Publ. Math. Debrecen
74/1-2 (2009), 135–157

Constant Jacobi osculating rank of a g.o. space. A method to obtain explicitly the Jacobi operator

By TERESA ARIAS-MARCO (Badajoz) and ANTONIO M. NAVEIRA (Valencia)

Dedicated to Professor Oldřich Kowalski on the occasion of his 72th birthday

Abstract. A Riemannian g.o. manifold is a homogeneous Riemannian manifold on which every geodesic is an orbit of a one-parameter group of isometries. The first counter-example of a Riemannian g.o. manifold which is not naturally reductive is Kaplan's six-dimensional example.

In this paper, we study the constant osculating rank of the curvature operator and of the Jacobi operator over g.o. spaces settling the concept of constant Jacobi osculating rank of a Riemannian g.o. space. Moreover, we show that an expression of the Jacobi operator valid for all geodesic of a given g.o. space exists on every Riemannian g.o. space with constant Jacobi osculating rank. In addition, we develop a method to obtain such expression when we work on a g.o. space which is also an H-type group. In particular, we apply this method on Kaplan's example.

Mathematics Subject Classification: 53C30, 22E25, 53C20, 53C21.
Key words and phrases: Riemannian g.o. space, H-type group, osculating rank of the curvature operator, Jacobi osculating rank of a Riemannian g.o. space, Jacobi operator.
The authors' work has been partially supported by D.G.I. (Spain) and FEDER Project MTM 2007-65852, the network MTM2006-27480-E/ and by a grant ACOMP07/088 from Agencia Valenciana de Ciencia y Tecnología. The first author's work has also been supported by a Predoctoral Research Grant from Programa de FPU del Ministerio de Educación y Ciencia (Spain) and by DFG Sonderforschungsbereich 647 (Germany).

1. Introduction

It is well-known (cf. [15, chapter X, Sections 2, 3]) that a Riemannian homogeneous space $(M,g) = G/H$ with its origin $p = \{H\}$ and with an ad(H)-invariant decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ is *naturally reductive* (with respect to this decomposition) if and only if the following holds:

$$\begin{gathered}\text{For any vector } X \in \mathfrak{m}\backslash\{0\}, \text{ the curve } \gamma(t) = \tau(\exp tX)(p) \\ \text{is a geodesic with respect to the Riemannian connection}\end{gathered} \tag{1}$$

where exp and $\tau(h)$ denote the Lie exponential map of G and the left transformation of G/H induced by $h \in G$ respectively. Moreover, the naturally reductive spaces have been studied by a number of authors as a natural generalization of Riemannian symmetric spaces.

Now, natural reductivity is still a special case of a more general property, which follows easily from (1):

$$\begin{gathered}\text{Each geodesic of } (M,g) = G/H, \text{ is an orbit of a} \\ \text{one-parameter group of isometries } \{\exp tZ\}, \ Z \in \mathfrak{g}.\end{gathered} \tag{2}$$

Riemannian homogeneous spaces $(M,g) = G/H$ with the property (2) will be called (Riemannian) g.o. spaces.

The extensive study of g.o. spaces only started with A. Kaplan's paper [14] in 1983, because he gave the first counter-example of a Riemannian g.o. manifold which is not naturally reductive. This is a six-dimensional Riemannian nilmanifold with a two-dimensional center, one of the so-called "generalized Heisenberg groups" or "H-type groups". Subsequently, the class of generalized Heisenberg groups has provided a large number of further counter-examples. (See [20], [6]). A classification of all g.o. spaces in dimension not greater than six is given by O. Kowalski and L. Vanhecke in [17]. All Riemannian g.o. manifolds of dimension ≤ 5 are proved to be naturally reductive. In dimension 6, new examples of g.o. spaces are given which are in no way naturally reductive. Moreover, in 2004, the first known 7-dimensional compact examples of Riemannian g.o. manifolds which are not naturally reductive were found by Z. Dušek, O. Kowalski, and S. Ž. Nikčević in [11]. For more information on the relation between naturally reductive spaces and g.o. spaces, and also for the references to related topics, see [1], [2], [11], [12], [16] and [17].

Let $c : I \to \mathbb{R}^n$ be a curve defined on an open interval I of $\mathbb{R}$ into $\mathbb{R}^n$. From the classical point of view, we say that $c(t)$ has *constant osculating rank r* if, for all $t \in I$, its higher order derivatives $c^{1)}(t), \ldots, c^{r)}(t)$ are linearly independent and $c^{1)}(t), \ldots, c^{r+1)}(t)$ are linearly dependent in $\mathbb{R}^n$.

On the other hand, K. Tsukada in [23] gave a criterion for the existence of totally geodesic submanifolds of naturally reductive spaces. That criterion is based on the curvature tensor and on a finite number of its derivatives with respect to the Levi–Civita connection. In particular, to prove that result he used two basic formulae proved exclusively for naturally reductive spaces by K. Tojo in [21]. From those formulae he obtained that the curvature tensor can be considered as a curve in the space of curvature tensors on $\mathfrak{m}$. Later, using the general theory, he pointed out that the curvature tensor has constant osculating rank, $r \in \mathbb{N}$, over naturally reductive spaces.

Some years later, K. Tsukada's result was applied by the second author and A. Tarrío in [19] to give a method for solving the Jacobi equation $Y'' + \mathcal{J}Y = 0$ on the naturally reductive manifold $V_1 = Sp(2)/SU(2)$. Here, $\mathcal{J}$ denotes the Jacobi operator. Given the generality of the method, the authors conjectured that it could also be applied to solve the Jacobi equation in several other examples of naturally reductive homogeneous spaces. Indeed, they were not wrong because this method has been successfully applied by the first author and S. Bartoll on the manifold $M^6 = U(3)/(U(1) \times U(1) \times U(1))$ in [4], [5] and by E. Macías-Virgós, the second author and A. Tarrío on the Wilking manifold $V_3 = (SO(3) \times SU(3))/\dot{U}(2)$, endowed with a particular bi-invariant metric, in [18].

In this paper, we study the constant osculating rank of the curvature operator and of the Jacobi operator over g.o. spaces. In Section 2, we settle the concepts of *Jacobi osculating rank* and *constant Jacobi osculating rank* of a Riemannian g.o. space. Moreover, we show that *every g.o. space has Jacobi osculating rank* and *on every Riemannian g.o. space with constant Jacobi osculating rank exists an expression of the Jacobi operator valid for all geodesic of the given g.o. space.* The aim of the last section is to develop a method to obtain such expression when we work on a g.o. space which is also an H-type group. In particular, we divide the last section in two subsections. In the first one, we recall some basic definitions and results regarding "H-type groups". In addition, we give a recursive expression for the n^{th} covariant derivative of the Jacobi operator at the origin of an H-type group. In the last subsection, we obtain that the Jacobi osculating rank of Kaplan's example is 4 and it is also constant because we proof that

$$\frac{1}{4}\mathcal{J}_0^{1)} + \frac{5}{4}\mathcal{J}_0^{3)} + \mathcal{J}_0^{5)} = 0.$$

Here, $\mathcal{J}_0^{n)}$ denotes the n^{th} covariant derivative of the Jacobi operator along an arbitrary geodesic at the origin.

Finally, we use this information to calculate the explicit expression of $\mathcal{J}_t$, the Jacobi operator along *anyone* geodesic, on Kaplan's example. That is

$$\mathcal{J}_t = c_0 + c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(t/2) + c_4 \sin(t/2),$$

where the coefficients c_0, c_1, c_2, c_3, c_4 are linear combinations of $\mathcal{J}_0$, $\mathcal{J}_0^{1)}$, $\mathcal{J}_0^{2)}$, $\mathcal{J}_0^{3)}$, $\mathcal{J}_0^{4)}$.

2. Jacobi osculating rank of a g.o. space

Let $G \subset I(M)$ be a connected Lie group which acts transitively on a Riemannian manifold M and let $p \in M$ be a fixed point. If we denote by H the isotropy group at p, then M can be identified with the homogeneous manifold G/H. In general, there may be more than one such group $G \subset I(M)$. If, for example, we take a connected Lie group G' such that $G \neq G' \subset I(M)$ and G' also acts transitively on M, then there is another expression of M as G'/H' (where H' is the new isotropy group).

For any fixed choice $M = G/H$, G acts effectively on G/H from the left. The Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is then called a *Riemannian homogeneous space.* Such space is always a *reductive homogeneous space* in the following sense (cf. [15, vol.II, p. 190]): we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad} : H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There is a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi : G \to G/H = M$. Using this natural identification and the scalar product g_p on T_pM, we obtain a scalar product $\langle\ ,\ \rangle$ on $\mathfrak{m}$ which is obviously $\mathrm{Ad}(H)$-invariant.

Definition 2.1. A Riemannian homogeneous space $(G/H, g)$ is called a g.o. space if each geodesic of $(G/H, g)$ (with respect to the Riemannian connection) is an orbit of a one-parameter subgroup $\{\exp(tZ)\}$, $Z \in \mathfrak{g}$, of the group of isometries G.

A homogeneous Riemannian manifold (M, g) is called a Riemannian g.o. manifold if each geodesic of (M, g) is an orbit of a one-parameter group of isometries.

Definition 2.2. Let $(G/H, g)$ be a g.o. space. A vector $Z \in \mathfrak{g}$ is called a geodesic vector if the curve $\tau(\exp(tZ))(p)$ is a geodesic. Here $\tau(h)$ denotes the left transformation of G/H induced by $h \in G$.

Let $Z \in \mathfrak{g}$, we denote by Z^* the corresponding fundamental vector field on M; that is

$$Z_q^* = \frac{d}{dt}_{|0}(\tau(\exp(tZ))(q))$$

for each $q \in M$. Moreover, it is easy to see from Definition 2.1 (cf. [17]) that

Proposition 2.1. *G/H is a Riemannian g.o. space if and only if the projections of all geodesic vectors fill in the set $T_pM\backslash\{0\}$.*

Now, we easily generalize to Riemannian g.o. spaces two important results over naturally reductive spaces proved by K. Tojo and K. Tsukada in [21] and [23], respectively.

