{\nu,k}(r)(r^2 - x^2)^k,$$

from Theorem 1 we obtain for all $0 < x \le r \le \pi$ the following sharp Kober-type inequality:

$$\begin{aligned} S_{-1/2,n}(x) + \alpha_{-1/2}(r)(r^2 - x^2)^{n+1} &\le \cos x \\ &\le S_{-1/2,n}(x) + \beta_{-1/2}(r)(r^2 - x^2)^{n+1}, \end{aligned} \tag{2.6}$$

where

$$S_{-1/2,n}(x) = \sum_{k=0}^{n} a_{-1/2,k}(r)(r^2 - x^2)^k,$$

n is a natural number and the coefficients $a_{-1/2,k}(r)$ are defined recursively by

$$a_{-1/2,0}(r) = \cos r,\ a_{-1/2,1}(r) = \frac{\sin r}{2r},$$

$$a_{-1/2,k+1}(r) = \frac{2k-1}{2(k+1)r^2} a_{-1/2,k}(r) - \frac{1}{4k(k+1)r^2} a_{-1/2,k-1}(r)$$

for all $k \in \{1, 2, \ldots, n\}$. Moreover, the constants

$$\alpha_{-1/2}(r) = a_{-1/2,n+1}(r) \quad \text{and} \quad \beta_{-1/2}(r) = \frac{1 - \sum_{k=0}^{n} a_{-1/2,k}(r) r^{2k}}{r^{2(n+1)}}$$

are the best possible. This result completes [33, Theorem 13] and improves the other known results from the literature. For more details on improved Kober's inequalities we refer to the paper [18]. It is worth mentioning that although the technique is similar our approach is much simpler than in [33]. This is because in [33] there are used spherical Bessel functions instead of Bessel functions, for which the monotonicity of their higher order derivative requires some complicated preliminary results. Our proof is based just on the simple monotonicity property of the function $\mathcal{J}_\nu$ established by the first author [9, Theorem 3].

2. Secondly, suppose that $\nu = 1/2$. Then, by using (1.3) and (1.4), from (2.1) we obtain for all $0 < x \le r \le j_{3/2,1}$ the following sharp Jordan-type inequality:

$$S_{1/2,n}(x) + \alpha_{1/2}(r)(r^2 - x^2)^{n+1} \le \frac{\sin x}{x} \le S_{1/2,n}(x) + \beta_{1/2}(r)(r^2 - x^2)^{n+1}, \quad (2.7)$$

where

$$S_{1/2,n}(x) = \sum_{k=0}^{n} a_{1/2,k}(r)(r^2 - x^2)^k,$$

n is a natural number and the coefficients $a_{1/2,k}(r)$ are defined recursively by

$$a_{1/2,0}(r) = \frac{\sin r}{r}, \quad a_{1/2,1}(r) = \frac{\sin r - r\cos r}{2r^3},$$

$$a_{1/2,k+1}(r) = \frac{2k+1}{2(k+1)r^2} a_{1/2,k}(r) - \frac{1}{4k(k+1)r^2} a_{1/2,k-1}(r)$$

for all $k \in \{1, 2, \ldots, n\}$. Here $j_{3/2,1} = 4.493409457$ in view of (1.4) is in fact the first positive zero of the equation $\tan x = x$. Moreover, the constants

$$\alpha_{1/2}(r) = a_{1/2,n+1}(r) \quad \text{and} \quad \beta_{1/2}(r) = \frac{1 - \sum_{k=0}^{n} a_{1/2,k}(r) r^{2k}}{r^{2(n+1)}}$$

are the best possible. We note that this result was obtained in [33, Theorem 6] and [34, Theorem 1], however our approach is simpler than in [33], [34]. Moreover, it is worth mentioning that in [34] the inequality (2.7) is proved just for the case when $0 < x \le r \le j_{1/2,1} = \pi$, while in [33] just for the case when $0 < x \le r \le j_{-1/2,1} = \pi/2$. Thus our result (2.1) not only extend (2.7) to Bessel functions, but even improves the range of validity. Now, if we choose $r = \pi/2$ in (2.7), then we get the following sharp Jordan-type inequalities

$$\sum_{k=0}^{n} b_k(\pi^2 - 4x^2)^k + b_{n+1}(\pi^2 - 4x^2)^{n+1} \le \frac{\sin x}{x}$$

$$\le \sum_{k=0}^{n} b_k(\pi^2 - 4x^2)^k + \frac{1 - \sum_{k=0}^{n} b_k \pi^{2k}}{\pi^{2(n+1)}}(\pi^2 - 4x^2)^{n+1},$$

where n is a natural number, as above, and the coefficients b_k are defined as follows

$$b_0 = \frac{2}{\pi},\ b_1 = \frac{1}{\pi^3},\ b_{k+1} = \frac{2k+1}{2(k+1)\pi^2}b_k - \frac{1}{16k(k+1)\pi^2}b_{k-1}$$

for all $k \in \{1, 2, \dots, n\}$. These results appear on three different papers of the same author, namely in [33, Theorem 7], [34, Corollary 1] and [32, Theorem 5].

