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Minimal solution of a Riccati type differential equation

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Abstract. We consider a Riccati type differential equation which appears in the oscillation theory of half-linear differential equations. We establish the existence of the so-called *minimal solution* of this equation and we investigate basic properties of this solution. In particular, we prove a Sturmian type theorem for minimal solutions of a pair of considered equations.

1. Introduction

The Riccati type differential equation which we investigate in this paper comes from the oscillation theory of half-linear differential equations. Recall that the half-linear differential equation is an equation of the form

$$(r(t)\Phi(x'))' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x,\ p > 1, \tag{1}$$

and that oscillation theory of this equation attracted considerable attention in the recent years, let us mention at least the books [1, 6] and the references given therein. It was shown that oscillatory properties of (1) are essentially the same as those of the linear Sturm–Liouville equation (which is the special case $p = 2$ in (1))

$$(r(t)x')' + c(t)x = 0 \tag{2}$$

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and hence equation (1) can be classified as oscillatory or nonoscillatory similarly as for (2). As pioneers of the half-linear oscillation theory are usually regarded Elbert and Mirzov with their papers [7] and [14], even if elements of the theory of half-linear equations had already appeared in Bihari's papers [2], [3], [4].

In the classical oscillation theory of (1), this equation is regarded as a perturbation of the (nonoscillatory) *one-term equation*

$$(r(t)\Phi(x'))' = 0 \tag{3}$$

and an important role is played there by the Riccati type differential equation (related to (1) by the substitution $w = r\Phi(x'/x)$)

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \quad q := \frac{p}{p-1}. \tag{4}$$

Comparing this equation with the classical Riccati equation (associated with (2) by the substitution $w = rx'/x$)

$$w' + c(t) + \frac{w^2}{r(t)} = 0, \tag{5}$$

the power q in (4) makes no essential difference with respect to the power 2 in (5), so (non)oscillation criteria for (1) derived in this way are similar to those for (2). Another reason for this similarity is the fact that the solution space of (3) is actually *linear*.

Recently, a more general approach (sometimes called the *perturbation principle*) to half-linear oscillation theory has been introduced. There, equation (1) is viewed as a perturbation of (nonoscillatory) half-linear equation *of the same form* (i.e., linearity of its solution space is lost)

$$(r(t)\Phi(x'))' + \tilde{c}(t)\Phi(x) = 0. \tag{6}$$

More precisely, let h be an eventually positive solution of (6) and let $w_h := r\Phi(h'/h)$ be the solution of the Riccati equation associated with (6). If w is a solution of (4) and $v = h^p(w - w_h)$, then v solves the first order equation

$$v' + (c(t) - \tilde{c}(t))h^p(t) + (p-1)r^{1-q}(t)h^{-q}(t)H(t,v) = 0, \tag{7}$$

where

$$H(t,v) := |v + G(t)|^q - q\Phi^{-1}(G(t))v - |G(t)|^q, \quad G(t) := r(t)h(t)\Phi(h'(t)),$$

$\Phi^{-1}(s) = |s|^{q-2}s$ being the inverse function of Φ. Of course, if $\tilde{c}(t) \equiv 0$ and $h(t) \equiv 1$, then (7) reduces to (4).We refer to [5] and to [6, Section 5.6] for a brief summary of basic ideas of this "perturbation" approach.

Motivated by the above mentioned facts, the main objective of our paper is the Riccati type differential equation of the form

$$v' + c(t) + r^{-1}(t)H(g(t), v) = 0, \tag{8}$$

where c, r, g are continuous functions, $r(t) > 0$, and

$$H(g, v) = |v + g(t)|^q - q\Phi^{-1}(g(t))v - |g(t)|^q.$$

We will show that similarly to (4), among all proper solutions of (8) (see the next section for the definition of this concept) there exists the so-called *minimal solution*, and we establish Sturmian type comparison theorem for minimal solutions of two equations of the form (8).