Let (M, g) be a Riemannian g.o. space and $Z \in \mathfrak{g}$ be an arbitrary geodesic vector. We put

$$\mathrm{e}^{-\nabla_X} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}\nabla_X^l, \quad X \in \mathfrak{m}.$$

Over g.o. spaces, ∇ denotes the Levi–Civita connection, so obviously the linear map ∇_X is skew-symmetric (i.e. $\nabla_X g{=}0$ for all $X \in \mathfrak{m}$). Therefore, the mapping $\mathrm{e}^{-\nabla_X} : (\mathfrak{m}, \langle\ ,\ \rangle) \to (\mathfrak{m}, \langle\ ,\ \rangle)$ is an isometry, $\mathrm{e}^{-\nabla_{tx}}$ is a one-parameter subgroup of the linear isometric transformation group $SO(\mathfrak{m})$ and we can obtain the following lemmas.

Lemma 2.1. *The parallel translation along the geodesic*

$$\gamma_x(t) = \tau(\exp(tZ))(p) \quad \textit{with} \quad \gamma_x(0) = p, \quad \gamma_x'(0) = x = Z_p^*$$

is given by

$$\tau(\exp(tZ))_* \circ \mathrm{e}^{-\nabla_{tx}} : T_pM(=\mathfrak{m}) \longrightarrow T_{\gamma_x(t)}M.$$

Proof. Since (M, g) is a g.o. space, we can take

$$Y(t) = \tau(\exp(tZ))_*(\mathrm{e}^{-\nabla_{tx}}(y))$$

as a vector field along the geodesic $\gamma_x(t) = \tau(\exp(tZ))(p)$ such that $Y(0) = y \in \mathfrak{m}$.

From Proposition 1.2 of Chapter VI of [15], we have

$$\begin{aligned}\nabla_{\gamma'(t)}Y(t) &= \nabla_{\gamma'(t)}(\tau(\exp(tZ))_*(\mathrm{e}^{-\nabla_{tx}}(y)))\\ &= \tau(\exp(tZ))_*(\nabla_x \circ (\mathrm{e}^{-\nabla_{tx}}(y))) + \tau(\exp(tZ))_*\left(\frac{d}{dt}(\mathrm{e}^{-\nabla_{tx}}(y))\right)\end{aligned}$$

$$= \tau(\exp(tZ))_* \left(\nabla_x \circ \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} t^l \nabla_x^l(y) + \sum_{l=1}^{\infty} \frac{(-1)^l}{l-1!} t^{(l-1)} \nabla_x^l(y) \right) = 0. \qquad \square$$

Let $\mathcal{R}$ be the curvature tensor defined by

$$\mathcal{R}(U,V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]},$$

where U and V are vector fields on M and let P_{t_0x} denote the parallel transport with respect to ∇ along the geodesic $\gamma_x(t) = \tau(\exp(tZ))(p)$ from p to $\gamma_x(t_0)$. We now define a $(1,3)$-tensor $\mathcal{R}_x(t)$ on T_pM and its n^{th} covariant derivative along $\gamma_x(t)$ as follows:

$$\mathcal{R}_x^{n)}(t)(u,v)w = P_{tx}^{-1} \circ \mathcal{R}_{\gamma_x(t)}^{n)}(P_{tx}u, P_{tx}v)P_{tx}w$$

for $u, v, w \in T_pM$. Here, we denote $\nabla^n_{\dot\gamma_x(t)}\mathcal{R}$ by $\mathcal{R}^{n)}_{\gamma_x(t)}$ and $\mathcal{R}_x(t)$ by $\mathcal{R}_x^{0)}(t)$.

From now on, we denote $\mathcal{R}_x^{n)}(0)$ by $\mathcal{R}_0^{n)}$ for brevity. Moreover, note that $\mathcal{R}_0^{n)}(u,v)w = \nabla_x^n\mathcal{R}(u,v)w$ for u, v, $w \in T_pM$, i.e.; $\mathcal{R}_0^{n)}$ also denotes the n^{th} covariant derivative of the curvature tensor along $\gamma_x(t)$ at the origin $p = \gamma_x(0)$.

Lemma 2.2. *The $(1,3)$-tensor $\mathcal{R}_x^{n)}(t)$ on $\mathfrak{m}$ obtained by the parallel translation of the n^{th} covariant derivative of the curvature tensor along $\gamma_x(t)$ is given by*

$$\mathcal{R}_x^{n)}(t) = \mathrm{e}^{\nabla_{tx}} \cdot \mathcal{R}_0^{n)} \tag{3}$$

where $x = Z_p^*$ and the dot indicates the action of $\mathrm{e}^{\nabla_{tx}}$ on the space $\mathcal{R}(\mathfrak{m})$ of curvature tensors on $\mathfrak{m}$.

Therefore, $\mathcal{R}_x(t)$ is in the orbit of $\mathcal{R}_0$ with respect to the linear isometric transformation group $SO(\mathfrak{m})$ and $\dim(\mathcal{R}_x(t)) \le \frac{n(n-1)}{2}$.

Proof. We denote briefly $dg_t = \tau(\exp(tZ))_*$. From Lemma 2.1 and due to the fact that $\mathrm{e}^{\nabla_{tx}}$ is an isometry, we have

$$\left\langle \mathcal{R}_x^{n)}(t)(u,v)w, \xi \right\rangle = \left\langle P_{tx}^{-1} \circ \mathcal{R}^{n)}_{\gamma_x(t)}(P_{tx}u, P_{tx}v)P_{tx}w, \xi \right\rangle$$
$$= \left\langle \mathrm{e}^{\nabla_{tx}} \circ (dg_t)^{-1} \circ \mathcal{R}^{n)}_{\gamma_x(t)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u), (dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w), \xi \right\rangle$$
$$= \left\langle (dg_t)^{-1} \circ \mathcal{R}^{n)}_{\gamma_x(t)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u), (dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w), \mathrm{e}^{\nabla_{-tx}}(\xi) \right\rangle$$
$$= g(\mathcal{R}^{n)}_{\gamma_x(t)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u), (dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w), (dg_t)\mathrm{e}^{-\nabla_{tx}}(\xi))$$
$$= g((\nabla^n_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}\mathcal{R})((dg_t)\mathrm{e}^{-\nabla_{tx}}(u), (dg_t)\mathrm{e}^{-\nabla_{tx}}(v))(dg_t)\mathrm{e}^{-\nabla_{tx}}(w), (dg_t)\mathrm{e}^{-\nabla_{tx}}(\xi))$$
$$= g((dg_t)\mathcal{R}_0^{n)}(\mathrm{e}^{-\nabla_{tx}}(u), \mathrm{e}^{-\nabla_{tx}}(v))\mathrm{e}^{-\nabla_{tx}}(w), (dg_t)\mathrm{e}^{-\nabla_{tx}}(\xi))$$
$$= \left\langle \mathrm{e}^{\nabla_{tx}}\mathcal{R}_0^{n)}(\mathrm{e}^{-\nabla_{tx}}(u), \mathrm{e}^{-\nabla_{tx}}(v))\mathrm{e}^{-\nabla_{tx}}(w), \xi \right\rangle. \qquad \square$$

Moreover for each $u \in \mathfrak{m}$, we denote by $\mathcal{R}(u)$ the smallest subspace of $\mathcal{R}(\mathfrak{m})$ which satisfies $\mathcal{R}_0 \in \mathcal{R}(u)$ and $\nabla_u \cdot \mathcal{R}(u) \subset \mathcal{R}(u)$. We define d_u by $d_u = \dim \mathcal{R}(u)$. Trivially, due to every element of $\mathcal{R}(u)$ belong to the orbit of $\mathcal{R}_0$ with respect to the linear transformation group $GL(\mathfrak{m})$ we have $d_u \leq n^2$.

We now recall some fundamental facts about a curve in the n-dimensional Euclidean space $\mathbb{R}^n$. Let $c : I \to \mathbb{R}^n$ be a curve defined on an open interval I of $\mathbb{R}$ into $\mathbb{R}^n$. We say that $c(t)$ has *constant osculating rank* r if, for all $t \in I$, its higher order derivatives $c^{1)}(t), \dots, c^{r)}(t)$ are linearly independent and $c^{1)}(t), \dots, c^{r+1)}(t)$ are linearly dependent in $\mathbb{R}^n$. It is a fundamental fact that if $c(t)$ has constant osculating rank r, there are smooth functions $a_1, \dots, a_r : I \to \mathbb{R}$ such that

$$c(t) = c(0) + a_1(t)c^{1)}(0) + \cdots + a_r(t)c^{r)}(0) \quad \text{for all } t \in I.$$

Moreover, P. Dombrowski proved in [9] the following property:

Proposition 2.2. *Let S denote a connected m-dimensional real analytic manifold and $f : S \to \mathbb{R}^n$ a real analytic map of S into $\mathbb{R}^n$, let ∇ denote the Levi–Civita connection of $\mathbb{R}^n$. Then, $f : S \to \mathbb{R}^n$ has constant osculating rank.*

Let us return to a g.o. space M. Let $U(t)$, $V(t)$, $W(t)$ be the vector fields along the geodesic $\gamma_x(t)$, arbitrary but fixed, such that $U(0) = u$, $V(0) = v$, $W(0) = w$ are unit vectors in $\mathfrak{m}$. Since $\mathrm{e}^{\nabla_{tx}}$ is a 1-parameter subgroup of the group of linear isometries of $\mathcal{R}(\mathfrak{m})$, it follows that $\mathcal{R}_x(t)$, or more explicitly

$$\mathcal{R}_x(t)(u,v)w = \mathrm{e}^{\nabla_{tx}} \cdot \mathcal{R}_0(\mathrm{e}^{\nabla_{-tx}}u, \mathrm{e}^{\nabla_{-tx}}v)\mathrm{e}^{\nabla_{-tx}}w,$$

is a curve in $\mathcal{R}(\mathfrak{m})$. Moreover, if S is an open interval I of the real space $\mathbb{R}$ and we use the natural identification of $\mathbb{R}^n$ with $\mathfrak{m}$, Proposition 2.2 yields that $\mathcal{R}_x(t)$ has constant osculating rank. Therefore, there is a number $r_\gamma \in \mathbb{N}$ and there are smooth functions $a_1, \dots, a_{r_\gamma} : I \to \mathbb{R}$ for every fixed but arbitrary geodesic $\gamma_x(t)$ such that

$$\mathcal{R}_x(t) = \mathcal{R}_0 + a_1(t)\mathcal{R}_0^{1)} + \cdots + a_{r_\gamma}(t)\mathcal{R}_0^{r_\gamma)} \quad \text{for all } t \in I. \tag{4}$$

Furthermore, by the classical constant osculating rank definition there are $\alpha_1, \dots, \alpha_{r_\gamma}$ constants depending of the fixed geodesic such that

$$\alpha_1 \mathcal{R}_x^{1)}(t) + \cdots + \alpha_{r_\gamma}\mathcal{R}_x^{r_\gamma)}(t) + \mathcal{R}_x^{r_\gamma+1)}(t) = 0. \tag{5}$$

Thus, since $\mathcal{R}_0^{i)} = \nabla_x^i \cdot \mathcal{R}_0$, for all $i \in \mathbb{N}$, $\mathcal{R}(x)$ coincides with the subspace of $\mathcal{R}(\mathfrak{m})$ spanned by $\mathcal{R}_0$, $\nabla_x \cdot \mathcal{R}_0, \dots, \nabla_x^{r_\gamma)} \cdot \mathcal{R}_0$. In particular, we have $r_\gamma = d_x - 1$ or d_x.