3. Now, let focus on the expansions (2.4) and (2.5) and suppose that $\nu = -1/2$. Then, by using again (1.2), for all $0 < x \leq r \leq \pi$ we obtain

$$\cos x = \sum_{k=0}^{n} a_{-1/2,k}(r)(r^2 - x^2)^k + \frac{\mathcal{J}_{n+1/2}(\zeta)}{4^{n+1}(n+1)!(1/2)_{n+1}}(r^2 - x^2)^{n+1},$$

which leads to the new power series expansion of the cosine function

$$\cos x = \sum_{k \geq 0} a_{-1/2,k}(r)(r^2 - x^2)^k,$$

where the coefficients $a_{-1/2,k}(r)$ are defined recursively by

$$a_{-1/2,0}(r) = \cos r, \quad a_{-1/2,1}(r) = \frac{\sin r}{2r},$$

$$a_{-1/2,k+1}(r) = \frac{2k-1}{2(k+1)r^2}a_{-1/2,k}(r) - \frac{1}{4k(k+1)r^2}a_{-1/2,k-1}(r)$$

for all $k \in \{1, 2, \dots\}$. Moreover, in this case $\beta_{-1/2}(r)$ becomes zero, i.e. the coefficients $a_{-1/2,k}(r)$ have the following property

$$\sum_{k=0}^{n} a_{-1/2,k}(r)r^{2k} = 1.$$

4. Suppose that $\nu = 1/2$. Then, in view of (1.3), for all $0 < x \leq r \leq j_{3/2,1}$ we obtain

$$\frac{\sin x}{x} = \sum_{k=0}^{n} a_{1/2,k}(r)(r^2 - x^2)^k + \frac{\mathcal{J}_{n+3/2}(\zeta)}{4^{n+1}(n+1)!(3/2)_{n+1}}(r^2 - x^2)^{n+1},$$

which leads to the new power series expansion of the sine function

$$\frac{\sin x}{x} = \sum_{k \geq 0} a_{1/2,k}(r)(r^2 - x^2)^k,$$

where the coefficients $a_{1/2,k}(r)$ are defined recursively by

$$a_{1/2,0}(r) = \frac{\sin r}{r}, \quad a_{1/2,1}(r) = \frac{\sin r - r\cos r}{2r^3},$$
$$a_{1/2,k+1}(r) = \frac{2k+1}{2(k+1)r^2}a_{1/2,k}(r) - \frac{1}{4k(k+1)r^2}a_{1/2,k-1}(r)$$

for all $k \in \{1, 2, \dots\}$. Moreover, in this case $\beta_{1/2}(r)$ becomes zero, i.e. the coefficients $a_{1/2,k}(r)$ have the following property

$$\sum_{k=0}^{n} a_{1/2,k}(r)r^{2k} = 1.$$

We note here that the above series expansions are in fact equivalent with [33, Theorem 8] and [33, Theorem 9], however, in [33] the results in the question were proved just for $0 < x \leq r \leq \pi/2$. See also [32, Theorem 7], where [33, Theorem 8] is reproduced for $r = \pi/2$. Now, choose in the above relations $r = \pi/2$. Then we obtain [32, Theorem 8], [33, Theorem 10]:

$$\frac{\sin x}{x} = \sum_{k\geq 0} b_k(\pi^2 - 4x^2)^k,$$

where

$$b_0 = \frac{2}{\pi}, \quad b_1 = \frac{1}{\pi^3}, \quad b_{k+1} = \frac{2k+1}{2(k+1)\pi^2}b_k - \frac{1}{16k(k+1)\pi^2}b_{k-1}$$

for all $k \in \{1, 2, \dots\}$.

