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On generalized Einstein metrics in Finsler geometry

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Abstract. In this paper we define the generalized Einstein metrics in Finsler geometry by Cartan connection and study the geometrical properties of such Finsler metrics. Further, by use of the theory of Y-Riemannian metrics, we discuss the curvature properties of the Y-Riemannian space induced from a vector field which is determined by β and α in a generalized Einstein (α, β)-space.

1. Introduction

In this paper we mainly study the analogues in Finsler geometry of some important results of Einstein metrics in Riemann geometry. For an n-dimensional Finsler manifold (M, F), let ${R_h}^i{}_{jk}$ denote the h-curvature tensor of the Cartan connection. Define $R_{ij} := {R_i}^r{}_{jr}$ and $R_{hijk} := g_{ir}{R_h}^r{}_{jk}$, where g_{ij} denote the fundamental tensor of F. We call a Finsler metric a *generalized Einstein metric* if $R_{ij} = (n-1)\lambda(x, y)g_{ij}$ for a scalar function $\lambda(x, y)$ on $TM \setminus \{0\}$ which is a positively homogeneous function of degree zero in y. By the definition, $\mathbf{Ric}(x, y) = R_{ij}(x, y)y^i y^j$, where $\mathbf{Ric}(x, y)$ denotes the Ricci curvature of F. A Finsler metric F is called an *Einstein metric* if $\mathbf{Ric} = (n-1)\mathbf{K}(x)F^2$ for a scalar function $\mathbf{K}(x)$ on M. Obviously, a generalized Einstein metric must be an Einstein metric if $\lambda = \lambda(x)$ is a scalar function on M. For the studies of Einstein metrics and Ricci curvature in Finsler geometry, see [7], [13].

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It is well-known that, for an n-dimensional Riemann manifold, if there exists a scalar function $\lambda(x)$ such that $R_{ij} = (n-1)\lambda g_{ij}$, then $\lambda = constant$ $(n \geq 3)$, where R_{ij} is just the Ricci tensor defined by Levi–Civita connection. If $n = 3$, then such a Riemann metric has constant sectional curvature. In this paper, we first generalize the above results in Riemann geometry to the case of generalized Einstein manifolds in Finsler geometry.

Theorem 1.1. *Let (M, F) be an n-dimensional Finsler manifold. Assume that F is a generalized Einstein metric with $R_{ij} = (n-1)\lambda(x, y)g_{ij}$. Then we have the following results:*

(a) $(n = 3)$ *λ is a constant and (M, F) has constant flag curvature $\mathbf{K} = \lambda$.*

(b) $(n \geq 4)$ *λ is a constant and (M, F) has constant flag curvature $\mathbf{K} = \lambda$ provided that (M, F) is of scalar flag curvature.*

Further, if F is a Landsberg metric, we have the following theorem, in which item (a) is based on Theorem 1.1 and Numata's theorem in [11].

Theorem 1.2. *Let (M, F) be an n-dimensional Landsberg space. Assume that F is a generalized Einstein metric with $R_{ij} = (n-1)\lambda(x, y)g_{ij}$. Then we have the following results:*

(a) $(n = 3)$ *the Finsler manifold (M, F) is a Riemannian manifold of constant curvature $\mathbf{K} = \lambda$ provided that $\lambda \neq 0$.*

(b) $(n \geq 3)$ *λ is a h-covariant constant, that is, $\lambda_{|k} = 0$, where "$_|$" denotes the horizontal covariant derivative with respect to the Cartan connection of F.*

We note that R_{ij} are not symmetric in i and j in general. However, besides Riemannian metrics, there are some interesting Finsler metrics , whose R_{ij} are symmetric in i and j. For example, we can prove that, for a Finsler metric of scalar flag curvature, R_{ij} are symmetric (see Proposition 5.1 below).

In the case of $n = 2$, Finsler metrics must be of scalar flag curvature and always satisfy $R_{ij} = (n-1)\lambda(x, y)g_{ij}$ for a $\lambda(x, y)$. However, $\lambda(x, y)$ is generally not a constant. Then Theorem 1.1 does not hold generally. Besides, we can find many Landsberg metrics with $R_{ij} = (n-1)\lambda(x, y)g_{ij}$, which are not Riemannian and $\lambda_{|k} \neq 0$. These examples can be found in 2-dimensional Finsler spaces with constant main scalar.

In 1986, M. Matsumoto introduced so-called Y-Riemannian metric and Cartan Y-connection defined from a Finsler metric F and a non-zero tangent vector field $Y(x)$ on the underlying manifold M when he studied minimal hypersurfaces in a Finsler space ([3], [10]). Concretely, given a non-zero tangent vector field $Y(x)$ on a Finsler manifold (M, F), we can introduce the Riemannian metric

$\widetilde{g}_{ij}(x) := g_{ij}(x, Y(x))$, which is called the Y-*Riemannian metric*. In particular, for an (α, β)-metric $F = F(\alpha, \beta)$, where F is a positively homogeneous function of degree one in α and β, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a differential 1-form on a manifold M, we can introduce a non-zero tangent vector field $B(x) = b^i(x)\frac{\partial}{\partial x^i}$ when $\beta \neq 0$, where $b^i = a^{ij}b_j$. Now we can investigate the B-Riemannian metric on an (α, β)-space and obtain the following theorem.