On the other hand, note that if we know (5), using the general theory about ordinary differential equations we can determine the functions $a_i(t)$, $i = 1, \ldots, r_\gamma$, of (4). We only need to use the next result where $Q(y) = y^{r_\gamma} + \alpha_{r_\gamma} y^{r_\gamma - 1} + \cdots + \alpha_1 y$ is the characteristic polynomial of (5).

Proposition 2.3. *Let (M, g) be a Riemannian g.o. space and let r_γ be the constant osculating rank of the curvature operator $\mathcal{R}_x(t)$ for an arbitrary geodesic $\gamma_x(t)$. Then, $\mathcal{R}_x(t)$ can be written as a matrix-linear combination of the following functions: for each λ, real root of $Q(y)$ with multiplicity k, the functions $e^{\lambda t}$, $te^{\lambda t}, \ldots, t^{k-1}e^{\lambda t}$ and, for each $\sigma \pm \imath\tau$ $(\tau \neq 0)$, conjugate complex roots of $Q(y)$ with multiplicity h, the functions $e^{\sigma t}\cos(\tau t)$, $e^{\sigma t}\sin(\tau t)$, $te^{\sigma t}\cos(\tau t)$, $te^{\sigma t}\sin(\tau t), \ldots, t^{h-1}e^{\sigma t}\cos(\tau t)$, $t^{h-1}e^{\sigma t}\sin(\tau t)$. Moreover, the coefficients of such a matrix-linear combination are linear combination of $\mathcal{R}_0$, $\mathcal{R}_0^{1)}, \ldots, \mathcal{R}_0^{r_\gamma)}$.*

A useful technique to describe the curvature along a geodesic γ in a Riemannian manifold (M, g), with Riemannian curvature tensor $\mathcal{R}$, is the use of the Jacobi operator $\mathcal{J}(\cdot) = \mathcal{R}(\cdot, \dot\gamma)\dot\gamma$. $\mathcal{J}$ determines a self-adjoint tensor field along γ.

In particular, from Lemma 2.2 the Jacobi operator on $\mathfrak{m}$ and its n^{th} covariant derivative along the geodesic $\gamma_x(t)$ are given by

$$\begin{aligned}
\mathcal{J}_x^{n)}(t)(u) &= \mathcal{R}_x^{n)}(t)(u, x)x = P_{tx}^{-1} \circ \mathcal{R}^{n)}_{\gamma_x(t)}(P_{tx}u, P_{tx}x)P_{tx}x \\
&= P_{tx}^{-1} \circ (\nabla^n_{\dot\gamma_x(t)}\mathcal{R})(P_{tx}u, \dot\gamma_x(t))\dot\gamma_x(t) \\
&= P_{tx}^{-1} \circ \mathcal{J}^{n)}(P_{tx}u) = \mathrm{e}^{\nabla_{tx}}\mathcal{J}_0^{n)}(\mathrm{e}^{-\nabla_{tx}}(u))
\end{aligned} \tag{6}$$

for $u \in \mathfrak{m}$. Obviously, if $t = 0$, $\mathcal{J}_0^{n)}(u) = (\nabla_x^n\mathcal{R})(u, x)x$, where, as before, $\mathcal{J}_0^{n)}$ denotes $\mathcal{J}_x^{n)}(0)$. In addition, we will write throughout the paper $\mathcal{J}_t^{n)}$ instead of $\mathcal{J}_x^{n)}(t)$ to shorten notation.

It is obvious that the property (3) can also be written as

$$\mathcal{R}_x^{n)}(t) = \mathrm{e}^{\nabla_{tx}} \cdot (\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}\mathcal{R}_0^{n-1)}). \tag{7}$$

Therefore, (6) becomes

$$\mathcal{J}_t^{n)}(u) = \mathrm{e}^{\nabla_{tx}}(\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}\mathcal{J}_0^{n-1)})(\mathrm{e}^{-\nabla_{tx}}(u)) \tag{8}$$

for $u \in \mathfrak{m}$ and if $t = 0$ we have

$$\mathcal{J}_0^{n)}(u) = (\nabla_x\mathcal{J}_0^{n-1)})(u). \tag{9}$$

The following result is basic to introduce the *Jacobi osculating rank of a g.o. space*.

Lemma 2.3. *The Jacobi operator* $\mathcal{J}_t^{n)}$ *on* $\mathfrak{m}$ *obtained by the parallel translation of the* n^{th} *covariant derivative of the Jacobi operator along* $\gamma_x(t)$ *satisfies the following identity*

$$\mathcal{J}_t^{n)}(u) = \mathrm{e}^{\nabla_{tx}} \nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}_0^{n-1)}(\mathrm{e}^{-\nabla_{tx}}(u))) - \mathrm{e}^{\nabla_{tx}} \mathcal{J}_0^{n-1)}(\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathrm{e}^{-\nabla_{tx}}(u)))$$

for $u \in \mathfrak{m}$ *where* $x = \dot{\gamma}_x(0) = Z_p^*$, $\mathcal{J}_0^{n)}$ *denotes the* n^{th} *covariant derivative of the Jacobi operator along* $\gamma_x(t)$ *at the origin* $p = \gamma_x(0)$ *and the dot indicates the action of* $\mathrm{e}^{\nabla_{tx}}$ *on the space* $\mathcal{R}(\mathfrak{m})$ *of curvature tensors on* $\mathfrak{m}$. *Moreover, in the particular case* $t = 0$, *the identity becomes*

$$\mathcal{J}_0^{n)}(u) = \nabla_x(\mathcal{J}_0^{n-1)}(u)) - \mathcal{J}_0^{n-1)}(\nabla_x u). \tag{10}$$

Proof. We denote briefly $dg_t = \tau(\exp(tZ))_*$. From Lemma 2.1, formula (6), the condition $\nabla_{\dot{\gamma}_x(t)}\dot{\gamma}_x(t) = 0$ and due to the fact that $\mathrm{e}^{\nabla_{tx}}$ is an isometry, we have

$$\begin{aligned}
&\left\langle \mathcal{J}_t^{n)}(u), \xi \right\rangle = \left\langle P_{tx}^{-1} \circ \mathcal{J}^{n)}(P_{tx}u), \xi \right\rangle \\
&= \left\langle P_{tx}^{-1} \circ \nabla_{\dot{\gamma}_x(t)}(\mathcal{J}^{n-1)}(P_{tx}u)), \xi \right\rangle - \left\langle P_{tx}^{-1} \circ \mathcal{J}^{n-1)}(\nabla_{\dot{\gamma}_x(t)}(P_{tx}u)), \xi \right\rangle \\
&= \left\langle \mathrm{e}^{\nabla_{tx}} \circ (dg_t)^{-1} \circ \nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}^{n-1)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), \xi \right\rangle \\
&\quad - \left\langle \mathrm{e}^{\nabla_{tx}} \circ (dg_t)^{-1} \circ \mathcal{J}^{n-1)}(\nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), \xi \right\rangle \\
&= g(\nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}^{n-1)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), (dg_t)\mathrm{e}^{\nabla_{-tx}}\xi) \\
&\quad - g(\mathcal{J}^{n-1)}(\nabla_{(dg_t)\mathrm{e}^{-\nabla_{tx}}(x)}((dg_t)\mathrm{e}^{-\nabla_{tx}}(u))), (dg_t)\mathrm{e}^{\nabla_{-tx}}\xi) \\
&= g((dg_t)\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}_0^{n-1)}(\mathrm{e}^{-\nabla_{tx}}(u))), (dg_t)\mathrm{e}^{\nabla_{-tx}}\xi) \\
&\quad - g((dg_t)\mathcal{J}_0^{n-1)}(\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathrm{e}^{-\nabla_{tx}}(u))), (dg_t)\mathrm{e}^{\nabla_{-tx}}\xi) \\
&= \left\langle \mathrm{e}^{\nabla_{tx}} \nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathcal{J}_0^{n-1)}(\mathrm{e}^{-\nabla_{tx}}(u))) - \mathrm{e}^{\nabla_{tx}} \mathcal{J}_0^{n-1)}(\nabla_{\mathrm{e}^{-\nabla_{tx}}(x)}(\mathrm{e}^{-\nabla_{tx}}(u))), \xi \right\rangle. \quad \square
\end{aligned}$$

It is obvious on every g.o. space that for every fixed but arbitrary geodesic $\gamma_x(t)$ the associated Jacobi operator, $\mathcal{J}_t$, has also constant osculating rank in the classical sense. Therefore, we can rewrite (4) and (5) substituting the curvature operator by the Jacobi operator. In particular at $t = 0$, we will always have $\alpha_1, \ldots, \alpha_{r_\gamma}$, constants depending of the fixed geodesic, such that

$$\alpha_1 \mathcal{J}_0^{1)} + \cdots + \alpha_{r_\gamma} \mathcal{J}_0^{r_\gamma)} + \mathcal{J}_0^{r_\gamma+1)} = 0. \tag{11}$$

Proposition 2.4. *If* (11) *is known for a fixed geodesic on a g.o. space, then it exists a relation between the first* $\mathtt{r}+1$ *covariant derivatives of the Jacobi operator at* $t = 0$ *for every* $\mathtt{r} \geq r_\gamma$.