2. Half-linear and Riccati type differential equations

First of all, observe that if $p = 2$ in (7), then this equation takes the form

$$v' + (c(t) - \tilde{c}(t))h^2(t) + \frac{v^2}{r(t)h^2(t)} = 0$$

which is the equation of the same form as (5) and this is the Riccati equation associated with the second order Sturm–Liouville equation resulting from (2) upon the transformation $x = h(t)u$, where h is a solution of (6) with $p = 2$. It is known (see, e.g., [6, Section 1.3]) that the linear transformation theory does not extend to (1) (since the function Φ is not additive), so from this point of view equation (7) can be regarded as an attempt to overcome this difficulty.

If $\tilde{c}(t) \equiv 0$, $\int^\infty r^{1-q}(t)\,dt < \infty$, and we take $h(t) = \int_t^\infty r^{1-q}(s)\,ds$, then $G(t) = -h(t)$ and (7) reads as

$$v' + c(t)h^p(t) + (p-1)h^{-q}(t)r^{1-q}(t)\left\{|v - h(t)|^q + q\Phi^{-1}(h(t))v - |h(t)|^q\right\} = 0,$$

and this Riccati type differential equation played an important role in the paper [13]. If $r(t) \equiv 1$ and $\tilde{c}(t) = \gamma_p t^{-p}$, $\gamma_p := \left(\frac{p-1}{p}\right)^p$, i.e., (6) reduces to the half-linear Euler equation with the critical coefficient

$$(\Phi(x'))' + \frac{\gamma_p}{t^p}\Phi((x)) = 0 \tag{9}$$

and with the solution $h(t) = t^{\frac{p-1}{p}}$. Then (7) takes the form

$$v' + (c(t) - \gamma_p t^{-p})t^{p-1} + \frac{p-1}{t}\left[\left|v + \left(\frac{p-1}{p}\right)^{p-1}\right|^q - v - \left(\frac{p-1}{p}\right)^p\right] = 0,$$

and this equation was an important tool in proving the main results of [11], [16].

In the next part of this section we recall the main results of the papers [8], [9], where the equation

$$(r(t)x')' + c(t)f(x, r(t)x') = 0 \tag{10}$$

is considered under the assumptions on f:

(i) The function f is continuous on $\Omega = \mathbb{R} \times \mathbb{R}_0$, where $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$;

(ii) It holds $xf(x, y) > 0$ if $xy \neq 0$;

(iii) The function f is homogeneous, i.e., $f(\lambda x, \lambda y) = \lambda f(x, y)$ for $\lambda \in \mathbb{R}_0$ and $(x, y) \in \Omega$;

(iv) The function f is sufficiently smooth in order to ensure the continuous dependence and the uniqueness of solutions of the initial value problem $x(t_1) = x_0$, $x'(t_1) = x_1$ at some $(x_0, x_1) \in \Omega$;

(v) Let

$$F(t) = tf(t, 1), \tag{11}$$

then

$$\int_{-\infty}^{\infty} \frac{dt}{1 + F(t)} < \infty \quad \text{and} \quad \lim_{|t|\to\infty} F(t) = \infty.$$

Let g be the differentiable function given by the formula

$$g(u) = \begin{cases} \int_{1/u}^{\infty} \dfrac{ds}{F(s)} & \text{if } u > 0, \\ -\int_{-\infty}^{1/u} \dfrac{ds}{F(s)} & \text{if } u < 0, \end{cases} \tag{12}$$

and $g(0) = 0$. Then g is strictly increasing and $\lim_{u\to\pm\infty} g(u) = \pm\infty$. If x is a solution of (10) such that $x(t) \neq 0$, then the function $u = g(rx'/x)$ solves the Riccati type differential equation

$$u' + c(t) + r^{-1}(t)H(u) = 0, \tag{13}$$

where the function H is given by

$$\int_{g(u)}^{\infty} \frac{ds}{H(s)} = \frac{1}{u}, \quad \text{if } u > 0, \qquad \int_{-\infty}^{g(u)} \frac{ds}{H(s)} = -\frac{1}{u}, \quad \text{if } u < 0$$