Theorem 1.3. *Let $F = F(\alpha, \beta)$ be an (α, β)-metric on an n-dimensional manifold. Suppose that F is a non-Riemannian Landsberg metric. Let $\widetilde{g}(x) := g(x, B(x))$ be the B-Riemannian metric. Then we have the following results:*

(ia) *$(n = 2)$ both $\widetilde{g}$ and α are flat and $F(\alpha, \beta)$ is flat-parallel.*

(ib) *$(n \geq 3)$ $\widetilde{g}$ has constant sectional curvature if and only if $F(\alpha, \beta)$ is flat-parallel. In this case, both $\widetilde{g}$ and α are flat.*

(ic) *$(n \geq 3)$ $\widetilde{g}$ is an Einstein metric if and only if α is an Einstein metric. In this case, both $\widetilde{g}$ and α are Ricci-flat. In particular, in the case of $n = 3$, both $\widetilde{g}$ and α are flat.*

If we further assume that $F(\alpha, \beta)$ is a generalized Einstein metric with $R_{ij} = (n-1)\lambda(x, y)g_{ij}$, then

(iia) *$(n = 2, 3)$ $\widetilde{g}$ is flat and $F(\alpha, \beta)$ is flat-parallel. In this case, $\lambda(x, y) = 0$.*

(iib) *$(n \geq 4)$ both $\widetilde{g}$ and α are Ricci-flat and $\lambda(x, B(x)) = 0$.*

Here, an (α, β)-metric is said to be flat-parallel if α is locally flat and β is parallel with respect to α. In this case, the (α, β)-metric is locally Minkowski metric.

We have known that a non-Riemannian (α, β)-metric is Landsbergian if and only if β is parallel with respect to α. In this case, it is Berwaldian (cf. [4], [16]). The proof of Theorem 1.3 will be partially based on this result.

Remark 1.4. In Theorem 1.2 and Theorem 1.3, we assume that the Finsler metrics are Landsberg metrics. If Landsberg metrics are Berwald metrics, as newly several mathematicians claim, then our proof can be simplified.

In [5], D. Bao and C. Robles studied Einstein Randers metrics. We can see that our results on generalized Einstein metrics in Theorems 1.1- Theorem 1.3 are similar to those on Einstein Randers metrics in Lemma 11 and Propositions 12, 16 in [5].

In section 5, we will give some remarks about the Theorems in this section. In Section 6, we will give some examples about generalized Einstein metrics.

2. Preliminaries

Let (M, F) be a Finsler manifold with the fundamental function $F = F(x, y)$. For a vector $y = y^i \frac{\partial}{\partial x^i}|_x \neq 0$, F induces an inner product g_y on T_xM as follows

$$g_y(u, v) = g_{ij}(x, y)u^i v^j,$$

where $g_{ij}(x, y) := \frac{1}{2}[F^2]_{y^i y^j}$, $u = u^i \frac{\partial}{\partial x^i}|_x$ and $v = v^i \frac{\partial}{\partial x^i}|_x$. Further, the *Cartan torsion* $\mathbf{C}$ is defined as follows ([9])

$$\mathbf{C}_y(u, v, w) := C_{ijk}u^i v^j w^k,$$

where

$$C_{ijk}(x, y) := \frac{1}{4}[F^2]_{y^i y^j y^k}(x, y)$$

and $w = w^k \frac{\partial}{\partial x^k}|_x$. Let $I_i(x, y) := g^{jk}(x, y)C_{ijk}(x, y)$. $\mathbf{I} := I_i(x, y)dx^i$ is called the *mean Cartan torsion*.

Further, the geodesic $x = x(t)$ of Finsler metric F is characterized by the following system of 2nd order ordinary differential equations:

$$\frac{d^2 x^i(t)}{dt^2} + 2G^i\big(x(t), x'(t)\big) = 0,$$

where

$$G^i := \frac{1}{4} g^{il}\Big\{\big(\partial_r \dot{\partial}_l (F^2)\big)y^r - \partial_l (F^2)\Big\},$$

where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$, $\partial_i = \frac{\partial}{\partial x^i}$. G^i are called the *geodesic coefficients* of F.

Let

$$G^j_i := \dot{\partial}_i G^j$$

and

$$\Gamma^i_{jk} := \frac{g^{il}}{2}\big\{\delta_j g_{kl} + \delta_k g_{jl} - \delta_l g_{jk}\big\},$$

where $\delta_k := \partial_k - G^r_k \dot{\partial}_r$. Let the symbols " $_|$ " and " $|$ " denote the horizontal and vertical covariant derivatives with respect to Cartan connection respectively. As we know, Cartan connection is determined by the triple $(\Gamma^i_{jk}, G^i_j, C^i_{jk})$, where $C^i_{jk} := g^{il}C_{ljk}$. The h-, hv- and v-curvature tensors of Cartan connection are given respectively by

$$R_h{}^i{}_{jk} = \{\delta_k \Gamma^i_{hj} + \Gamma^r_{hj}\Gamma^i_{rk} - (j/k)\} + C^i_{hr}R^r{}_{jk}, \tag{1}$$