Proof. We derive $(\mathtt{r}-r_\gamma)$-times the relation (11) using (9) and (10). Thus, we obtain

$$\alpha_1\mathcal{J}_0^{1+m)}+\cdots+\alpha_{r_\gamma}\mathcal{J}_0^{r_\gamma+m)}+\mathcal{J}_0^{r_\gamma+1+m)}=0 \quad \text{for } m=1,\dots,\mathtt{r}-r_\gamma.$$

Finally, adding the previous relations we find $\beta_1,\dots,\beta_{\mathtt{r}}$ constants depending of γ such that

$$\beta_1\mathcal{J}_0^{1)}+\cdots+\beta_{\mathtt{r}}\mathcal{J}_0^{\mathtt{r})}+\mathcal{J}_0^{\mathtt{r}+1)}=0 \quad \text{for } \mathtt{r}\geq r_\gamma. \tag{12}$$

□

Therefore, if we put $\mathtt{r}=\max\{r_\gamma : \text{for all } \gamma \text{ geodesic of a given g.o. space}\}$, we can *always* find a relation of type (12) on every geodesic of a given g.o. space. From now on, we say that $\mathtt{r}\in\mathbb{N}$ is the *Jacobi osculating rank* of a given g.o. space. Of course, $\mathtt{r}\leq n^2$ and the constants of each relation of type (12) still depend of the corresponding geodesic.

Lemma 2.4. *Let (M,g) be a g.o. space with Jacobi osculating rank $\mathtt{r}$. Let $\gamma_x(t)$ a geodesic and $\beta_1,\dots,\beta_{\mathtt{r}}$ constants depending of γ such that $\beta_1\mathcal{J}_0^{1)}+\cdots+\beta_{\mathtt{r}}\mathcal{J}_0^{\mathtt{r})}+\mathcal{J}_0^{\mathtt{r}+1)}=0$. Then,*

$$\beta_1\mathcal{J}_0^{k+1)}+\cdots+\beta_{\mathtt{r}}\mathcal{J}_0^{k+\mathtt{r})}+\mathcal{J}_0^{k+\mathtt{r}+1)}=0 \quad \textit{for } k=0,1,2,\dots \tag{13}$$

Proof. Let us assume that (13) is true for $k=i$ and we will prove the result for $k=i+1$. Using (9) and (10) we have

$$\begin{aligned}\mathcal{J}_0^{i+\mathtt{r}+2)}&=\nabla_x\mathcal{J}_0^{i+\mathtt{r}+1)}=-\nabla_x\left(\beta_1\mathcal{J}_0^{i+1)}+\cdots+\beta_{\mathtt{r}}\mathcal{J}_0^{i+\mathtt{r})}\right)\\&=-\beta_1\nabla_x\mathcal{J}_0^{i+1)}-\cdots-\beta_{\mathtt{r}}\nabla_x\mathcal{J}_0^{i+\mathtt{r})}=-\beta_1\mathcal{J}_0^{i+2)}-\cdots-\beta_{\mathtt{r}}\mathcal{J}_0^{i+1+\mathtt{r})}.\end{aligned}$$ □

Proposition 2.5. *Let (M,g) be a g.o. space with Jacobi osculating rank $\mathtt{r}$. Let $\gamma_x(t):I\to M$ a geodesic and $\beta_1,\dots,\beta_{\mathtt{r}}$ constants depending of γ such that $\beta_1\mathcal{J}_0^{1)}+\cdots+\beta_{\mathtt{r}}\mathcal{J}_0^{\mathtt{r})}+\mathcal{J}_0^{\mathtt{r}+1)}=0$. Then,*

$$\beta_1\mathcal{J}_t^{1)}+\cdots+\beta_{\mathtt{r}}\mathcal{J}_t^{\mathtt{r})}+\mathcal{J}_t^{\mathtt{r}+1)}=0 \quad \textit{for all } t\in I. \tag{14}$$

Moreover by Proposition 2.3, we can determine smooth functions $a_1,\dots,a_{\mathtt{r}}:I\to\mathbb{R}$ to obtain the Jacobi operator along $\gamma_x(t)$. More explicitly, we obtain that

$$\mathcal{J}_t=\mathcal{J}_0+a_1(t)\mathcal{J}_0^{1)}+\cdots+a_{\mathtt{r}}(t)\mathcal{J}_0^{\mathtt{r})}. \tag{15}$$

PROOF. Using the expansion in Taylor's series of the Jacobi operator $\mathcal{J}_t$, it is clear that

$$\mathcal{J}_t^{n)} = \sum_{i=n}^{\infty} \frac{t^{i-n}}{(i-n)!} \mathcal{J}_0^{i)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{J}_0^{k+n)} \quad \text{for } n = 1, \dots, \mathtt{r}.$$

Therefore, we conclude by the previous Lemma that

$$\beta_1 \mathcal{J}_t^{1)} + \dots + \beta_r \mathcal{J}_t^{\mathtt{r})} + \mathcal{J}_t^{\mathtt{r}+1)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\beta_1 \mathcal{J}_0^{k+1)} + \dots + \beta_{\mathtt{r}} \mathcal{J}_0^{k+\mathtt{r})} + \mathcal{J}_0^{k+\mathtt{r}+1)} \right) = 0.$$

□

Finally, note that if the constants β_i, $i = 1, \dots, \mathtt{r}$, of (14) do *not depend* of the geodesic then we can find an expression of type (14) valid *for all* geodesic of a given g.o. space. Moreover, we obtain an expression of type (15) for the Jacobi operator valid *for all* geodesic of a given g.o. space. When this happens, we say that the given g.o. space has *constant Jacobi osculating rank*.

Remark 2.1. Although a g.o. space has constant Jacobi osculating rank, the operator $\mathcal{J}_t$ with respect to every fixed geodesic $\gamma_x(t)$ has still its own (as a curve) constant osculating rank r_γ that could be less than or equal to the Jacobi osculating rank $\mathtt{r}$ of the g.o. space.

It is clear by the previous results that every g.o. space has Jacobi osculating rank. Now, the question is if every g.o. space has also constant Jacobi osculating rank. In the next section we will prove that Kaplan example is a g.o. space with constant Jacobi osculating rank and in a forthcoming paper we will provide an example of a g.o. space with non-constant Jacobi osculating rank.

On the other hand, note that every g.o. space with constant Jacobi osculating rank $\mathtt{r}$ satisfies a relation of type (14) between the first $\mathtt{r}+1$ covariant derivatives of the Jacobi operator along anyone geodesic. These relations can be compared with the relation $\nabla\mathcal{R} = 0$ that symmetric spaces satisfy.

3. Method to obtain the Jacobi osculating rank of a g.o. space of *H*-type

In the previous section, we showed that on every Riemannian g.o. space with constant Jacobi osculating rank exists an expression of the Jacobi operator valid for all geodesic of a given g.o. space. The aim of this section is to develop a method to obtain such expression when we work on a g.o. space which is also an H-type group. In particular, we will apply this method on Kaplan's example.

We divide this section in two subsections. In the first one, we will recall some basic definitions and results regarding "H-type groups". In addition, we will give a recursive expression for the n^{th} covariant derivative of the Jacobi operator at the origin of an H-type group. In the last subsection, we will calculate the Jacobi osculating rank of Kaplan's example and we will check that Kaplan's example has constant Jacobi osculating rank. Finally, we will use it to obtain the explicit expression of $\mathcal{J}_t$, the Jacobi operator along *anyone* geodesic, on Kaplan's example.

The computer support, for example using the software MATHEMATICA 6.0, can be so useful in Section 3.2 to obtain the objectives without loss the transparency of the calculations.

3.1. Preliminaries about H-type groups.

Definition 3.1. Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with an inner product $\langle\ ,\ \rangle$. Let $\mathfrak{z}$ be the center of $\mathfrak{n}$ and let $\mathfrak{v}$ be its orthogonal complement. For each vector $A \in \mathfrak{z}$, the operator $j(A) : \mathfrak{v} \to \mathfrak{v}$ is defined by the relation

$$\langle j(A)X, Y\rangle = \langle A, [X, Y]\rangle \quad \text{for all } X, Y \in \mathfrak{v}. \tag{16}$$

The algebra $\mathfrak{n}$ is called a generalized Heisenberg algebra (H-type algebra) if, for each $A \in \mathfrak{z}$, the operator $j(A)$ satisfies the identity

$$j(A)^2 = -|A|^2 Id_{\mathfrak{v}} \tag{17}$$

where $|\ \ |^2$ denotes the quadratic form of the inner product $\langle\ ,\ \rangle$. A connected, simply connected Lie group whose Lie algebra is an H-type algebra is diffeomorphic to $\mathbb{R}^n$ and it is called an H-type group. It is endowed with a left-invariant metric.

In particular, the Lie algebra structure on $\mathfrak{n}$ is defined by extending the skew-symmetric bilinear map $[\ ,\ ] : \mathfrak{v} \times \mathfrak{v} \to \mathfrak{z}$ to a bracket $[A + X, B + Y] = [X, Y]$ where $A, B \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$. Moreover, it is well-known that

$$[X, j(A)X] = A|X|^2 \tag{18}$$

for all $A \in \mathfrak{z}$ and $X \in \mathfrak{v}$. (See [13], [6, p. 24]).

H-type groups were intrinsically described in [13] and [6, p. 28]. They obtained that the Riemannian connection is given by

$$\begin{aligned} \nabla_X Y &= \frac{1}{2}[X, Y], \\ \nabla_A X &= \nabla_X A = -\frac{1}{2} j(A)X, \\ \nabla_A B &= 0, \end{aligned} \tag{19}$$

where $A, B \in \mathfrak{z}$ and $X, Y \in \mathfrak{v}$. A straightforward computation now shows that the Riemannian curvature tensor is given by

$$\begin{aligned}
\mathcal{R}(X,Y)Z &= \frac{1}{4}(2j([X,Y])Z - j([Y,Z])X - j([Z,X])Y),\\
\mathcal{R}(X,Y)A &= \frac{1}{4}([Y,j(A)X] - [X,j(A)Y]),\\
\mathcal{R}(X,A)Y &= -\frac{1}{4}[X,j(A)Y],\\
\mathcal{R}(X,A)B &= -\frac{1}{4}j(A)j(B)X,\\
\mathcal{R}(A,B)X &= \frac{1}{4}(j(A)j(B)X - j(B)j(A)X),\\
\mathcal{R}(A,B)C &= 0,
\end{aligned} \tag{20}$$

where $A, B, C \in \mathfrak{z}$ and $X, Y, Z \in \mathfrak{v}$.