5. Finally, choose $r = j_{\nu,1}$ in (2.4) and (2.5). Then, for all $0 < x \leq j_{\nu,1}$, we obtain the following new expansion of the Bessel functions of the first kind:

$$J_\nu(x) = \sum_{k=1}^{n} \left(\frac{x}{j_{\nu,1}}\right)^\nu \frac{J_{\nu+k}(j_{\nu,1})}{2^k k! j_{\nu,1}^k}(j_{\nu,1}^2 - x^2)^k + \left(\frac{x}{\zeta}\right)^\nu \frac{J_{\nu+n+1}(\zeta)}{2^{n+1}(n+1)!\zeta^{n+1}}(j_{\nu,1}^2 - x^2)^{n+1}, \tag{2.8}$$

where $\zeta \in (x, j_{\nu,1})$. This leads to the new series expansion

$$J_\nu(x) = \sum_{n\geq 1} \left(\frac{x}{j_{\nu,1}}\right)^\nu \frac{J_{\nu+n}(j_{\nu,1})}{2^n n! j_{\nu,1}^n}(j_{\nu,1}^2 - x^2)^n. \tag{2.9}$$

First observe that this new series expansion is consistent with the fact that $j_{\nu,1}$ is a simple zero of J_ν. More precisely, from (2.9) it follows that $J_\nu(j_{\nu,1}) = 0$ and $J'_\nu(j_{\nu,1}) = -J_{\nu+1}(j_{\nu,1}) \neq 0$, which shows that $j_{\nu,1}$ is indeed a simple zero of J_ν.

Here we used the known fact that the zeros of J_ν and $J_{\nu+1}$ are interlaced and then we have $J_{\nu+1}(j_{\nu,1}) \neq 0$. Secondly, it is worth mentioning here that the new formulas (2.4) and (2.5), and consequently (2.8) and (2.9), follows easily from the classical Taylor theorem with Lagrange's form of the remainder. To see this, consider the function $x \mapsto \varphi_\nu(x) = \mathcal{J}_\nu(\sqrt{x}) = 2^\nu\Gamma(\nu+1)x^{-\nu/2}J_\nu(\sqrt{x})$, which is continuously differentiable $n+1$ times on the whole real line and satisfies the differentiation formula $4(\nu+1)\varphi_\nu'(x) = -\varphi_{\nu+1}(x)$ for all $x \in \mathbb{R}$ and $\nu > -1$, corresponding to formula (1.1). Then clearly we have

$$\varphi_\nu^{(n)}(x) = \frac{(-1)^n}{4^n(\nu+1)_n}\varphi_{\nu+n}(x)$$

for all $x \in \mathbb{R}$, $n \in \{0,1,2,\dots\}$ and $\nu > -1$. Consequently, from Taylor's theorem with Lagrange's form of the remainder we conclude that there exists $\xi \in (x,r)$ such that

$$\varphi_\nu(x) = \sum_{k=0}^{n} \frac{(-1)^k\varphi_{\nu+k}(r)}{4^k k!(\nu+1)_k}(x-r)^k + \frac{(-1)^{n+1}\varphi_\nu(\xi)}{4^{n+1}(n+1)!(\nu+1)_{n+1}}(x-r)^{n+1},$$

which leads to the power series expansion

$$\varphi_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n\varphi_{\nu+n}(r)}{4^n n!(\nu+1)_n}(x-r)^n.$$

Now, changing x with x^2 and r with r^2 we reobtain the expansions (2.4) and (2.5). Since Taylor's theorem (with the integral formulation of the remainder term) is also valid if the corresponding function has complex values, from the above discussion we conclude that the new series expansions (2.5) and (2.9) are in fact valid for wider range of x and ν, i.e. for $x, \nu \in \mathbb{C}$ such that $\nu \neq -1, -2, \dots$.

3. Jordan-type inequalities for modified Bessel functions and a new power series representation of modified Bessel functions

3.1. The second main result and its proof. In this section we are going to present the hyperbolic counterpart of the results from the previous section. Corresponding to Theorem 1 we have the following results for the function $\mathcal{I}_\nu$. We note that the proof of this theorem is similar to that of Theorem 1, however we have included below its proof only for the sake of completeness.

Theorem 2. *If $\nu > -1$ and $0 < x \leq r$, then the following sharp Jordan-type inequalities hold*

$$P_{\nu,n}(x) + \gamma_\nu(r)(r^2 - x^2)^{n+1} \leq \mathcal{I}_\nu(x) \leq P_{\nu,n}(x) + \delta_\nu(r)(r^2 - x^2)^{n+1}, \qquad (3.1)$$

where

$$P_{\nu,n}(x) = \sum_{k=0}^{n} c_{\nu,k}(r)(r^2 - x^2)^k,$$

n is an even natural number and the coefficients $c_{\nu,k}(r)$ are defined explicitly by

$$c_{\nu,k}(r) = \frac{(-1)^k \mathcal{I}_{\nu+k}(r)}{4^k k! (\nu+1)_k} \qquad (3.2)$$