with $H(0) = 0$. Conversely, having a function $H(u) > 0$ for $u \neq 0$, with $H(0) = 0$, such that

$$\int_{-\infty} \frac{ds}{H(s)} < \infty, \quad \int^{\infty} \frac{ds}{H(s)} < \infty$$

one can associate with (13) equation (10) with f satisfying (i)–(v). More precisely, the function g is given as the solution of the differential equation

$$g'(u) = \frac{1}{u^2} H(g(u)), \quad g(0) = 0, \tag{14}$$

and the function $f : \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}$ is given by the formula

$$f(1,u) := \frac{1}{g'(u)}, \quad f(t,s) := \begin{cases} tf(1, s/t), & t \neq 0 \\ 0 & t = 0 \end{cases} \tag{15}$$

We finish this section with presenting some properties of the function $H(g,v)$ in (8). Directly one can verify that the derivative $H_v(g,v) = 0$ if and only if $v = 0$ and that H is strictly convex with respect to the second variable. Also, $H(g,v)$ is Lipschitzian in v, hence the initial value problem for (8) has a unique solution. This means that the graphs of solutions of (8) cannot intersect.

3. Minimal solution

Suppose that (1) is nonoscillatory, i.e., there exists a solution of this equation which is eventually of one sign. Then the associated Riccati equation (4) possesses the so-called *proper solution*, i.e., a solution which is defined on some interval $[t_0, \infty)$. Mirzov [15] and independently Elbert and Kusano [10] showed that among all proper solutions of (4) there exists the so-called *minimal solution* $\tilde{w}$, which is the proper solution with the property that any other proper solution w of (4) satisfies $w(t) > \tilde{w}(t)$ on the interval of its existence. The minimal solution of (4) defines then the so-called *principal solution* of (1) via the formula

$$\tilde{x}(t) = C \exp\left\{ \int^t \Phi^{-1}(\tilde{w}(s)/r(s))\, ds \right\}, \quad 0 \neq C \in \mathbb{R},$$

Note that the principal solution of (1) plays an important role in the oscillation theory of these equations, see [6, Section 4.2].

To establish the existence of the minimal solution of (8), we need the following auxiliary result.

Lemma 1. *Consider equation* (8) *in an interval* $[t_0, t_0 + T]$, $T > 0$ *arbitrary. There exists* $v_0 < 0$ *such that any solution of* (8) *with* $v(t_0) < v_0$ *satisfies*

$$\lim_{t \to t_1-} v(t) = -\infty$$

for some $t_1 \in [t_0, t_0 + T]$.

Proof. Denote

$$\hat{c} = \min_{t_0, t_0+T} c(t), \qquad \hat{r} = \max_{t_0, t_0+T} r(t).$$

Since the function g is continuous, it attains for $t \in [t_0, t_0 + T]$ values in some bounded closed interval, denote it $[A, B]$, and for this interval let

$$\hat{H}(v) := \min_{\tau \in [A,B]} H(\tau, v).$$

Consequently, for $t \in [t_0, t_0 + T]$ and $v \in \mathbb{R}$ we have $\hat{H}(v) \le H(g(t), v)$. Together with (8), consider the equation

$$u' + \hat{c} + \hat{r}^{-1}\hat{H}(u) = 0. \tag{16}$$

Then by the standard theorem for differential inequalities (see, e.g. [12]), if $v(t_0) < u(t_0)$, then $v(t) < u(t)$ for $t > t_0$ for which $v(t)$ exists.