Moreover, a geodesic, $t \to \gamma(t)$, through the origin p of an H-type group is described by means of two vector-valued functions $t \to X(t) \in \mathfrak{v}$, $t \to A(t) \in \mathfrak{z}$ as follows: $\gamma(t) = \exp(X(t) + A(t))$ such that $X(0) = 0$, $A(0) = 0$ and the unit tangent vector of γ at the origin p is given by $\dot\gamma_0 = \dot X_0 + \dot A_0$ where $\dot f_0$ denotes $(df/dt)_{t=0}$ of any real or vector-valued function $f(t)$. (See [13], [14], [6, p. 30] and [22] for more detailed results about geodesics over H-type groups).

Now we define the mappings $\zeta_{(n,A)} : \mathfrak{z} \to \mathfrak{z}$, $\nu_{(n,A)} : \mathfrak{z} \to \mathfrak{v}$, $\zeta_{(n,X)} : \mathfrak{v} \to \mathfrak{z}$ and $\nu_{(n,X)} : \mathfrak{v} \to \mathfrak{v}$ in a recurrent way for each $n \in \mathbb{N}$ by

$$\begin{aligned}
\zeta_{(0,A)}(B) &= \frac{1}{4}|\dot X_0|^2 B,\\
\nu_{(0,A)}(B) &= \frac{1}{2}j(B)j(\dot A_0)\dot X_0 - \frac{1}{4}j(\dot A_0)j(B)\dot X_0,\\
\zeta_{(0,X)}(Y) &= \frac{1}{4}[\dot X_0, j(\dot A_0)Y] - \frac{1}{2}[Y, j(\dot A_0)\dot X_0],\\
\nu_{(0,X)}(Y) &= \frac{1}{4}|\dot A_0|^2 Y + \frac{3}{4}j([Y,\dot X_0])\dot X_0,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\zeta_{(n,A)}(B) &= \frac{1}{2}([\dot X_0, \nu_{(n-1,A)}(B)] + \zeta_{(n-1,X)}(j(B)\dot X_0)),\\
\nu_{(n,A)}(B) &= \frac{1}{2}(\nu_{(n-1,X)}(j(B)\dot X_0) - j(\dot A_0)\nu_{(n-1,A)}(B) - j(\zeta_{(n-1,A)}(B))\dot X_0),\\
\zeta_{(n,X)}(Y) &= \frac{1}{2}([\dot X_0, \nu_{(n-1,X)}(Y)] + \zeta_{(n-1,X)}(j(\dot A_0)Y) - \zeta_{(n-1,A)}([\dot X_0, Y])),\\
\nu_{(n,X)}(Y) &= \frac{1}{2}(\nu_{(n-1,X)}(j(\dot A_0)Y) - j(\dot A_0)\nu_{(n-1,X)}(Y)\\
&\quad - j(\zeta_{(n-1,X)}(Y))\dot X_0 - \nu_{(n-1,A)}([\dot X_0, Y])),
\end{aligned} \tag{22}$$

where $B \in \mathfrak{z}$ and $Y \in \mathfrak{v}$.

Proposition 3.1. *The n^{th} covariant derivative of the Jacobi operator at the origin $p = \gamma(0)$ of an H-type group is given by*

$$\mathcal{J}_0^{n)}(B) = \zeta_{(n,A)}(B) + \nu_{(n,A)}(B),$$
$$\mathcal{J}_0^{n)}(Y) = \zeta_{(n,X)}(Y) + \nu_{(n,X)}(Y). \tag{23}$$

where $B \in \mathfrak{z}$ and $Y \in \mathfrak{v}$.

Proof. For $n = 0$, using (20), (18), (17) and (21) we get that

$$\begin{aligned}\mathcal{J}_0(B) &= \mathcal{R}(B, \dot{X}_0)\dot{X}_0 + \mathcal{R}(B, \dot{X}_0)\dot{A}_0 + \mathcal{R}(B, \dot{A}_0)\dot{X}_0 + \mathcal{R}(B, \dot{A}_0)\dot{A}_0 \\ &= \frac{1}{4}|\dot{X}_0|^2 B + \frac{1}{2}j(B)j(\dot{A}_0)\dot{X}_0 - \frac{1}{4}j(\dot{A}_0)j(B)\dot{X}_0 \\ &= \zeta_{(0,A)}(B) + \nu_{(0,A)}(B),\end{aligned}$$

$$\begin{aligned}\mathcal{J}_0(Y) &= \mathcal{R}(Y, \dot{X}_0)\dot{X}_0 + \mathcal{R}(Y, \dot{X}_0)\dot{A}_0 + \mathcal{R}(Y, \dot{A}_0)\dot{X}_0 + \mathcal{R}(Y, \dot{A}_0)\dot{A}_0 \\ &= \frac{1}{4}[\dot{X}_0, j(\dot{A}_0)Y] - \frac{1}{2}[Y, j(\dot{A}_0)\dot{X}_0] + \frac{1}{4}|\dot{A}_0|^2 Y + \frac{3}{4}j([Y, \dot{X}_0])\dot{X}_0 \\ &= \zeta_{(0,X)}(Y) + \nu_{(0,X)}(Y).\end{aligned}$$

Finally, assuming that (23) is true for $n - 1$, we prove the result for n using (10), (19) and (22). This finishes the proof, the detailed verification of the last statement being left to the reader. $\square$

3.2. Kaplan's example. Let $\mathfrak{n}$ be a vector space of dimension 6 equipped with a scalar product and let $\{E_1, E_2, E_3, E_4, E_5, E_6\}$ form an orthonormal basis. The elements E_5 and E_6 span the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{n}$. The structure of a Lie algebra on $\mathfrak{n}$ is given by the following relations:

$$\begin{aligned}&[E_1, E_2] = 0, \\ &[E_1, E_3] = E_5, \qquad [E_2, E_3] = E_6, \\ &[E_1, E_4] = E_6, \qquad [E_2, E_4] = -E_5, \qquad [E_3, E_4] = 0, \\ &[E_k, E_5] = 0, \qquad \text{for } k = 1, \ldots, 4, \\ &[E_k, E_6] = 0, \qquad \text{for } k = 1, \ldots, 4, \qquad [E_5, E_6] = 0.\end{aligned} \tag{24}$$

Moreover, from (16) we easily obtain that

$$\begin{aligned}&j(E_5)E_1 = E_3, \quad j(E_5)E_2 = -E_4, \quad j(E_5)E_3 = -E_1, \quad j(E_5)E_4 = E_2, \\ &j(E_6)E_1 = E_4, \quad j(E_6)E_2 = E_3, \quad j(E_6)E_3 = -E_2, \quad j(E_6)E_4 = -E_1.\end{aligned} \tag{25}$$

The condition (17) for the operators $j(A)$ can be easily verified from (25). Thus, the relation (24) defines an H-type algebra.

The H-type group corresponding to $\mathfrak{n}$ is named *Kaplan's example*. We denote it briefly by N.

Moreover, Z. Dušek in [10] expresses N as an homogeneous space G/H where $H \cong SU(2)$ and $G = N \rtimes H$. Here the group G is not the full isometry group of N, but the group N is a g.o. space with respect to this group.

In the remainder of this section, our purpose is to obtain an expression of type (15). It is mean, we want to determine explicitly the Jacobi operator along an arbitrary geodesic γ with initial vector $x = \dot\gamma(0)$. From now on, we consider that

$$x = \sum_{i=1}^{6} x_i E_i \quad \text{with} \quad |x|^2 = \sum_{i=1}^{6} (x_i)^2 = 1. \tag{26}$$

Thus, using the notation of Section 3.1, $\dot X_0 = \sum_{i=1}^{4} x_i E_i$ and $\dot A_0 = \sum_{\alpha=5}^{6} x_\alpha E_\alpha$. There is no loss of generality in assuming that $x \in \mathfrak{m}$ is a unit vector. Nevertheless, it will be sometimes convenient to ignore it. Furthermore, we denote by $\{Q_i\}$, $i = 1, \ldots, 6$, the orthonormal frame field obtained by parallel translation of the basis $\{E_i\}$, $i = 1, \ldots, 6$, along the geodesic γ.

More precisely, to determine explicitly the Jacobi operator along an arbitrary geodesic, firstly, we have to calculate which is the Jacobi osculating rank $\mathtt{r}$ of N. It is mean, we must first obtain the relation of type (12) that it is satisfied on N.

Let us start with the following technical lemma on N.

Lemma 3.1. *The operator* $[\dot X_0, Y] \in \mathfrak{z}$, $Y \in \mathfrak{v}$, *is given by*

$$\begin{aligned} &[\dot X_0, E_1] = -x_3E_5 - x_4E_6, && [\dot X_0, E_2] = x_4E_5 - x_3E_6, \\ &[\dot X_0, E_3] = x_1E_5 + x_2E_6, && [\dot X_0, E_4] = -x_2E_5 + x_1E_6. \end{aligned} \tag{27}$$

The operator $j(\dot A_0) : \mathfrak{v} \to \mathfrak{v}$ *is defined by*

$$\begin{aligned} &j(\dot A_0)(E_1) = x_5E_3 + x_6E_4, && j(\dot A_0)(E_2) = x_6E_3 - x_5E_4, \\ &j(\dot A_0)(E_3) = -x_5E_1 - x_6E_2, && j(\dot A_0)(E_4) = -x_6E_1 + x_5E_2. \end{aligned} \tag{28}$$

The operator $j(\cdot)(\dot X_0) : \mathfrak{z} \to \mathfrak{v}$ *is given by*

$$\begin{aligned} j(E_5)(\dot X_0) &= -x_3E_1 + x_4E_2 + x_1E_3 - x_2E_4, \\ j(E_6)(\dot X_0) &= -x_4E_1 - x_3E_2 + x_2E_3 + x_1E_4. \end{aligned} \tag{29}$$