$$P_h{}^i{}_{jk} = \dot{\partial}_k \Gamma^i_{hj} - C^i_{hk|j} + C^i_{hr}L^r{}_{jk}, \tag{2}$$

$$S_h{}^i{}_{jk} = \dot{\partial}_k C^i_{hj} + C^r_{hj}C^i_{rk} - (j/k) = C^r_{hj}C^i_{rk} - (j/k), \tag{3}$$

where ${R^i}_{jk} := \delta_k G^i_j - (j/k) = {R_0}^i{}_{jk}$ and ${L^i}_{jk} = \dot{\partial}_k G^i_j - \Gamma^i_{kj} = {P_0}^i{}_{jk}$ and $T_{ij} - (i/j)$ means $T_{ij} - T_{ji}$ for a general tensor T_{ij}. We have the following identities ([3], [9])

$$R_{ijkl} = -R_{jikl} = -R_{ijlk}, \tag{4}$$

$$L_{ijk} = C_{ijk|0}, \quad P_{hijk} = \dot{\partial}_h L_{ijk} + L_{hjr} C^r_{ik} - (h/i), \tag{5}$$

where $C_{ijk|0} = C_{ijk|r} y^r, L_{ijk} := g_{ir} {L^r}_{jk}$, $R_{ijkl} := g_{jr} {R_i}^r{}_{kl}$ and $P_{ijkl} := g_{jr} {P_i}^r{}_{kl}$. Further, it is easy to see that $L_{ijk} = -\frac{1}{2} y^s g_{rs} {G_i}^r{}_{jk}$, where ${G_h}^i{}_{jk} = \dot{\partial}_h \dot{\partial}_j \dot{\partial}_k G^i$. A Finsler metric is called a *Berwald metric* if ${G_h}^i{}_{jk} = 0$. A Finsler metric is called a *Landsberg metric* if $L_{ijk} = 0$ ([9], [16]). A Landsberg metric has good geometric properties (see [1]).

For Cartan connection, we have the following Bianchi identities ([3], [9]).

$$C^h_{ir} {R^r}_{jk} - {R_i}^h{}_{jk} + (i, j, k) = 0, \tag{6}$$

$${P_m}^h{}_{ir} {R^r}_{jk} + {R_m}^h{}_{ij|k} + (i, j, k) = 0, \tag{7}$$

$${R^h}_{ij}|_k - {R_k}^h{}_{ij} + \left\{ {R^h}_{ir} C^r_{jk} + {L^h}_{ir} {L^r}_{jk} + {L^h}_{jk|i} - (i/j) \right\} = 0, \tag{8}$$

where $T_{ijk} + (i, j, k)$ means $T_{ijk} + T_{jki} + T_{kij}$ for a general tensor T_{ijk}.

For a tangent plane $P = span\{y, X\} \subseteq T_x M$, we define the *flag curvature* $\mathbf{K}$ of P by

$$\mathbf{K}(x, y, X) := \frac{R_{hijk}(x, y) y^h X^i y^j X^k}{\{g_{hj}(x, y) g_{ik}(x, y) - (j/k)\} y^h X^i y^j X^k}.$$

This definition is independent of some well-known connections. If the curvature $\mathbf{K}(x, y, X)$ is independent of X, that is $\mathbf{K} = \mathbf{K}(x, y)$ is just a scalar function on $TM \backslash \{0\}$, then the Finsler space is said to be of *scalar flag curvature.* If $\mathbf{K}$ is a constant, then the space is said to be of *constant flag curvature.* On the studies of scalar flag curvature, see [8], [12], [14], [15], etc.

Let $Y = Y^i(x) \partial / \partial x^i$ be a non-zero tangent vector filed on a domain D of the manifold M and $\widetilde{g} := g(x, Y(x))$ be the Y-Riemannian metric induced from the vector field Y. Denote by $B\widetilde{\Gamma}$ the Barthel linear connection associated to Cartan connection by the vector field Y. Let $\widetilde{\Gamma}^i_{jk}$ be the connection coefficients and $\widetilde{T}^i_{jk}$ and ${\widetilde{R}_h}{}^i{}_{jk}$ be the torsion and the curvature tensor of the connection $B\widetilde{\Gamma}$ respectively. Then

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{jk}(x, Y) + C^i_{jr}(x, Y) Y^r_k(x, Y), \tag{9}$$

$$\widetilde{T}^i_{jk} = C^i_{jr}(x, Y) Y^r_k(x, Y) - (j/k), \tag{10}$$

$$\widetilde{R}_h{}^i{}_{jk} = R_h{}^i{}_{jk}(x,Y) + \big\{P_h{}^i{}_{jk}(x,Y)Y^r_k(x,Y) - (j/k) \\ + S_h{}^i{}_{rs}(x,Y)Y^s_k(x,Y)Y^r_j(x,Y)\big\}, \tag{11}$$

where $Y^i_j(x,y) := \partial_j Y^i + G^i_j(x,y)$ ([3], [10]). It is easy to see that if $Y^i_j(x,Y)=0$ for some vector field Y, then the Barthel connection is just the Levi–Civita connection of the induced Riemann metric $\widetilde{g}$.