for all $k \in \{0, 1, \ldots, n+1\}$ or recursively by

$$c_{\nu,0}(r) = \mathcal{I}_\nu(r), \quad c_{\nu,1}(r) = -\frac{1}{4(\nu+1)}\mathcal{I}_{\nu+1}(r),$$

$$c_{\nu,k+1}(r) = \frac{\nu+k}{(k+1)r^2} c_{\nu,k}(r) + \frac{1}{4k(k+1)r^2} c_{\nu,k-1}(r) \qquad (3.3)$$

for all $k \in \{1, 2, \ldots, n\}$. Moreover, the Jordan-type inequality (3.1) is reversed when n is odd and in both of cases the constants

$$\gamma_\nu(r) = c_{\nu,n+1}(r) \quad \text{and} \quad \delta_\nu(r) = \frac{1 - \sum_{k=0}^{n} c_{\nu,k}(r) r^{2k}}{r^{2(n+1)}}$$

are the best possible. In addition, there exist $\zeta \in (x, r)$ depending on m such that

$$\mathcal{I}_\nu(x) = \sum_{k=0}^{m} c_{\nu,k}(r)(r^2 - x^2)^k + \frac{(-1)^{m+1}\mathcal{I}_{\nu+m+1}(\zeta)}{4^{m+1}(m+1)!(\nu+1)_{m+1}}(r^2 - x^2)^{m+1}, \qquad (3.4)$$

which leads to the power series expansion

$$\mathcal{I}_\nu(x) = \sum_{m \geq 0} c_{\nu,m}(r)(r^2 - x^2)^m. \qquad (3.5)$$

Proof. Our strategy is as in the proof of Theorem 1. However, we distinguish here two cases. First we suppose that n is even. Let $m \in \{1, 2, \ldots, n+1\}$ and consider the functions $h_m, g_m : [0, r] \to \mathbb{R}$, defined by

$$h_1(x) = \mathcal{I}_\nu(x) - \sum_{k=0}^{n} c_{\nu,k}(r)(r^2 - x^2)^k,$$

$$h_m(x) = \frac{(-1)^{m-1}\mathcal{I}_{\nu+m-1}(x)}{4^{m-1}(\nu+1)_{m-1}} - \sum_{k=m-1}^{n} k(k-1)\ldots(k-m+2)c_{\nu,k}(r)(r^2-x^2)^{k-m+1}$$

for all $m \geq 2$ and

$$g_1(x) = (r^2 - x^2)^{n+1},$$

$$g_m(x) = (n+1)n(n-1)\ldots(n-m+3)(r^2-x^2)^{n-m+2}, \quad m \geq 2.$$

Now consider the function $Q_2 : (0, r) \to \mathbb{R}$, defined by $Q_2(x) = h_1(x)/g_1(x)$. First observe that the inequality (3.1) is equivalent to $\gamma_\nu(r) \leq Q_2(x) \leq \delta_\nu(r)$. Thus, to prove (3.1), in what follows we show that Q_2 is decreasing. In view of the differentiation formula

$$\mathcal{I}'_\nu(x) = \frac{x}{2(\nu+1)}\mathcal{I}_{\nu+1}(x),$$

it is easy to verify that for all $s \in \{0, 1, 2, \ldots, n\}$ we have $h_{s+1}(r) = g_{s+1}(r) = 0$ and consequently for each $x \in (0, r)$ and $s \in \{1, 2, \ldots, n\}$ one has

$$\frac{h'_s(x)}{g'_s(x)} = \frac{h_{s+1}(x)}{g_{s+1}(x)} = \frac{h_{s+1}(x) - h_{s+1}(r)}{g_{s+1}(x) - g_{s+1}(r)}.$$

Using again the monotone form of l'Hospital's rule [2, Lemma 2.2] $n+1$ times it is clear that to show that Q_2 is decreasing it is enough to prove that the function

$$x \mapsto \frac{h'_{n+1}(x)}{g'_{n+1}(x)} = -c_{\nu,n+1}(r)\frac{\mathcal{I}_{\nu+n+1}(x)}{\mathcal{I}_{\nu+n+1}(r)}$$

is decreasing on $(0, r)$. But, it is well-known that for all $\nu > -1$ the function $x \mapsto \mathcal{I}_\nu(x)$ is increasing on $(0, \infty)$. Now changing ν with $\nu + n + 1$ we obtain that the function $x \mapsto \mathcal{I}_{\nu+n+1}(x)$ is increasing too on $(0, \infty)$ and with this the monotonicity of Q_2 is proved.