Finally, the mappings $\zeta_{(0,X)}$, $\nu_{(0,X)}$, $\zeta_{(0,A)}$, $\nu_{(0,A)}$ *are defined by*

$$\begin{aligned}
\zeta_{(0,X)}(E_1) &= \frac{1}{4}((-x_1x_5 - 3x_2x_6)E_5 + (3x_2x_5 - x_1x_6)E_6),\\
\zeta_{(0,X)}(E_2) &= \frac{1}{4}((-x_2x_5 + 3x_1x_6)E_5 + (-3x_1x_5 - x_2x_6)E_6),\\
\zeta_{(0,X)}(E_3) &= \frac{1}{4}((-x_3x_5 - 3x_4x_6)E_5 + (3x_4x_5 - x_3x_6)E_6),\\
\zeta_{(0,X)}(E_4) &= \frac{1}{4}((-x_4x_5 + 3x_3x_6)E_5 + (-3x_3x_5 - x_4x_6)E_6),\\
\nu_{(0,X)}(E_1) &= \frac{1}{4}((-3(x_3^2 + x_4^2) + x_5^2 + x_6^2)E_1 + 3(x_1x_3 + x_2x_4)E_3\\
&\quad + 3(x_1x_4 - x_2x_3)E_4),\\
\nu_{(0,X)}(E_2) &= \frac{1}{4}((-3(x_3^2 + x_4^2) + x_5^2 + x_6^2)E_2 + 3(x_2x_3 - x_1x_4)E_3\\
&\quad + 3(x_1x_3 + x_2x_4)E_4),\\
\nu_{(0,X)}(E_3) &= \frac{1}{4}(3(x_1x_3 + x_2x_4)E_1 + 3(x_2x_3 - x_1x_4)E_2\\
&\quad + (-3(x_1^2 + x_2^2) + x_5^2 + x_6^2)E_3),\\
\nu_{(0,X)}(E_4) &= \frac{1}{4}(3(x_1x_4 - x_2x_3)E_1 + 3(x_1x_3 + x_2x_4)E_2\\
&\quad + (-3(x_1^2 + x_2^2) + x_5^2 + x_6^2)E_4),\\
\zeta_{(0,A)}(E_\alpha) &= \frac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)E_\alpha, \quad \alpha = 5, 6,\\
\nu_{(0,A)}(E_5) &= \frac{1}{4}((-x_1x_5 - 3x_2x_6)E_1 + (3x_1x_6 - x_2x_5)E_2\\
&\quad + (-x_3x_5 - 3x_4x_6)E_3 + (-x_4x_5 + 3x_3x_6)E_4),\\
\nu_{(0,A)}(E_6) &= \frac{1}{4}((3x_2x_5 - x_1x_6)E_1 + (-3x_1x_5 - x_2x_6)E_2\\
&\quad + (3x_4x_5 - x_3x_6)E_3 + (-3x_3x_5 - x_4x_6)E_4). \qquad (30)
\end{aligned}$$

Proof. We need only consider the linearity of all involve operators and formulas (21), (24) and (25). □

In general, the n^{th} covariant derivative of the Jacobi operator at the origin of a Riemannian manifold is given by the matrix $\mathcal{J}_0^{n)} = \left(\mathcal{J}_{ij}^{n}(0)\right)$ where $\mathcal{J}_{ij}^{n}(0) = \langle \mathcal{J}_t^{n)}(Q_i), Q_j\rangle(0) = \langle \mathcal{J}_0^{n)}(E_i), E_j\rangle$. On an H-type group, we also know by Proposition 3.1 and the orthogonality between the center $\mathfrak{z}$ and $\mathfrak{v}$ that

$$\begin{aligned}
\mathcal{J}_{\alpha\beta}^{n)}(0) &= \langle \zeta_{(n,A)}(E_\alpha), E_\beta\rangle, & \mathcal{J}_{\alpha j}^{n)}(0) &= \langle \nu_{(n,A)}(E_\alpha), E_j\rangle,\\
\mathcal{J}_{i\beta}^{n)}(0) &= \langle \zeta_{(n,X)}(E_i), E_\beta\rangle, & \mathcal{J}_{ij}^{n)}(0) &= \langle \nu_{(n,X)}(E_i), E_j\rangle, \qquad (31)
\end{aligned}$$

where the indexes α, β identify the elements that span $\mathfrak{z}$ and i, j identify the elements of $\mathfrak{v}$. In particular, on Kaplan's example $i, j = 1, \ldots, 4$ and $\alpha, \beta = 5, 6$. Moreover, we obtain the explicit expressions of $\mathcal{J}_0^{n)}$, $n = 0, \ldots, 5$, on N by formulas (31), (22), (24), (25), Lemma 3.1 and the linearity of all involve operators. They can be seen in [3, p. 74–83]. Anyway, we write here the expression of $\mathcal{J}_0^{0)}$ and the elements $\mathcal{J}_{11}^{n)}(0)$, $\mathcal{J}_{12}^{n)}(0)$, $\mathcal{J}_{56}^{n)}(0)$, $\mathcal{J}_{66}^{n)}(0)$, $n = 1, 2, 3, 4, 5$, for the illustration of the next result's proof. In addition, we show how to obtain $\mathcal{J}_{11}^{1}(0)$.

The explicit expression of $\mathcal{J}_0^{0)} = \left(\mathcal{J}_{ij}^0(0)\right)$, $i, j = 1, \ldots, 6$, is given by

$$\begin{aligned}
&\mathcal{J}_{11}^0(0) = \mathcal{J}_{22}^0(0) = \frac{1}{4}(-3x_3^2 - 3x_4^2 + x_5^2 + x_6^2), \quad \mathcal{J}_{12}^0(0) = \mathcal{J}_{34}^0(0) = \mathcal{J}_{56}^0(0) = 0,\\
&\mathcal{J}_{13}^0(0) = \mathcal{J}_{24}^0(0) = \frac{3}{4}(x_1x_3 + x_2x_4), \quad \mathcal{J}_{14}^0(0) = -\mathcal{J}_{23}^0(0) = \frac{3}{4}(-x_2x_3 + x_1x_4),\\
&\mathcal{J}_{15}^0(0) = \frac{1}{4}(-x_1x_5 - 3x_2x_6), \quad \mathcal{J}_{16}^0(0) = \frac{1}{4}(3x_2x_5 - x_1x_6),\\
&\mathcal{J}_{25}^0(0) = \frac{1}{4}(-x_2x_5 + 3x_1x_6), \quad \mathcal{J}_{26}^0(0) = \frac{1}{4}(-3x_1x_5 - x_2x_6),\\
&\mathcal{J}_{35}^0(0) = \frac{1}{4}(-x_3x_5 - 3x_4x_6), \quad \mathcal{J}_{36}^0(0) = \frac{1}{4}(3x_4x_5 - x_3x_6),\\
&\mathcal{J}_{33}^0(0) = \mathcal{J}_{44}^0(0) = \frac{1}{4}(-3x_1^2 - 3x_2^2 + x_5^2 + x_6^2), \quad \mathcal{J}_{45}^0(0) = \frac{1}{4}(-x_4x_5 + 3x_3x_6),\\
&\mathcal{J}_{46}^0(0) = \frac{1}{4}(-3x_3x_5 - x_4x_6), \quad \mathcal{J}_{55}^0(0) = \mathcal{J}_{66}^0(0) = \frac{1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2). \qquad (32)
\end{aligned}$$

The elements $\mathcal{J}_{11}^n(0)$ of $\mathcal{J}_0^{n)}$, $n = 1, 2, 3, 4, 5$, are

$$\begin{aligned}
\mathcal{J}_{11}^1(0) &\overset{(31)}{=} \langle \nu_{(1,X)}(E_1), E_1\rangle \overset{(22)}{=} \Big\langle \frac{1}{2}(\nu_{(0,X)}(j(\dot{A}_0)E_1) - j(\dot{A}_0)\nu_{(0,X)}(E_1)\\
&\quad - j(\zeta_{(0,X)}(E_1))\dot{X}_0 - \nu_{(0,A)}\big([\dot{X}_0, E_1]\big)), E_1\Big\rangle\\
&\overset{\text{Lem. 3.1}}{=} \frac{1}{2}(\langle \nu_{(0,X)}(x_5E_3 + x_6E_4), E_1\rangle - \Big\langle j(\dot{A}_0)\Big(\frac{1}{4}((-3(x_3^2 + x_4^2) + x_5^2 + x_6^2)E_1\\
&\quad + 3(x_1x_3 + x_2x_4)E_3 + 3(x_1x_4 - x_2x_3)E_4)\Big), E_1\Big\rangle - \langle j\Big(\frac{1}{4}((-x_1x_5\\
&\quad - 3x_2x_6)E_5 + (3x_2x_5 - x_1x_6)E_6)\Big)\dot{X}_0, E_1\Big\rangle + \langle \nu_{(0,A)}(x_3E_5 + x_4E_6), E_1\rangle)\\
&= \frac{1}{2}(x_5\langle \nu_{(0,X)}(E_3), E_1\rangle + x_6\langle \nu_{(0,X)}(E_4), E_1\rangle\\
&\quad - \frac{1}{4}((-3(x_3^2 + x_4^2) + x_5^2 + x_6^2)\langle j(\dot{A}_0)(E_1), E_1\rangle\\
&\quad + 3(x_1x_3 + x_2x_4)\langle j(\dot{A}_0)(E_3), E_1\rangle + 3(x_1x_4 - x_2x_3)\langle j(\dot{A}_0)(E_4), E_1\rangle)\\
&\quad - \frac{1}{4}((-x_1x_5 - 3x_2x_6)\langle j(E_5)\dot{X}_0, E_1\rangle + (3x_2x_5 - x_1x_6)\langle j(E_6)\dot{X}_0, E_1\rangle)
\end{aligned}$$