3. Proofs of Theorem 1.1 and Theorem 1.2

In order to prove Theorem 1.1, we first give the following lemmas. Because $L^i{}_{jk} = P_0{}^i{}_{jk}$, by (5), it is easy to show the following lemma.

Lemma 3.1. *Let (M,F) be a Finsler manifold. Then F is a Landsberg metric if and only if the hv-curvature tensor $P_h{}^i{}_{jk} = 0$.*

Furthermore, we can obtain the following

Lemma 3.2. *Let (M,F) be a 3-dimensional Finsler manifold. Fix $x \in M$ and $y \in T_xM$. By the local diffeomorphism between T_xM and M at x, we can choose a small enough neighborhood at $x \in M$ with the local coordinate system (x^r) such that at $x \in M$, $g_{rs} := g_y(\partial/\partial x^r, \partial/\partial x^s) = 0$ for $r \neq s$. Let $\rho = g^{rs}R_{rs}$. Then we have the following result at the fixed point $(x,y) \in TM$ for arbitrarily fixed $i \neq j$:*

$$R_{ijij} - g_{ii}R_{jj} - g_{jj}R_{ii} + \frac{1}{2}\rho g_{ii}g_{jj} = 0.$$

Proof. In the whole of the proof, all values should be taken at the fixed point $(x,y) \in TM$.

Note that the dimension $n = 3$. Now we fix in the following arbitrary i,j,k satisfying $i \neq j, j \neq k, i \neq k$. By the choice of the local coordinate system and the the relations in (4), we have $g^{rs} = 0$ and $g_{rs} = 0$ for $r \neq s$, and $R_{iiji} = 0$, $R_{ijjj} = 0$. Then we easily get

$$R_{ij} = \sum_r R_i{}^r{}_{jr} = \sum_{r,s} g^{rs}R_{isjr} = g^{ii}R_{iiji} + g^{jj}R_{ijjj} + g^{kk}R_{ikjk} \\ = g^{kk}R_{ikjk} = \frac{1}{g_{kk}}R_{ikjk},$$

Similarly,

$$R_{ii} = \sum_r R_i{}^r{}_{ir} = \sum_{r,s} g^{rs}R_{isir} = g^{jj}R_{ijij} + g^{kk}R_{ikik} = \frac{1}{g_{jj}}R_{ijij} + \frac{1}{g_{kk}}R_{ikik}.$$

Therefore,

$$R_{ijij} - g_{ii}R_{jj} - g_{jj}R_{ii} = -\frac{1}{g_{kk}}\{g_{kk}R_{ijij} + (i,j,k)\}$$

and

$$\{R_{ijij} - g_{ii}R_{jj} - g_{jj}R_{ii}\}/g_{ii}g_{jj} = -\frac{1}{det(g)}\{g_{kk}R_{ijij} + (i,j,k)\}. \tag{12}$$

where $det(g) = g_{ii}g_{jj}g_{kk}$ is the determinant of the matrix (g_{rs}).

On the other hand, we have

$$\begin{aligned}\rho = \sum_r g^{rr}R_{rr} &= g^{ii}\left\{\frac{1}{g_{jj}}R_{ijij} + \frac{1}{g_{kk}}R_{ikik}\right\} + (i,j,k)\\ &= \frac{2}{det(g)}\{g_{kk}R_{ijij} + (i,j,k)\}.\end{aligned} \tag{13}$$

Comparing (12) and (13) we obtain

$$R_{ijij} - g_{ii}R_{jj} - g_{jj}R_{ii} + \frac{1}{2}\rho g_{ii}g_{jj} = 0.$$

Now we have completed the proof. □

Lemma 3.3. *Let (M,F) be a 3-dimensional Finsler manifold with $R_{ij} = 2\lambda(x,y)g_{ij}$ for some scalar $\lambda(x,y)$. Then (M,F) is of scalar flag curvature $\mathbf{K} = \lambda$.*

Proof. To compute the flag curvature of any plane spanned by two vectors $y, X \in T_xM$ with $g_y(y,X) = 0$, we choose $Y \in T_xM$ satisfying $g_y(y,Y) = 0$ and $Y \neq X$. Select a local coordinate system (x^r) such that $y = \partial/\partial x^1|_x$, $X = \partial/\partial x^2|_x$ and $Y = \partial/\partial x^3|_x$. By Lemma 3.2 we have

$$R_{1212} - g_{11}R_{22} - g_{22}R_{11} + \frac{1}{2}\rho g_{11}g_{22} = 0.$$

Therefore,

$$\frac{R_{1212}}{g_{11}g_{22}} = \frac{R_{11}}{g_{11}} + \frac{R_{22}}{g_{22}} - \frac{1}{2}\rho = 4\lambda - \frac{1}{2}\rho = \lambda,$$

where we have used the fact $\rho = g^{rs}R_{rs} = 6\lambda$. This result is independent of the vector X, which implies that the Finsler manifold is of scalar flag curvature $\mathbf{K} = \lambda$. This completes the proof. □