To complete the proof of (3.1) all that remains is to prove that the constants $\gamma_\nu(r)$ and $\delta_\nu(r)$ in (3.1) are the best possible. For this observe that

$$\lim_{x \searrow 0} Q_2(x) = \frac{1 - \sum_{k=0}^{n} c_{\nu,k}(r)r^{2k}}{r^{2(n+1)}} = \delta_\nu(r)$$

and by using the l'Hospital's rule $n+1$ times

$$\lim_{x \nearrow r} Q_2(x) = \lim_{x \nearrow r} \frac{h'_1(x)}{g'_1(x)} = \lim_{x \nearrow r} \frac{h'_2(x)}{g'_2(x)} = \ldots = \lim_{x \nearrow r} \frac{h'_{n+1}(x)}{g'_{n+1}(x)} = c_{\nu,n+1}(r) = \gamma_\nu(r).$$

Now, suppose that n is odd. A similar argument to that presented above yields that in this case that the function

$$x \mapsto \frac{h'_{n+1}(x)}{g'_{n+1}(x)} = c_{\nu,n+1}(r)\frac{\mathcal{I}_{\nu+n+1}(x)}{\mathcal{I}_{\nu+n+1}(r)}$$

is increasing on $(0, r)$. Consequently Q_2 is increasing, and thus we have $\gamma_\nu(r) \geq Q_2(x) \geq \delta_\nu(r)$ for all $\nu > -1$ and $0 < x \leq r$, i.e. the inequality (3.1) is reversed.

Finally, to prove the recursive relation for the coefficients $c_{\nu,k}(r)$ recall the formula [20, p. 79]

$$I_{\nu-1}(x) - I_{\nu+1}(x) = (2\nu/x)I_\nu(x),$$

from which we easily obtain

$$4\nu(\nu+1)\mathcal{I}_{\nu-1}(x) - x^2\mathcal{I}_{\nu+1}(x) = 4\nu(\nu+1)\mathcal{I}_\nu(x).$$

Changing in the above relation ν with $\nu + k$ and using (3.2) we easily obtain (3.3).

Now, let us focus on (3.4). Using the Cauchy mean value theorem there exist a constant $\xi_1 \in (x, r)$ such that

$$Q_2(x) = \frac{h_1(x)}{g_1(x)} = \frac{h_1(x) - f_1(r)}{g_1(x) - g_1(r)} = \frac{h'_1(\xi_1)}{g'_1(\xi_1)},$$

but as we have mentioned above we have

$$\frac{h'_1(\xi_1)}{g'_1(\xi_1)} = \frac{h_2(\xi_1)}{g_2(\xi_1)} = \frac{h_2(\xi_1) - h_2(r)}{g_2(\xi_1) - g_2(r)}.$$

Thus using again the Cauchy mean value theorem m times we obtain that there exist $\xi_{m+1} \in (\xi_m, r)$, where $\xi_i \in (\xi_{i-1}, r)$ for all $i \in \{2, 3, \dots, m\}$, such that

$$\begin{aligned} Q_2(x) &= \frac{h'_1(\xi_1)}{g'_1(\xi_1)} = \frac{h'_2(\xi_2)}{g'_2(\xi_2)} = \dots = \frac{h'_m(\xi_m)}{g'_m(\xi_m)} \\ &= \frac{h'_{m+1}(\xi_{m+1})}{g'_{m+1}(\xi_{m+1})} = (-1)^{m+1}c_{\nu,m+1}(r)\frac{\mathcal{I}_{\nu+m+1}(\xi_{m+1})}{\mathcal{I}_{\nu+m+1}(r)}. \end{aligned}$$

Now denoting ξ_{m+1} with ζ and by using (3.2) we obtain (3.4). On the other hand when m tends to infinity ξ_{m+1} tends to r, and thus $\mathcal{I}_{\nu+m+1}(\xi_{m+1})/\mathcal{I}_{\nu+m+1}(r)$ tends to 1. Moreover, it is easy to show that the expression $(-1)^{m+1}c_{\nu,m+1}(r)$ as well as $(-1)^{m+1}c_{\nu,m+1}(r)(r^2 - x^2)^{m+1}$ tends to zero as m tends to infinity. With other words, the reminder term in (3.4) tends to zero as m tends to infinity, which leads to the series expansion (3.5). □