Now consider equation (16). We have

$$\int_{u(t_0)}^{u(t)} \frac{ds}{-\hat{c} - \hat{r}^{-1}\hat{H}(s)} = t - t_0.$$

Since $\hat{H}(u) = H(g(t_0), u)$ for some $t_0 \in [a, b]$, we have $\hat{H}(u) \to \infty$ as $u \to -\infty$ and there exists $\tilde{u}$ such that $-\hat{c} - \hat{r}^{-1}\hat{H}(s) < 0$ for $u < \tilde{u}$, i.e., $u(t)$ is decreasing and

$$\int_{u(t)}^{u(t_0)} \frac{ds}{\hat{c} + \hat{r}^{-1}\hat{H}(s)} = t - t_0$$

if $u(t_0) < \tilde{u}$. Hence

$$\infty > \int_{-\infty}^{u(t_0)} \frac{ds}{\hat{c} + \hat{r}^{-1}\hat{H}(s)} > \int_{u(t)}^{u(t_0)} \frac{ds}{\hat{c} + \hat{r}^{-1}\hat{H}(s)} = t - t_0.$$

Now, if $u(t_0) \to -\infty$, the first integral in the previous formula tends to 0, which means that $t \to t_0$, i.e., $t - t_0 < T$ for $u(t_0)$ sufficiently negative. Hence $u(t)$ has to blow down to $-\infty$ inside of the interval $[t_0, t_0 + T]$ and inequality for solutions of (8) and (16) implies that a solution v of (8) starting with sufficiently negative initial value $v(t_0)$ has the same property. □

In the remaining part of this section we assume that there exists $t_0 \in T$ such that

$$\text{(8) possesses a solution defined on } [t_0, \infty). \tag{17}$$

Similarly to (4), such a solution we will call *the proper* solution of (8).

Definition 1. A proper solution $\tilde{v}$ of (8) is said to be *minimal*, if any other proper solution v of (8) satisfies $v(t) > \tilde{v}(t)$ on the interval of existence of v.

Denote

$$\mathcal{V} = \{v \in \mathbb{R},\ \text{the solution of (8) given by } v(t_0) = v \text{ is proper}\}.$$

By our assumption $\mathcal{V} \neq \emptyset$ and by Lemma 1 the set $\mathcal{V}$ is bounded below. Let

$$v_0 = \inf \mathcal{V}. \tag{18}$$

Theorem 1. *Suppose that* (17) *holds and let $\tilde{v}$ be the solution of* (8) *given by the initial condition $\tilde{v}(t_0) = v_0$, where v_0 is given by* (18). *Then $\tilde{v}$ is a proper solution, i.e., it exists on $[t_0, \infty)$ and it is the minimal solution of* (8).

Proof. By contradiction, suppose that $\tilde{v}$ is not proper, i.e., $\tilde{v}(T_1-) = -\infty$ for some $T_1 > t_0$. Let $T_2 > T_1$ be arbitrary. For $t \in [t_0, T_2]$ the function g attains the values in an interval $[A, B]$. Denote

$$\begin{aligned} \hat{H}(u) &= \min_{\tau \in [A,B]} \left\{ |u+\tau|^q - q\Phi^{-1}(\tau)\, u - |\tau|^q \right\}, \\ \tilde{H}(u) &= \max_{\tau \in [A,B]} \left\{ |u+\tau|^q - q\Phi^{-1}(\tau)\, u - |\tau|^q \right\}. \end{aligned} \tag{19}$$

Then we have for $t \in [t_0, T_2]$

$$\hat{H}(u) \leq H(g(t), u) \leq \tilde{H}(u). \tag{20}$$

Consider the Riccati type equations

$$u' + c(t) + r^{-1}(t)\hat{H}(u) = 0, \tag{21}$$

$$u' + c(t) + r^{-1}(t)\tilde{H}(u) = 0. \tag{22}$$

These equations are of the same form as (13), so one can associate with them the second order differential equations

$$(r(t)z')' + c(t)\hat{f}(z, r(t)z') = 0, \tag{23}$$

$$(r(t)z')' + c(t)\tilde{f}(z, r(t)z') = 0, \tag{24}$$

the functions $\hat{f}$, $\tilde{f}$ are related to $\hat{H}$, $\tilde{H}$ as described in Section 2 by relations (14) and (15).