Lemma 3.4. *Let (M,F) be an n-dimensional Finsler manifold with the dimension $n \geq 3$ and $R_{ij} = (n-1)\lambda(x,y)g_{ij}$ for some scalar $\lambda(x,y)$. Then $\lambda(x,y)$ is a constant and (M,F) is of constant flag curvature $\mathbf{K} = \lambda$ provided that (M,F) is of scalar flag curvature.*

Proof. By the Bianchi identity (8) we have

$$R_k{}^h{}_{ij} = R^h{}_{ij\cdot k} + R^r{}_{ij} C^h_{rk} + \big\{ L^h{}_{ir} L^r{}_{jk} + L^h{}_{jk|i} - (i/j) \big\}, \tag{14}$$

where we define the symbol $T_{i\cdot j} := \frac{\partial T_i}{\partial y^j}$ as an example. The same is in the following. By contraction on h and j in (14) we have

$$R_{ki} = R^r{}_{ir\cdot k} + R^r{}_{is} C^s_{rk} + \big\{ L^s{}_{ir} L^r{}_{sk} - I_{r|0} L^r{}_{ik} + I_{k|0|i} - L^r{}_{ik|r} \big\}. \tag{15}$$

Now suppose the Finsler space (M, F) is of scalar flag curvature $\tau(x, y)$. Then we have

$$R^i{}_{jk} = h^i_k \tau_j - h^i_j \tau_k,$$

where $\tau_j = \frac{1}{3} F^2 \tau_{\cdot j} + \tau y_j$. Therefore we have

$$\begin{aligned} R^r{}_{ir} &= \frac{1}{3}(n-2) F^2 \tau_{\cdot i} + (n-1)\tau y_i, \\ R^r{}_{ir\cdot k} &= \frac{2}{3}(n-2)\tau_{\cdot i} y_k + \frac{1}{3}(n-2) F^2 \tau_{\cdot i\cdot k} + (n-1)\tau_{\cdot k} y_i + (n-1)\tau g_{ik}, \\ R^r{}_{is} C^s_{rk} &= \frac{1}{3} F^2 (I_k \tau_{\cdot i} - C^r_{ik} \tau_{\cdot r}) + \tau I_k y_i. \end{aligned}$$

Now plug the above relations into (15) and we get

$$\begin{aligned} R_{ki} - (n-1)\tau g_{ik} &= \frac{2}{3}(n-2)\tau_{\cdot i} y_k + \frac{1}{3}(n-2) F^2 \tau_{\cdot i\cdot k} + (n-1)\tau_{\cdot k} y_i \\ &+ \frac{1}{3} F^2 \big(I_k \tau_{\cdot i} - C^r_{ik} \tau_{\cdot r} \big) + \tau I_k y_i + \big\{ L^s{}_{ir} L^r{}_{sk} - I_{r|0} L^r{}_{ik} + I_{k|0|i} - L^r{}_{ik|r} \big\}. \end{aligned} \tag{16}$$

Contracting (16) by y^i and y^k, we have

$$\big\{ R_{ki} - (n-1)\tau g_{ik} \big\} y^i y^k = 0 \tag{17}$$

Since $R_{ij} = (n-1)\lambda(x, y) g_{ij}$, by (17), we obtain

$$\lambda = \tau. \tag{18}$$

Further, contracting (16) by y^k and using (18) we get

$$\frac{1}{3}(n-2) F^2 \tau_{\cdot i} = 0. \tag{19}$$

Therefore we have $\tau_{\cdot i} = 0$ in dimension $n > 2$. Now by Proposition 3.2.2.1 in [3], we see that $\tau = constant$ when the dimension $n > 2$. So $\lambda(x, y)$ is a constant and (M, F) is of constant flag curvature $\mathbf{K} = \lambda$. This completes the proof. □

Proof of Theorem 1.1. Theorem 1.1(a) follows directly from Lemma 3.3 and Lemma 3.4. Theorem 1.1(b) follows directly from Lemma 3.4. □

Proof of Theorem 1.2. (a) If $\lambda \neq 0$, then the Finlser manifold (M, F) is a Riemannian manifold of constant curvature λ by Theorem 1.1(a) and Numata's theorem in [11].

(b) Since F is a Landsberg metric, we get $P_h{}^i{}_{jk} = 0$ by Lemma 3.1. Then by the Bianchi identity (7) we obtain

$$R_i{}^h{}_{jk|l} + (j, k, l) = 0.$$

Therefore

$$R_i{}^r{}_{jr|k} + R_i{}^r{}_{rk|j} + R_i{}^r{}_{kj|r} = 0.$$

Because $g_{ij|k} = 0$ and $R_{ij} = (n-1)\lambda g_{ij}$, by (4), we have

$$(n-1)\big(\lambda_{|k} g_{ij} - \lambda_{|j} g_{ik}\big) + g^{rs} R_{iskj|r} = 0$$

and

$$g^{ij}\big\{(n-1)\big(\lambda_{|k} g_{ij} - \lambda_{|j} g_{ik}\big) - g^{rs} R_{sikj|r}\big\} = 0.$$

From this, using $g^{ij} R_{sikj} = R_s{}^j{}_{kj} = R_{sk}$, we can get $(n-2)\lambda_{|k} = 0$. Hence $\lambda_{|k} = 0$ when $n > 2$. That is, λ is a h-covariant constant in case of $n \geq 3$. Now we have completed the proof. □

4. Proof of Theorem 1.3

In [16], Z. Shen proved that an (α, β)-metric $F(\alpha, \beta)$ is Landsbergian if and only if β is parallel with respect to α. In this case, $F(\alpha, \beta)$ is Berwaldian (also see [4]). Then G^i_{jk} is independent of y and $G^i_{jk}(x) = \Gamma^i_{jk}(x)$, where Γ^i_{jk} are the connection coefficients of Cartan connection. Further, we have $G^i = \hat{G}^i$, where $\hat{G}^i$ denotes the geodesic coefficients of α.