3.2. Particular cases and remarks. 1. First let us focus on inequality (3.1) and suppose that $\nu=-1/2$. Then by using (1.5), (1.6) and the notation

$$P_{\nu,n}(x)=\sum_{k=0}^{n}c_{\nu,k}(r)(r^2-x^2)^k,$$

from Theorem 2 for all $0<x\leq r$ we get the following Kober-type inequality:

$$\begin{aligned}P_{-1/2,n}(x)+\gamma_{-1/2}(r)(r^2-x^2)^{n+1}&\leq\cosh x\\ &\leq P_{-1/2,n}(x)+\delta_{-1/2}(r)(r^2-x^2)^{n+1},\end{aligned}\tag{3.6}$$

where

$$P_{-1/2,n}(x)=\sum_{k=0}^{n}c_{-1/2,k}(r)(r^2-x^2)^k,$$

n is an even natural number and the coefficients $c_{-1/2,k}(r)$ are defined recursively by

$$c_{-1/2,0}(r)=\cosh r,\quad c_{-1/2,1}(r)=-\frac{\sinh r}{2r},$$
$$c_{-1/2,k+1}(r)=\frac{2k-1}{2(k+1)r^2}c_{-1/2,k}(r)+\frac{1}{4k(k+1)r^2}c_{-1/2,k-1}(r)$$

for all $k\in\{1,2,\dots,n\}$. Moreover, the constants

$$\gamma_{-1/2}(r)=c_{-1/2,n+1}(r)\quad\text{and}\quad\delta_{-1/2}(r)=\frac{1-\sum_{k=0}^{n}c_{-1/2,k}(r)r^{2k}}{r^{2(n+1)}}$$

are the best possible. Note that, when n is odd, the inequality (3.6) is reversed.

2. Secondly, suppose that $\nu=1/2$. Then, by using (1.6) and (1.7) from (3.1), we obtain for all $0<x\leq r$ the following sharp Jordan-type inequality:

$$\begin{aligned}P_{1/2,n}(x)+\gamma_{1/2}(r)(r^2-x^2)^{n+1}\leq\frac{\sinh x}{x}&\leq P_{1/2,n}(x)\\ &+\delta_{1/2}(r)(r^2-x^2)^{n+1},\end{aligned}\tag{3.7}$$

where

$$P_{1/2,n}(x)=\sum_{k=0}^{n}c_{1/2,k}(r)(r^2-x^2)^k,$$

n is an even natural number and the coefficients $c_{1/2,k}(r)$ are defined recursively by

$$c_{1/2,0}(r)=\frac{\sinh r}{r},\quad c_{1/2,1}(r)=\frac{\sinh r-r\cosh r}{2r^3},$$

$$c_{1/2,k+1}(r)=\frac{2k+1}{2(k+1)r^2}c_{1/2,k}(r)+\frac{1}{4k(k+1)r^2}c_{1/2,k-1}(r)$$

for all $k\in\{1,2,\ldots,n\}$. Moreover, the constants

$$\gamma_{1/2}(r)=c_{1/2,n+1}(r)\quad\text{and}\quad\delta_{1/2}(r)=\frac{1-\sum_{k=0}^{n}c_{1/2,k}(r)r^{2k}}{r^{2(n+1)}}$$

are the best possible. We note that when n is odd then the inequality (3.7) is reversed.

3. Finally, choose $r=j_{\nu,1}$ in (3.4) and (3.5). Then, for all $0<x\leq j_{\nu,1}$, we obtain the following new expansion of the modified Bessel functions of the first kind:

$$\begin{aligned}I_\nu(x)=&\sum_{k=0}^{m}\left(\frac{x}{j_{\nu,1}}\right)^{\nu}\frac{(-1)^kI_{\nu+k}(j_{\nu,1})}{2^kk!j_{\nu,1}^k}(j_{\nu,1}^2-x^2)^k\\&+\left(\frac{x}{\zeta}\right)^{\nu}\frac{(-1)^{m+1}I_{\nu+m+1}(\zeta)}{2^{m+1}(m+1)!\zeta^{m+1}}(j_{\nu,1}^2-x^2)^{m+1},\end{aligned}$$

where $\zeta\in(x,j_{\nu,1})$. This leads to the new series expansion

$$I_\nu(x)=\sum_{m\geq0}\left(\frac{x}{j_{\nu,1}}\right)^{\nu}\frac{(-1)^mI_{\nu+m}(j_{\nu,1})}{2^mm!j_{\nu,1}^m}(j_{\nu,1}^2-x^2)^m.\tag{3.8}$$

Observe that, analogously to the results of the previous section, (3.4), (3.5) and the above formulas are valid for all $x,\nu\in\mathbb{C}$ such that $\nu\neq-1,-2,\ldots$. This is an immediate consequences of Taylor's theorem with Lagrange's form of the remainder applied to the function $x\mapsto\gamma_\nu(x)=\mathcal{I}_\nu(\sqrt{x})=2^\nu\Gamma(\nu+1)x^{-\nu/2}I_\nu(\sqrt{x})$. We note that this simple idea to use instead of $\mathcal{I}_\nu$ the function γ_ν is also useful in the problem of finding a finite sum formula for the probability density function of the non-central chi-squared distribution [3].