Consider the solutions $\hat{z}$, $\tilde{z}$ of (23), (24), respectively, given by the initial condition

$$\hat{z}(T_2) = 0, \quad \hat{z}'(T_2) = -1, \quad \tilde{z}(T_2) = 0, \quad \tilde{z}'(T_2) = -1,$$

and let

$$\hat{u} = \hat{g}\,(r\hat{z}'/\hat{z}), \quad \tilde{u} = \tilde{g}\,(r\tilde{z}'/\tilde{z})$$

with $\hat{g}$, $\tilde{g}$ defined again via corresponding $\hat{f}$ and $\tilde{f}$ using formula (12). Then (since $\hat{g}(-\infty) = -\infty = \tilde{g}(-\infty)$)

$$\hat{u}(T_2-) = -\infty = \tilde{u}(T_2-)$$

and $\tilde{u}(T_1) \geq \hat{u}(T_1)$. Indeed, if, by contradiction, $\tilde{u}(T_1) < \hat{u}(T_1)$, then the solution $\bar{u}$ of (23) given by $\bar{u}(T_1) = \tilde{u}(T_1)$ satisfies $\bar{u}(t) \geq \tilde{u}(t)$, $t \in [T_1,\ T_2)$, so its graph either intersects that of $\hat{u}$ what is a contradiction with the unique solvability of (23), or $\bar{u}(T_2-) = -\infty$, again a contradiction, since the solutions of (23) satisfying $\hat{z}(T_2) = 0$ are determined up to a multiplicative factor (because of the homogeneity of the solution space), hence they determine the unique solution of (21).

Now consider the solution v of (8) with $v(T_1) \in [\hat{u}(T_1), \tilde{u}(T_1)]$. Since (20) holds, we have

$$\hat{u}(t) \leq v(t) \leq \tilde{u}(t) \quad \text{for} \quad t \in [T_1, T_2)\,,$$

i.e., $v\,(T_2-) = -\infty$. Moreover, the unique solvability of (8) implies that v exists on $[t_0, T_1]$ and $v(t_0) > v_0$. Indeed, if $v(t_1+) = \infty$ for some $t_1 \in [t_0, T_1)$ then the graph of v intersects the graph of any proper solution of (8) on $[t_0, \infty)$. Also, $v(t_0) \leq v_0$ implies the intersection of graphs of v and $\tilde{v}$ at some $t \in [t_0, T_1)$. Consequently, we have constructed a solution v of (8) starting with $v(t_0) > v_0$ which is not proper. This is contradiction with the definition of v_0. □

The next statement is a Sturmian type comparison theorem for minimal solutions of two equations of the form (8).

Theorem 2. *Together with* (8) *we consider the equation*

$$u' + C(t) + R^{-1}(t)H(g, u) = 0, \tag{25}$$

with $c(t) \leq C(t)$ and $0 < R(t) < r(t)$ for large t (i.e., (25) *is a majorant of* (8) *in the classical Sturmian setting for $p = 2$). Suppose that* (25) *possessed a proper solution and let $\tilde{u}$ be its minimal solution which is defined for $t \geq t_0$. Then* (8) *possesses a proper solution as well and for its minimal solution $\tilde{v}$ we have $\tilde{v}(t) < \tilde{u}(t)$ for $t \geq t_0$.*

Proof. Let u be a proper solution of (25) and consider the solution v of (8) given by the initial condition $v\,(t_1) = u\,(t_1)$ for some (sufficiently large) t_1.