Now we consider the induced Riemann metric $\widetilde{g}$ by the tangent vector field $B(x) = b^i \frac{\partial}{\partial x^i}$. Since β is parallel with respect to α and $G^i = \hat{G}^i$, we have $Y^i_j = 0$ in (9) and (11). As we mentioned in Section 2, the Barthel connection induced by B is just the Levi–Civita connection of the induced Riemann metric $\widetilde{g}$ by B. By (9) we have $\widetilde{\Gamma}^i_{jk}(x) = \Gamma^i_{jk}(x)$. So $\widetilde{\Gamma}^i_{jk}(x) = \hat{G}^i_{jk}(x)$ and β is parallel with respect to the Riemann metric $\widetilde{g}$.

Thus we have the following facts: (i) $\Gamma^i_{jk}(x) = \hat{G}^i_{jk} = \widetilde{\Gamma}^i_{jk}(x)$ for F, α and $\widetilde{g}$; (ii) β is parallel with respect to the Riemann metrics α and $\widetilde{g}$. Then Theorem 1.3

(ia), (ib) and (ic) follow from these facts and the theories of Einstein metrics in Riemann geometry.

Next assume that $F(\alpha, \beta)$ is a generalized Einstein metric. By $Y^i_j = 0$ and (11) we have

$$\widetilde{R}_h{}^i{}_{jk}(x) = R_h{}^i{}_{jk}(x, B(x)).$$

Since $R_{ij} = (n-1)\lambda(x, y)g_{ij}$ we see that

$$\widetilde{R}_i{}^r{}_{jr}(x) = (n-1)\lambda(x, B(x))\widetilde{g}_{ij}(x),$$

which implies that the Riemann metric $\widetilde{g}$ is an Einstein metric. By the theories of Einstein metrics in Riemann geometry and (ia), (ic), we can get (iia) and (iib). In the case of (iia), since $F(\alpha, \beta)$ is flat-parallel, $F(\alpha, \beta)$ is locally Minkowskian, which implies that $\lambda(x, y) = 0$. Now we have completed the proof. □

5. Remarks

In this section we give some remarks and discussions on Theorems in Section 1. Firstly, we prove the following proposition.

Proposition 5.1. *If a Finsler metric F is of scalar flag curvature, then $R_{ijkl} = R_{klij}$. In this case, we have $R_{ij} = R_{ji}$.*

Proof. By the Bianchi identity (6) we have

$$R_{ijkl} + R_{kjli} + R_{ljik} = C_{ijr}R^r{}_{kl} + (i, k, l),$$
$$R_{jkli} + R_{lkij} + R_{ikjl} = C_{jkr}R^r{}_{li} + (j, l, i),$$
$$R_{klij} + R_{iljk} + R_{jlki} = C_{klr}R^r{}_{ij} + (k, i, j),$$
$$R_{lijk} + R_{jikl} + R_{kilj} = C_{lir}R^r{}_{jk} + (l, j, k).$$

Adding up the four identities above and using the relations (4), we get

$$2(R_{ljik} - R_{iklj}) = C_{ijr}R^r{}_{kl} + C_{jkr}R^r{}_{li} + C_{ljr}R^r{}_{ik} + (i, j, k, l). \tag{20}$$

where $T_{ijkl} + (i, j, k, l)$ means $T_{ijkl} + T_{jkli} + T_{klij} + T_{lijk}$ for a general tensor T_{ijkl} as an example. Notice that

$$C_{ijr}R^r{}_{kl} + (i, j, k, l) = C_{jkr}R^r{}_{li} + (i, j, k, l),$$
$$C_{ljr}R^r{}_{ik} + (i, j, k, l) = 0,$$

(20) is reduced to the following

$$R_{ljik} - R_{iklj} = \frac{1}{2}\{C_{ijr}R^r{}_{kl} + (i,j,k,l)\}. \tag{21}$$

Now suppose F is of scalar flag curvature $\mathbf{K} = \lambda(x,y)$. Then we have

$$R^i{}_{jk} = h^i_k\lambda_j - h^i_j\lambda_k, \tag{22}$$

where $\lambda_j = \frac{1}{3}F^2\lambda_{.j} + \lambda y_j$, $\lambda_{.j} = \dot{\partial}_i\lambda$, $h^i_k = \delta^i_k - \frac{y^i y_k}{F^2}$ and $y_k = y^r g_{kr}$. Plugging (22) into (21) and using $y^r C_{ijr} = 0$ we obtain

$$R_{ljik} - R_{iklj} = \frac{1}{2}\{C_{ijl}\lambda_k - C_{ijk}\lambda_l + (i,j,k,l)\} = 0.$$

This completes the proof. □

Proposition 5.1 shows that R_{ij} is a symmetric tensor in Finsler spaces of scalar flag curvature. However, R_{ij} also may be symmetric in non-trivial Finsler spaces which are not of scalar flag curvature (see examples in Section 6).