4. Jordan-type inequalities for generalized Bessel functions and a new power series representation of generalized Bessel functions

In this section, motivated by the work [15] we reformulate the main results of the previous section for generalized Bessel functions of the first kind. The generalized Bessel function of the first kind v_ν is defined [10] as a particular solution of the generalized Bessel differential equation

$$x^2y''(x)+bxy'(x)+\left[cx^2-\nu^2+(1-b)\nu\right]y(x)=0,$$

where $b, \nu, c \in \mathbb{R}$, and v_ν has the infinite series representation

$$v_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n c^n}{n!\Gamma\left(\nu + n + \frac{b+1}{2}\right)} \cdot \left(\frac{x}{2}\right)^{2n+\nu} \quad \text{for all } x \in \mathbb{R}.$$

This function permits us to study the classical Bessel function J_ν, the modified Bessel function I_ν, the spherical Bessel function and the modified spherical Bessel functions together. For $b = c = 1$ the function v_ν reduces to the function J_ν, while for $b = 1$ and $c = -1$ reduces to I_ν. Now the generalized and normalized (with conditions $u_\nu(0) = 1$ and $u'_\nu(0) = -c/(4\kappa)$) Bessel function of the first kind is defined [10] as follows

$$u_\nu(x) = 2^\nu \Gamma(\kappa) \cdot x^{-\nu/2} v_\nu(x^{1/2}) = \sum_{n\geq 0} \frac{(-c/4)^n}{(\kappa)_n} \frac{x^n}{n!} \quad \text{for all } x \in \mathbb{R},$$

where $\kappa := \nu + (b+1)/2 \neq 0, -1, -2, \ldots$. This function is related in fact to an obvious transform of the well-known hypergeometric function ${}_0F_1$, i.e. $u_\nu(x) = {}_0F_1(\kappa, -cx/4)$, and satisfies the following differential equation

$$xy''(x) + \kappa y'(x) + (c/4)y(x) = 0.$$

Now let us consider the function $\lambda_\nu : \mathbb{R} \to \mathbb{R}$, defined by

$$\lambda_\nu(x) = u_\nu(x^2) = \sum_{n\geq 0} \frac{(-c/4)^n}{(\kappa)_n} \frac{x^{2n}}{n!}.$$

It is worth mentioning that if $c = b = 1$, then λ_ν reduces to the function $\mathcal{J}_\nu$, while u_ν reduces to φ_ν, moreover, if $c = -1$ and $b = 1$, then λ_ν becomes $\mathcal{I}_\nu$ and u_ν becomes γ_ν. For more details on the function λ_ν, like geometric properties, inequalities and integral representations the interested reader is referred to the papers [4], [5], [6], [7], [8], [10], [12].

The main results of the above sections, namely Theorems 1 and 2, can be unified as follows, which improves significantly [7, Theorem 14] and [11, Theorem 2.2]. We note that when $b = c = 1$ then Theorem 3 reduces to Theorem 1, while for $b = 1$ and $c = -1$ Theorem 3 becomes Theorem 2.

Theorem 3. *Let $\kappa > 0$. Then for all $c \in [0, 1]$ and $0 < x \leq r \leq j_{\kappa,1}$ the following sharp Jordan-type inequalities hold*

$$G_{\nu,m}(x) + \varsigma_\nu(r)(r^2 - x^2)^{m+1} \leq \lambda_\nu(x) \leq G_{\nu,m}(x) + \tau_\nu(r)(r^2 - x^2)^{m+1}, \quad (4.1)$$

where

$$G_{\nu,m}(x)=\sum_{i=0}^{m}d_{\nu,i}(r)(r^2-x^2)^i,$$

m is a natural number and the coefficients $d_{\nu,i}(r)$ are defined explicitly by

$$d_{\nu,i}(r)=\left(\frac{c}{4}\right)^i\frac{\lambda_{\nu+i}(r)}{i!(\kappa)_i}$$

for all $i\in\{0,1,\ldots,m+1\}$ or recursively by

$$d_{\nu,0}(r)=\lambda_\nu(r),\quad d_{\nu,1}(r)=\frac{c}{4\kappa}\lambda_{\nu+1}(r),$$
$$d_{\nu,i+1}(r)=\frac{\kappa+i-1}{(i+1)r^2}d_{\nu,i}(r)-\frac{c}{4i(i+1)r^2}d_{\nu,i-1}(r)$$