Then inequalities between c, C, r, and R imply that $v(t) \geq u(t)$ for $t \geq t_0$. Since $H(g,u) \geq 0$ for $u \in \mathbb{R}$, the solution v can not blow up to ∞ at some finite time t, we have that v is a proper solution of (8). By contradiction, suppose that the minimal solutions $\tilde{u}$, $\tilde{v}$ satisfy $\tilde{v}(t_2) > \tilde{u}(t_2)$ for some $t_2 > t_0$. Consider the solution v of (8) given by $v(t_2) = \tilde{u}(t_2)$. Then by the same argument as in the previous part of the proof we have $v(t) > \tilde{u}(t)$ for $t \geq t_2$. At the same time, since $v(t_1) < \tilde{v}(t_1)$, we have $v(t) < \tilde{v}(t)$. This means that we have found a proper solution v of (8) which is less then minimal solution of this equation. This leads to a contradiction. □

Remark 1. The previous theorem is a comparison result with respect to c and r, while the function g is the same in (8), (25). The reason is that the behavior of H with respect to the first variable g is relatively complicated, since

$$\frac{\partial}{\partial g} H(g,u) = q\left[\Phi^{-1}(u+g) - (q-1)\,|g|^{q-2}\,u - \Phi^{-1}(g)\right]$$

and it is difficult to compute explicitly the roots of the equation $\frac{\partial}{\partial g} H(g,u) = 0$.

The last statement deals with the case when the function g is bounded.

Theorem 3. *Suppose that $\int_{t_0}^{\infty} r^{1-q}(t)\,dt = \infty$ for some $t_0 \in \mathbb{R}$, $c(t) \geq 0$, and g is bounded for $t \in [t_0,\infty)$. Then the minimal solution $\tilde{v}$ of (8) satisfies $\tilde{v}(t) \geq 0$ for $t \in [t_0,\infty)$.*

Proof. Suppose, by contradiction, that $\tilde{v}(T) < 0$ for some T. We proceed similarly as in the proof of Theorem 1. The function g is bounded, so it attains values in some bounded interval $[A,B]$. Consider equation (21) with the function $\hat{H}$ given by (19) and its solution satisfying $u(T) = \tilde{v}(T) < 0$. Then again $v(t) \leq u(t)$ for $t \geq T$. Since $c(t) \geq 0$, we have

$$u' + r^{-1}(t)\hat{H}(u) \leq 0. \tag{26}$$

The function u is decreasing (use that $\hat{H}(u) > 0$ for $u \neq 0$), hence from (26) for $t > T$

$$\int_{u(t)}^{u(T)} \frac{ds}{\hat{H}(s)} \geq \int_T^t r^{-1}(s)\,ds. \tag{27}$$

Letting $t \to \infty$, the integral on the left-hand side of (27) is convergent, while the integral on the right-hand side is divergent, which means that u cannot be a proper solution of (21) and hence $\tilde{v}$ is not a proper solution of (8) as well, a contradiction. □

Remark 2. (i) The last statement gives an alternative proof of the fact that $h(t) = t^{\frac{p-1}{p}}$ is the principal solution of (9) (proved in [11] by a different method). Indeed, let x be a solution of (9) linearly independently of h, $w_x = r\Phi(x'/x)$, $w_h = r\Phi(h'/h)$, $w_x(t) \neq w_h(t)$, and $v = h^p(w_x - w_h)$. Then v satisfies the equation

$$v' + \frac{p-1}{t}\left[\left|v + \left(\frac{p-1}{p}\right)^{p-1}\right|^q - v + \left(\frac{p-1}{p}\right)^p\right] = 0.$$

Moreover, the unique solvability of Riccati type equation (4) associated with (9) implies that $w_x(t) \neq w_h(t)$. By Theorem 3 $v(t) \geq 0$ for $t \in [t_0, \infty)$ which means $w_x(t) > w_h(t)$ i.e., w_h is the minimal solution of the Riccati equation associated with (9) and hence h is the principal solution of (9).

(ii) Generally, *any* condition which guaranties that v is the minimal solution of (8) is a sufficient condition for $w = h^{-p}v + w_h$ to be the minimal solution of (4) and then

$$x(t) = C\exp\left\{\int^t \Phi^{-1}(w(s)/r(s))ds\right\}, \quad 0 \neq C \in \mathbb{R},$$

is the principal solution of (1), which, as we have already mentioned before, plays the important role in the oscillation theory of (1).

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