$$
\begin{aligned}
&+ x_3\langle \nu_{(0,A)}(E_5), E_1\rangle + x_4\langle \nu_{(0,A)}(E_6), E_1\rangle)\\
&\overset{\text{Lem. 3.1}}{=} \frac{1}{2}\Big(x_5\Big(\frac{3}{4}(x_1x_3 + x_2x_4)\Big) + x_6\Big(\frac{3}{4}(x_1x_4 - x_2x_3)\Big)\\
&+ \frac{-3}{4}(x_1x_3 + x_2x_4)(-x_5) + \frac{-3}{4}(x_1x_4 - x_2x_3)(-x_6)\\
&- \frac{1}{4}((-x_1x_5 - 3x_2x_6)(-x_3) + (3x_2x_5 - x_1x_6)(-x_4))\\
&+ x_3\Big(\frac{1}{4}(-x_1x_5 - 3x_2x_6)\Big) + x_4\Big(\frac{1}{4}(3x_2x_5 - x_1x_6)\Big)\Big)\\
&= \frac{1}{2}(x_1(x_3x_5 + x_4x_6) + 3x_2(x_4x_5 - x_3x_6)),
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{J}^2_{11}(0) = {}& \frac{1}{4}(x_1^2(2x_3^2 + 2x_4^2 - x_5^2 - x_6^2) + x_2^2(2x_3^2 + 2x_4^2 - 3x_5^2 - 3x_6^2)\\
&+ 2((x_3^2 + x_4^2)^2 + 2(x_4x_5 - x_3x_6)^2)),\\
\mathcal{J}^3_{11}(0) = {}& \frac{1}{8}(x_1(x_3x_5 + x_4x_6)(-7x_1^2 - 7x_2^2 - 7x_3^2 - 7x_4^2 - x_5^2 - x_6^2)\\
&+ x_2(x_3x_6 - x_4x_5)(9x_1^2 + 9x_2^2 + 9x_3^2 + 9x_4^2 + 15x_5^2 + 15x_6^2)),\\
\mathcal{J}^4_{11}(0) = {}& \frac{1}{16}(x_1^2(-8x_1^2x_3^2 - 16x_3^4 - 8x_1^2x_4^2 - 16x_4^4 + 7x_1^2x_5^2 - x_3^2x_5^2 - 17x_4^2x_5^2 + x_5^4\\
&+ 7x_1^2x_6^2 - 17x_3^2x_6^2 - x_4^2x_6^2 + 2x_5^2x_6^2 + x_6^4) + x_2^2(-8x_2^2x_3^2 - 16x_3^4 - 8x_2^2x_4^2\\
&- 16x_4^4 + 9x_2^2x_5^2 + x_3^2x_5^2 - 15x_4^2x_5^2 + 15x_5^4 + 9x_2^2x_6^2 - 15x_3^2x_6^2 + x_4^2x_6^2\\
&+ 30x_5^2x_6^2 + 15x_6^4) + x_1^2x_2^2(-16x_3^2 - 16x_4^2 + 16x_5^5 + 16x_6^2) + x_3^2(-8x_3^4\\
&- 8x_3^2x_5^2 - 24x_3^2x_6^2 - 16x_5^2x_6^2 - 16x_6^4) + x_4^2(-8x_4^4 - 24x_4^2x_5^2 - 16x_5^4\\
&- 8x_4^2x_6^2 - 16x_5^2x_6^2) + x_3^2x_4^2(-32x_1^2 - 32x_2^2 - 24x_3^2 - 24x_4^2 - 32x_5^2\\
&- 32x_6^2) + x_3x_4x_5x_6(32x_1^2 + 32x_2^2 + 32x_3^2 + 32x_4^2 + 32x_5^2 + 32x_6^2)),\\
\mathcal{J}^5_{11}(0) = {}& \frac{-1}{8}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)\\
&(x_1(x_3x_5 + x_4x_6) + 3x_2(x_4x_5 - x_3x_6)) + \frac{5}{4}(x_1(x_3x_5 + x_4x_6)(-7x_1^2\\
&- 7x_2^2 - 7x_3^2 - 7x_4^2 - x_5^2 - x_6^2) + x_2(x_3x_6 - x_4x_5)(9x_1^2 + 9x_2^2 + 9x_3^2\\
&+ 9x_4^2 + 15x_5^2 + 15x_6^2))).
\end{aligned} \tag{33}
$$

The entries $\mathcal{J}^n_{12}(0)$ of $\mathcal{J}^{n)}_0$, $n = 1, 2, 3, 4, 5$, are

$$\mathcal{J}^1_{12}(0) = x_1(-x_4x_5 + x_3x_6) + x_2(x_3x_5 + x_4x_6),$$

$$\mathcal{J}^2_{12}(0) = \frac{1}{2}(x_5x_6(-2x_3^2 + x_4^2) + 2x_3x_4(x_5^2 - x_6^2) + x_1x_2(x_5^2 + x_6^2)),$$

$$\mathcal{J}^3_{12}(0) = (x_1(x_4x_5 - x_3x_6) - x_2(x_3x_5 + x_4x_6))(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2),$$

$$\begin{aligned}\mathcal{J}^4_{12}(0) = \frac{1}{8}(&x_1x_2(x_5^2 + x_6^2)(-x_1^2 - x_2^2 - x_3^2 - x_4^2 - 7x_5^2 - 7x_6^2)\\ &+ x_3x_4(-8x_5^4 + 8x_6^4 + 8(x_6^2 - x_5^2)(x_1^2 + x_2^2 + x_3^2 + x_4^2))\\ &+ x_5x_6(8x_3^4 - 8x_4^4 + 8(x_3^2 - x_4^2)(x_1^2 + x_2^2 + x_5^2 + x_6^2))),\end{aligned}$$

$$\begin{aligned}\mathcal{J}^5_{12}(0) = \frac{-1}{4}&(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)((x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)\\ &(x_1(-x_4x_5 + x_3x_6) + x_2(x_3x_5 + x_4x_6)) + 5((x_1(x_4x_5 - x_3x_6)\\ &- x_2(x_3x_5 + x_4x_6))(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2))).\end{aligned} \tag{34}$$

The elements $\mathcal{J}^n_{56}(0)$ of $\mathcal{J}_0^{n)}$, $n = 1, 2, 3, 4, 5$, are

$$\mathcal{J}^1_{56}(0) = \mathcal{J}^3_{56}(0) = \mathcal{J}^5_{56}(0) = 0,$$

$$\mathcal{J}^2_{56}(0) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2 + x_4^2)x_5x_6,$$

$$\mathcal{J}^4_{56}(0) = \frac{-1}{8}(x_1^2 + x_2^2 + x_3^2 + x_4^2)x_5x_6(7x_1^2 + 7x_2^2 + 7x_3^2 + 7x_4^2 + x_5^2 + x_6^2). \tag{35}$$

Finally, the entries $\mathcal{J}^n_{66}(0)$ of $\mathcal{J}_0^{n)}$, $n = 1, 2, 3, 4, 5$, are

$$\mathcal{J}^1_{66}(0) = \mathcal{J}^3_{66}(0) = \mathcal{J}^5_{66}(0) = 0,$$

$$\mathcal{J}^2_{66}(0) = \frac{-1}{4}(x_1^2 + x_2^2 + x_3^2 + x_4^2)(2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + x_5^2 - x_6^2),$$

$$\begin{aligned}\mathcal{J}^4_{66}(0) = \frac{1}{16}&(x_1^2 + x_2^2 + x_3^2 + x_4^2)(8x_1^4 + 8x_2^4 + 8x_3^4 + 8x_4^4 + x_5^4 - x_6^4 + 16x_3^2x_4^2\\ &+ 15x_3^2x_5^2 + 15x_4^2x_5^2 + x_3^2x_6^2 + x_4^2x_6^2 + x_2^2(16x_3^2 + 16x_4^2 + 15x_5^2 + x_6^2)\\ &+ x_1^2(16x_2^2 + 16x_3^2 + 16x_4^2 + 15x_5^2 + x_6^2)).\end{aligned} \tag{36}$$

Theorem 3.1. *The Kaplan's example N has constant Jacobi osculating rank* $\mathtt{r} = \mathbf{4}$. *In fact, the derivatives of the Jacobi operator at the origin satisfy the identity*

$$\frac{1}{4}|x|^4\mathcal{J}_0^{1)} + \frac{5}{4}|x|^2\mathcal{J}_0^{3)} + \mathcal{J}_0^{5)} = 0, \tag{37}$$

where x denotes the unit initial tangent vector of an arbitrary geodesic on N.

Proof. Let us consider the following linear homogeneous system of equations

$$\begin{aligned}
&1)\quad A\mathcal{J}^1_{11}(0)+B\mathcal{J}^2_{11}(0)+C\mathcal{J}^3_{11}(0)+D\mathcal{J}^4_{11}(0)=0,\\
&2)\quad A\mathcal{J}^1_{12}(0)+B\mathcal{J}^2_{12}(0)+C\mathcal{J}^3_{12}(0)+D\mathcal{J}^4_{12}(0)=0,\\
&3)\quad A\mathcal{J}^1_{56}(0)+B\mathcal{J}^2_{56}(0)+C\mathcal{J}^3_{56}(0)+D\mathcal{J}^4_{56}(0)=0,\\
&4)\quad A\mathcal{J}^1_{66}(0)+B\mathcal{J}^2_{66}(0)+C\mathcal{J}^3_{66}(0)+D\mathcal{J}^4_{66}(0)=0.
\end{aligned}\tag{38}$$

Using the information provide by (35) and (36) we conclude from the equations 3) and 4) that $B=D=0$ is the only possible solution valid for all geodesic in N. Finally, we study the simplified equations 1) and 2) using (33) and (34). We obtain that $A=C=0$. Therefore, $\mathcal{J}^{1)}_0$, $\mathcal{J}^{2)}_0$, $\mathcal{J}^{3)}_0$ and $\mathcal{J}^{4)}_0$ are linear independent.

On the other hand by (33), it is a straightforward computation to check that

$$\frac{1}{4}|x|^4\mathcal{J}^1_{11}(0)+\frac{5}{4}|x|^2\mathcal{J}^3_{11}(0)+\mathcal{J}^5_{11}(0)=0.$$

Analogously using $\mathcal{J}^{1)}_0$ of [3, p. 74], $\mathcal{J}^{3)}_0$ of [3, p. 76] and $\mathcal{J}^{5)}_0$ of [3, p. 81], we easily check that the following more general relation is satisfied

$$\frac{1}{4}|x|^4\mathcal{J}^1_{ij}(0)+\frac{5}{4}|x|^2\mathcal{J}^3_{ij}(0)+\mathcal{J}^5_{ij}(0)=0,\quad i,j=1,\dots,6.$$

Therefore, we obtain (37) and we state that $\mathtt{r}=4$. Moreover, this relation is valid for all geodesic in N due to $|x|^2=1$ (see (26)). Thus, the Jacobi osculating rank of N is constant. □

Now, due to $\mathtt{r}=4$ and Proposition 2.5, there are four smooth functions $a_1,\dots,a_4: I\to\mathbb{R}$ that provide an expression of type (15) for the Jacobi operator valid for all geodesic in N. In the following, we will determine these functions.