As we know, for an n-dimensional Riemann manifold ($n \geq 3$), if $R_{ij} = (n-1)\lambda(x)g_{ij}$, then $\lambda = \mathit{constant}$. However, Theorem 1.1 and Theorem 1.2 do not assure that $\lambda(x,y)$ must be a constant in Finsler spaces of dimension $n \geq 4$ which are not of scalar flag curvature (cf. Example 6.2).

Let (M,F) be an n-dimensional Finsler space with constant flag curvature $\mathbf{K} = \lambda$. Then by (16), $R_{ij} = (n-1)\lambda g_{ij}$ is equivalent to

$$\lambda I_k y_i + L^s{}_{ir}L^r{}_{sk} - I_{r|0}L^r{}_{ik} + I_{k|0|i} - L^r{}_{ik|r} = 0. \tag{23}$$

It is easily seen that (23) holds in Landsberg spaces with vanishing flag curvature $\lambda = 0$. In this case, $R_{ij} = 0$. Besides, we can verify that (23) always holds in the Finsler manifold of dimension 2.

Now we consider two kinds of generalized Einstein metrics. one of them is the metrics satisfying $R_{hijk} = \lambda(x,y)(g_{hj}g_{ik} - g_{hk}g_{ij})$. Another one is the metrics satisfying $R_{ijkl} = \frac{1}{n-1}(R_{ik}g_{jl} - R_{il}g_{jk})$. For the former case, when $n > 2$, Akbar-Zadeh shows in [2] that $\lambda(x,y)$ is a constant. Further, $S_h{}^i{}_{jk} = 0$ and $P_h{}^i{}_{jk} = P_h{}^i{}_{kj}$ if $\lambda \neq 0$. For the latter case, we have the following proposition.

Proposition 5.2. *Let (M,F) be a Finsler manifold of dimension $n \geq 3$. Suppose that R_{ijkl} satisfies $R_{ijkl} = \frac{1}{n-1}(R_{ik}g_{jl} - R_{il}g_{jk})$. Define $\rho = g^{ij}R_{ij}$. Then*

(i) *ρ is a constant and F has constant flag curvature $\mathbf{K}$ with $\mathbf{K} = \frac{\rho}{n(n-1)}$.*

(ii) *if* $\rho \neq 0$, *then* $S_h{}^i{}_{jk} = 0$ *and* $P_h{}^i{}_{jk} = P_h{}^i{}_{kj}$.

(iii) *if* $\rho \neq 0$ *and* F *is reversible , then* F *is a Riemannian metric of constant curvature.*

This Proposition can be easily proved by using Akbar-Zadeh's result in [2] and Brickell's result in [6].

6. Examples

In this section, we will give some examples on generalized Einstein metrics. We are also interested in finding the examples satisfying $P_h{}^i{}_{jk} = P_h{}^i{}_{kj}$.

Firstly, we introduce the *Weyl tensor* $\mathbf{W}$ which is defined by

$$W^i_{jk} = R^i{}_{jk} + \left\{y^i H_{jk} + \delta^i_j H_k - (j/k)\right\}/(n+1), \tag{24}$$

where $H_{jk} = R^r{}_{kr.j}$, $H_j = (nH_{0j} + H_{j0})/(n-1)$ ([3]). It is well-known that a Finsler space is of scalar flag curvature if and only if $W^i_{jk} = 0$.

Example 6.1. Consider $M = M_0 \times R^{n-2} (n > 2)$. Let (M, F) be an n-dimensional Finsler space with the metric F given by

$$F(x, y) = \sqrt{F_0^2 + \beta_3^2 + \cdots + \beta_n^2},$$

where (M_0, F_0) is a 2-dimensional Finsler space with $F_0 = F_0(x^1, x^2, y^1, y^2)$ and $\beta_3 = a_3(x^3)y^3, \cdots, \beta_n = a_n(x^n)y^n$ are non-zero 1-forms on R^{n-2}. Let $\mathbf{K}_0$ be the scalar flag curvature of F_0 and $\mathbf{K}$ be the flag curvature of F. Then the metric $F(x, y)$ has the following properties:

(a_1) F is of scalar flag curvature $\Longleftrightarrow \mathbf{K}_0 = 0 \Longleftrightarrow \mathbf{K} = 0$.

(a_2) $R_{ij} = (n-1)\lambda g_{ij}$ for some scalar $\lambda = \lambda(x, y) \Longleftrightarrow \mathbf{K}_0 = 0 \Longleftrightarrow \mathbf{K} = 0$. In this case, $\lambda = 0$.

(a_3) $P_h{}^i{}_{jk}$ are symmetric in j and k.

(a_4) F is a Landsberg metric if and only if F_0 is a Landsberg metric.