for all $i\in\{1,2,\ldots,n\}$. Moreover, if $c\leq 0$, $0<x\leq r$ and m is even, then the Jordan-type inequality (4.1) holds true, while if $c\leq 0$, $0<x\leq r$ and m is odd, then the Jordan-type inequality (4.1) is reversed. In each of cases the constants

$$\varsigma_\nu(r)=d_{\nu,m+1}(r)\quad\text{and}\quad\tau_\nu(r)=\frac{1-\sum_{k=0}^{m}d_{\nu,k}(r)r^{2k}}{r^{2(m+1)}}$$

are the best possible. In addition, for all $c\leq 1$ there exist $\zeta\in(x,r)$ depending on m such that

$$\lambda_\nu(x)=\sum_{i=0}^{m}d_{\nu,i}(r)(r^2-x^2)^i+\frac{c^{m+1}\lambda_{\nu+m+1}(\zeta)}{4^{m+1}(m+1)!(\kappa)_{m+1}}(r^2-x^2)^{m+1},$$

which leads to the power series expansion

$$\lambda_\nu(x)=\sum_{m\geq 0}d_{\nu,m}(r)(r^2-x^2)^m.$$

Proof. First observe that if $c\geq 0$, then $\mathcal{J}_{\kappa-1}(t\sqrt{c})=\lambda_\nu(t)$, while for $c\leq 0$ we have $\mathcal{I}_{\kappa-1}(t\sqrt{-c})=\lambda_\nu(t)$. Thus, if we suppose that $c\in[0,1]$ and we change in (2.1) ν with $\kappa-1$, x with $x\sqrt{c}$ and r with $r\sqrt{c}$, then we obtain (4.1). Similarly, if we suppose that $c\leq 0$ and we change in (3.1) ν with $\kappa-1$, x with $x\sqrt{-c}$ and r with $r\sqrt{-c}$, then we obtain (4.1). □

During the course of writing this paper we have found the work [15], where among other things, motivated by [7, Theorem 14] the inequality (4.1) is proved for $m=n-1$. See also [18] for more details. However, in the case when $c\in[0,1]$, it is assumed that $\kappa \geq 1/2$ and $r \leq \pi/2$. As we have seen above these conditions can be relaxed to $\kappa > 0$ and $r \leq j_{\kappa,1}$. Moreover, it is important to note here that since

$$1.570796327 = \pi/2 = j_{-1/2,1} < 2.4048255577 = j_{0,1} < j_{\kappa,1},$$

the interval $(0, j_{\kappa,1}]$ is larger than the interval $(0, \pi/2]$. We note that, since in [15] the approach is somewhat different to that given in the papers [32, 33, 34], and in this paper, it is not clear what is the connection between the inequality (2.7) and inequality (4.1) for $m = n - 1$. This is because in [32, 33, 34] the corresponding coefficients are defined recursively, while in [15] explicitly. However, from our discussion it is clear that in fact the inequality (2.7) is a particular case of (4.1), just taking $\nu = -1/2$ and $b = c = 1$ in Theorem 3. Finally, note that when $b = 2$ and $c = 1$ then v_ν becomes $x \mapsto (2/\sqrt{\pi})j_\nu(x)$, where

$$x \mapsto j_\nu(x) = \sqrt{\pi/(2x)}J_{\nu+1/2}(x)$$

is the spherical Bessel function of the first kind [1, p. 437], and in this case λ_ν reduces to the function

$$x \mapsto \mathcal{J}_{\nu+1/2}(x) = 2^{\nu+1/2}\Gamma(\nu+3/2)x^{-(\nu+1/2)}J_{\nu+1/2}(x).$$

Similarly, when $b = 2$ and $c = -1$ then v_ν reduces to $x \mapsto (2/\sqrt{\pi})i_\nu(x)$, where

$$x \mapsto i_\nu(x) = \sqrt{\pi/(2x)}I_{\nu+1/2}(x)$$

is the modified spherical Bessel function of the first kind [1, p. 443] and λ_ν in this case becomes

$$x \mapsto \mathcal{I}_{\nu+1/2}(x) = 2^{\nu+1/2}\Gamma(\nu+3/2)x^{-(\nu+1/2)}I_{\nu+1/2}(x).$$

If we choose $b = 2$ and $c = \pm 1$ in Theorem 3 then we obtain the corresponding results to Theorems 1 and 2 for spherical and modified spherical Bessel functions of the first kind.

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