Theorem 3.2. *The Jacobi operator along anyone geodesic γ of N with unit initial tangent vector can be written in the form*

$$\mathcal{J}_t=\mathcal{J}_0+a_1(t)\mathcal{J}^{1)}_0+a_2(t)\mathcal{J}^{2)}_0+a_3(t)\mathcal{J}^{3)}_0+a_4(t)\mathcal{J}^{4)}_0\tag{39}$$

where

$$\begin{aligned}
a_1(t)&=\frac{1}{3}(8\sin(t/2)-\sin(t)), & a_2(t)&=5+\frac{1}{3}(\cos(t)-16\cos(t/2)),\\
a_3(t)&=\frac{1}{3}(8\sin(t/2)-4\sin(t)), & a_4(t)&=4+\frac{4}{3}(\cos(t)-4\cos(t/2)).
\end{aligned}\tag{40}$$

Proof. From Proposition 2.3, Proposition 2.5 and due to we assume that $|x|^2=1$, we only have to solve the following homogeneous linear ordinary differential equation of order 5: $\frac{1}{4}\mathcal{J}^{1)}_t+\frac{5}{4}\mathcal{J}^{3)}_t+\mathcal{J}^{5)}_t=0$. Following the general theory about ordinary differential equations we have that $y^5 + \frac{5}{4}y^3 + \frac{1}{4}y$ is its characteristic polynomial whose roots are $\{0, \pm\frac{\imath}{2}, \pm\imath\}$. Thus, the Jacobi operator is given by

$$\mathcal{J}_t = c_0 + c_1\cos(t) + c_2\sin(t) + c_3\cos(t/2) + c_4\sin(t/2), \tag{41}$$

where c_l, $l = 1, \ldots, 4$ are arbitrary parameters. Now, it remains only to find the value of c_l. It is easy to obtain from (41) the relations

$$\mathcal{J}_0 = c_0 + c_1 + c_3, \quad \mathcal{J}_0^{1)} = c_2 + \frac{c_4}{2}, \quad \mathcal{J}_0^{2)} = -\left(c_1 + \frac{c_3}{4}\right),$$
$$\mathcal{J}_0^{3)} = -\left(c_2 + \frac{c_4}{8}\right), \quad \mathcal{J}_0^{4)} = c_1 + \frac{c_3}{16}.$$

Then, we have

$$c_0 = \mathcal{J}_0 + 5\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}, \quad c_1 = \frac{1}{3}(\mathcal{J}_0^{2)} + 4\mathcal{J}_0^{4)}), \quad c_2 = \frac{-1}{3}(\mathcal{J}_0^{1)} + 4\mathcal{J}_0^{3)}),$$
$$c_3 = \frac{-16}{3}(\mathcal{J}_0^{2)} + \mathcal{J}_0^{4)}), \quad c_4 = \frac{8}{3}(\mathcal{J}_0^{1)} + \mathcal{J}_0^{3)}).$$

We conclude the proof substituting these values in (41). □

Remark 3.1. The resolution of the Jacobi equation on N. The resolution of the Jacobi equation on a Riemannian manifold can be quite a difficult task. In the Euclidean space the solution is trivial. For the symmetric spaces, the problem is reduced to a system of differential equations with constant coefficients. In [7] and [8], I. Chavel obtained a partial solution of this problem for the naturally reductive manifolds $V_1 = Sp(2)/SU(2)$ and $V_2 = SU(5)/(Sp(2) \times S^1)$. The method used by I. Chavel, which allowed him to solve the Jacobi equation in some particular directions of the geodesic, is based on the use of the canonical connection. Nevertheless, his method does not seem to apply in a simple way to solve the Jacobi equation along an unit geodesic of an arbitrary direction. For naturally reductive compact homogeneous spaces, W. Ziller [24] solves the Jacobi equation working with the canonical connection; but the solution can be considered of qualitative type (it does not allow us to obtain in an easy way the Jacobi fields neither for any particular example nor for an arbitrary direction of the geodesic). The methods used by I. Chavel and W. Ziller for solving the Jacobi equation are special cases of a more general procedure (see Lemma of [6, p. 51]). In particular, this procedure is valid on any g.o. space and any generalized Heisenberg group. Although it is not always possible obtain explicit results using this method, the Jacobi equation $Y_t'' + \mathcal{J}_t Y_t = 0$ along an arbitrary geodesic $\gamma(t)$ and with respect to the Levi–Civita connection ∇ has been solved by this method on H-type groups by J. Berndt, F. Tricerri and L. Vanhecke in [6, p. 52]. Thus, using this result the Jacobi fields on N could be obtained explicitly.

Now, as a direct consequence of the study made in this paper, we can also use on g.o. spaces the new method known on naturally reductive spaces namely, the method based in the constant Jacobi osculating rank. This method allows us to obtain explicitly the Jacobi fields for a particular example and for an arbitrary direction of the geodesic. Moreover, it was presented by the second author and A. Tarrío in [19] on the naturally reductive space $Sp(2)/SU(2)$ and it was used by the first author and S. Bartoll in [5] on the naturally reductive space $U(3)/(U(1)\times U(1)\times U(1))$. In particular, as a direct consequence of Theorem 3.2, we can apply this new method to solve the Jacobi equation on N. Moreover, the first author calculated explicitly in [3, p. 70–73] the Jacobi field $(Y_t)_1$ on N when the unit tangent vector $\dot{\gamma}_0$ is $\dot{X}_0 = \sum_{i=1}^{4} x_i E_i$ (i.e., $x_5 = x_6 = 0$). Firstly, she calculated using the method based in the constant Jacobi osculating rank of N and later using Theorem of [6, p. 52]. Finally, she checked that both obtained results are equivalents.

References

[1] D. Alekseevsky and A. Arvanitoyeorgos, Metrics with homogeneous geodesics on flag manifolds, *Comment. Math. Univ. Carolinae* **43**, no. 2 (2002), 189–199.

[2] D. Alekseevsky and A. Arvanitoyeorgos, Riemannian flag manifolds with homogeneous geodesics, *Trans. Amer. Math. Soc.* **359**, no. 8 (2007), 3769–3789.

[3] T. Arias-Marco, Study of homogeneous D'Atri spaces of the Jacobi operator on g.o. spaces and the locally homogeneous connections on 2-dimensional manifolds with the help of Mathematica©, *Universitat de València, Valencia, Spain*, 2007, http://www.tdx.cat/TDX-0911108-110640.

[4] T. Arias-Marco and S. Bartoll, The Osculating rank of the Jacobi Operator in M^6. A Method for Solving the Jacobi Equation, Actas de las VIII Jornadas de Matemática Aplicada del DMA – IMPA, *Ed.UPV*, *Valencia, Spain*, 2007.

[5] T. Arias-Marco and S. Bartoll, Constant Jacobi osculating rank of U(3)/U(1)×U(1)×U(1), preprint.

[6] J. Berndt, F. Tricerri and L. Vanhecke, Generalized Heisenberg Groups and Damek–Ricci Harmonic Spaces, Lecture Notes in Mathematics 1598, *Springer-Verlag, Berlin, Heidelberg, New York*, 1995.

[7] I. Chavel, On normal Riemannian homogeneous spaces of rank one, *Bull. Amer. Math. Soc.* **73** (1967), 477–481.

[8] I. Chavel, Isotropic Jacobi fields and Jacobi's equations on Riemannian homogeneous spaces, *Comment. Math. Helvetici* **42** (1967), 237–248.

[9] P. Dombrowski, Differentiable maps into Riemannian manifolds of constant stable osculating rank. II, *J. Reine Angew. Math.* **289** (1977), 144–173.

[10] Z. Dušek, Explicit geodesic graphs on some H-type groups, *Rendiconti del Circolo Matematico di Palermo, Serie II, Suppl.*, no. 69 (2002), 77–88.

[11] Z. Dušek, O. Kowalski and S.Ž. Nikčević, New examples of Riemannian g.o. manifolds in dimension 7, *Differential Geom. and its Appl.* **21**, no. 1 (2004), 65–78.

[12] C. S. Gordon, Homogeneous Riemannian manifolds whose geodesics are orbits, *Prog. Nonlinear Differential Equations Appl.* **20** (1996), 155–174.

[13] A. Kaplan, Riemannian nilmanifolds attached to Clifford modules, *Geometriae Dedicata* **11**, no. 2 (1981), 127–136.

[14] A. Kaplan, On the geometry of groups of Heisenberg type, *Bull. London Math. Soc.* **15**, no. 1 (1983), 35–42.

[15] S. Kobayashi and K. Nomizu, Foundations of differential geometry I, II, *Interscience Publishers*, *New York*, 1963, 1969.

[16] O. Kowalski, F.Prüfer and L. Vanhecke, D'Atri spaces, *Prog. Nonlinear Differential Equations Appl.* **20** (1996), 241–284.

[17] O. Kowalski and L. Vanhecke, Riemannian manifolds with homogeneous geodesics, *Boll. Un. Math. Ital.* **(7) 5-B** (1991), 189–246.

[18] E. Macías-Virgós, A. M. Naveira and A. Tarrío, The constant osculating rank of the Wilking manifold V_3, *C. R. Acad. Sci. Paris, Ser. I.* **346** (2008), 67–70.

[19] A. M. Naveira and A. Tarrío, A method for the resolution of the Jacobi equation $Y'' + RY' = 0$ on the manifold Sp(2)/SU(2), *Monatsh. Math.* **158 no.3** (2008), 231–246. 154 (2008), no. 3, 231–246.

[20] C. Riehm, Explicit spin representations and Lie algebras of Heisenberg type, *J. London Math. Soc.* **29**92), no. 1 (1984), 46–62.

[21] K. Tojo, Totally geodesic submanifolds of naturally reductive homogeneous spaces, *Tsukuba J. Math.* **20**, no. 1 (1996 –190).

[22] F. Tricerri and L. Vanhecke, Homogeneous structures on Riemannian manifolds, Lecture Note Series, *London Mathematical Society*, Vol. 83, 1982.

[23] K. Tsukada, Totally geodesic submanifolds of Riemannian manifolds and curvature-invariant subspaces, *Kodai Math. J.* **19** (1996), 395–437.

[24] W. Ziller, The Jacobi equation on naturally reductive compact Riemannian homogeneous spaces, *Comment. Math. Helvetici* **52**, no. 4 (1977), 573–590.

TERESA ARIAS-MARCO
DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE EXTREMADURA
AV. DE ELVAS S/N
06071 BADAJOZ
SPAIN

E-mail: ariasmarco@unex.es

ANTONIO M. NAVEIRA
DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA
UNIVERSIDAD DE VALENCIA E.G.
AV. VICENTE ANDRÉS ESTELLÉS 1
46100 BURJASSOT, VALENCIA
SPAIN

E-mail: naveira@uv.es

(Received June 26, 2008)