Proof. (a_1) Suppose that F is of scalar flag curvature, that is, $W^i_{jk} = 0$. By $F^2 = F_0^2 + \beta_3^2 + \cdots + \beta_n^2$ and the definition of $R_h{}^i{}_{jk}$ in (1), it is easily seen that $R_h{}^i{}_{jk}$ are equal in (M_0, F_0) and (M, F) for $i, j, k = 1, 2$, and $R_h{}^i{}_{jk} = 0$ if one of the indices h, j, k, i is greater than 2. Therefore, $R^i{}_{jk}$ $(= y^r R_r{}^i{}_{jk})$ and $H_{jk} = R^r{}_{kr.j}$ are also equal in (M_0, F_0) and (M, F) for $i, j, k = 1, 2$.

Consider arbitrary $j,k=1,2$ and some fixed $i\geq 3$. Then it is easily seen that ${R^i}_{jk}=0$, and then $W^i_{jk}=0$ implies that $H_{jk}=H_{kj}$ by (24). Now we consider again arbitrary $i,j,k=1,2$. Note that we also have $W^i_{jk}=0$ in the Finsler space (M_0,F_0) ($\dim M_0=2$). Then since $W^i_{jk}=0$ in (M_0,F_0), $W^i_{jk}=0$ in (M,F) and (24) we have

$$\frac{1}{3}\delta^i_j(2H_{0k}+H_{k0})-(j/k)=\frac{1}{n^2-1}\delta^i_j(nH_{0k}+H_{k0})-(j/k),$$

which can be simplified to

$$\delta^i_j[(2n+1)H_{0k}+(n+2)H_{k0}]-(j/k)=0.$$

By contraction on i and j (note that $i,j,k=1,2$) we have

$$(2n+1)H_{0k}+(n+2)H_{k0}=0. \tag{25}$$

Since H_{0k} and H_{k0} are equal in (M_0,F_0) and (M,F), we see by (3.5.2.6) in [3] that

$$H_{0k}=\tau y_k,\quad H_{k0}=\tau y_k+\tau_{;2}F_0m_k, \tag{26}$$

where we put the scalar curvature $\mathbf{K}_0=\tau$, $F_0\tau_{.i}=\tau_{;2}m_i$, and (l,m) is the Berwald frame on (M_0,F_0). Plugging (26) into (25), we have

$$3(n+1)\tau y_k+(n+2)\tau_{;2}F_0m_k=0.$$

Thus we have $\mathbf{K}_0=\tau=0$. If $\mathbf{K}_0=0$, it is easily verified that F has vanishing flag curvature, that is, $\mathbf{K}=0$.

(a_2) Suppose $R_{ij}=(n-1)\lambda g_{ij}$. Since $R_{33}=0$ and $g_{33}=(a_3(x^3))^2$, we have $\lambda=0$ and then $\mathbf{K}_0=0$. It is clear that $R_{ij}=0$ if $\mathbf{K}_0=0$.

(a_3) By a similar proof as that in (a_1), ${P_h}^i{}_{jk}$ are are equal in (M_0,F_0) and (M,F) for $h,j,k,i=1,2$. So in this case, ${P_h}^i{}_{jk}={P_h}^i{}_{kj}$. In other cases, ${P_h}^i{}_{jk}=0$ if one of h,i,j,k is greater than 2.

(a_4) This assertion is easily verified as that in (a_3). □

Example 6.1 shows that there are many non-Landsberg spaces with vanishing flag curvature such that $R_{ij}=0$.

Example 6.2. Consider $M=M_1\times M_2\times\cdots\times M_n$. Let (M,F) be a $2n$-dimensional Finsler space with the metric F given by

$$F(x,y)=\sqrt{F_1^2+F_2^2+\cdots+F_n^2},$$

where all of $(M_i, F_i)(i = 1 \cdots n)$ are 2-dimensional Finsler space with

$$F_1 = F_1(x^1, x^2, y^1, y^2), \cdots, F_n = F_n(x^{2n-1}, x^{2n}, y^{2n-1}, y^{2n})$$

and they are of the same constant flag curvature $\mathbf{K}_i = \lambda$. Then the metric F has the following properties:

(a_1) F is of scalar curvature if and only if $\lambda = 0$. Hence, if $\lambda \neq 0$, then F is not of scalar flag curvature.

(a_2) $R_{ij} = (n-1)\lambda g_{ij}$.

(a_3) $P_h{}^i{}_{jk}$ are symmetric in j and k.

(a_4) F is a Landsberg metric if and only if all F_i's are Landsberg metrics.

The proof of Example 6.2 is similar to the proof of Example 6.1. Example 6.2 shows that, in the case of dimension $n \geq 4$, we can find many Finsler metrics which are not of scalar flag curvatures such that $R_{ij} = (n-1)\lambda g_{ij}$ for some constants λ.

Finally, the following two open problems are interesting and important for further study:

(i) Find the non-Riemannian Finsler metrics of non-zero constant flag curvature λ with $R_{ij} = (n-1)\lambda g_{ij}$ in the case of dimension $n \geq 3$.

(ii) Find the non-Riemannian Landsberg spaces with $R_{ij} = (n-1)\lambda(x,y)g_{ij}$, $\lambda(x,y) \neq 0$, in the case of dimension $n \geq 4$.

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