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On finite p-groups with cyclic characteristic series

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Dedicated to Professor Z. Daróczy on the occasion of his 70th birthday

Abstract. Let G be a finite p-group having a characteristic cyclic series (c.c.s.) and let Φ be its Frattini subgroup. It is shown that the automorphism group of G is either a p-group or is the semidirect product of a normal p-Sylow subgroup of G by an elementary abelian group of exponent $p-1$ and of order $(p-1)^r$, where $1 \leq r \leq s$ and $s = |G/\Phi|$. It is also shown that G has a c.c.s. containing Φ.

1. Introduction

The group G is said to have a characteristic cyclic series (c.c.s.) if there is a chain of characteristic subgroups

$$G = L_k \subseteq L_{k-1} \subseteq \cdots \subseteq L_0 = \{1\}, \tag{1}$$

such that each L_{i+1}/L_i is cyclic. We consider finite p-groups, having cyclic characteristic series. If G is a finite p-group and it has a c.c.s. (1) then it has a characteristic composition series

$$G = N_0 \subseteq N_1 \subseteq N_2 \subseteq \cdots \subseteq N_n = \{1\} \tag{2}$$

with $|N_{i-1}/N_i| = p$ $(i = 1, 2, \ldots, n)$.

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Some particular classes of p-groups clearly have c.c.s.:

(a) cyclic p-groups,

(b) the p-groups whose normal subgroups are characteristic,

(c) the p-groups of maximal class.

In these cases the characteristic series (2) is the refinement of the only central series of the group (see [3] and [4]).

As it is known the Sylow subgroup P_{p^m} of the symmetric group $\mathcal{S}_{p^m}$ (p is an arbitrary prime) is a m-fold wreath product of cyclic groups of order p. Weir in [11] described the characteristic subgroups of P_{p^m}. This group has cyclic characteristic series which are refinements of the unique central series. In the case $m > 2$ it is not of maximal class and it has normal subgroups which are not characteristic. The group P_{p^m} is generated by m elements and for odd p the automorphism group of P_{p^m} is a semidirect product of a p-group by an elementary abelian group of exponent $p-1$ and of order $(p-1)^m$ (see [8]).

Let G be a finite p-group and $A(G)$ be the group of those automorphisms which fix every normal subgroup of G. In [3] it is shown that either $A(G)$ is a p-group or it is the semidirect product of a normal p-group by C_{p-1}, where C_{p-1} denotes the cyclic group of order $p-1$.

Durbin and McDonald [4] proved that $\mathrm{Aut}(G)$, the automorphism group of G, is supersolvable if G has a c.c.s. For finite p-groups G they showed that $\mathrm{Aut}(G)$ has a normal Sylow p-subgroup P with a p'-complement B and the exponent of $\mathrm{Aut}(G)$ divides $p^t(p-1)$ for some $t \geq 0$.

Baartmans and Woeppel [1] proved that if G is a p-group of maximal class of order p^n, where $n \geq 4$ and p is odd, then $\mathrm{Aut}(G)$ has a normal Sylow p-subgroup P and P has a p'-complement B, so that $\mathrm{Aut}(G)$ is a semidirect product of P with B. Furthermore, B is isomorphic to a subgroup of $C_{p-1} \times C_{p-1}$.

In [1] the authors remarked that the above theorem holds for any finite p-group G with a characteristic cyclic series. Our Theorem 1 gives the precise formulation of that remark.

Further results on automorphism groups of finite p-groups can be found in the survey paper [7].

2. Results

Theorem 1. *Let G be a finite p-group having a c.c.s. and Φ be the Frattini subgroup of G. Then the automorphism group $\mathrm{Aut}(G)$ of G is either a p-group (for $p = 2$ this always holds) or it is the semidirect product of the normal p-Sylow* *subgroup P by an elementary abelian group B of exponent $p-1$ and of order $(p-1)^r$, where $1 \leq r \leq s$ and $s = |G/\Phi|$.*

Proof of Theorem 1. If an automorphism acts trivially on each factor N_{i-1}/N_i, $(1 \leq i \leq n)$ of the series (2) then we say that it *stabilizes* that series.

Denote by P the group of automorphism which stabilizes the series (2) and by B the the restriction of the p'-automorphisms of G to G/Φ.

We will prove that for a finite p-group having c.c.s. there is a splitting exact sequence:

$$1 \mapsto P \mapsto \mathrm{Aut}(G) \mapsto B \mapsto 1.$$

To complete the proof we need the following well known statements.

Lemma 1 ([10] and [6] p. 179)**.** *Let G be finite p-group. If P is a subgroup of* $\mathrm{Aut}(G)$ *which stabilizes a normal series of G of length r then P is p-group of class $r-1$.*

Lemma 2 (Burnside (see [6] p. 174))**.** *Let α be an automorphism of the p-group G whose order is not divisible by p. If α induces an identity on $G/\Phi(G)$ then α is the identity on G.*

Each $\varphi \in \mathrm{Aut}(G)$ induces an automorphism on every factorgroup of the series (2). Let φ_j $(j = 1, 2, \ldots n)$ be the restriction of φ to the factorgroup N_{j-1}/N_j. Obviously $\mathrm{Aut}(N_{j-1}/N_j)$ is either identity or C_{p-1}.

It is clear that $\sigma : \varphi \mapsto (\varphi_1, \varphi_2, \ldots, \varphi_n)$ is a homomorphism of $\mathrm{Aut}(G)$ into the group

$$\mathrm{Aut}(N_0/N_1) \times \cdots \times \mathrm{Aut}(N_{n-1}/N_n).$$

Since the kernel P of σ stabilizes the series (2) by Lemma 1 P is the normal p-Sylow group of $\mathrm{Aut}(G)$.

By the theorem of Schur ([6] p. 221) there exist a normal complement B of P in $\mathrm{Aut}(G)$ such that $\mathrm{Aut}(G)/P \simeq B$ and $\mathrm{Aut}(G) = PB$, where $p \nmid |B|$. B is a subgroup of $\sigma(\mathrm{Aut}(G))$ therefore it is a subgroup of $C_{p-1} \times \cdots \times C_{p-1}$.
Let $\bar{\rho}$ be a restriction of the map $\rho : \mathrm{Aut}(G) \to \mathrm{Aut}(G/\Phi)$ to $B \subset \mathrm{Aut}(G)$. By Lemma 2 the kernel of $\bar{\rho}$ is identity, thus $\bar{\rho}$ is an isomorphism from B into $\mathrm{Aut}(G/\Phi)$, and as a consequence $\mathrm{Aut}(G)$ is a semidirect product of P with B.

Let $s = |G/\Phi|$. Considering G/Φ as an s-dimensional vector space over the field Z_p, the p' group B may be represented faithfully on the B-module G/Φ. By Maschke's Theorem ([9] p. 467) the B-module G/Φ can be written as a direct sum of irreducible B-modules. These irreducible B-modules have dimension 1, thus the elements of B act on direct components of G/Φ as a "diagonal map", i.e.

they act on the cyclic direct components of G/Φ either trivially or it is a power map of order $p-1$. Thus, the group B of exponent $p-1$ and of order $(p-1)^r$, where $0 \le r \le s$. □

Remark. Since the kernel of the homomorphism ρ fixes the cosets of G by Φ, the subgroup $\mathrm{Ker}(\rho)$ defines a partition on set of minimal generators of G and each class contains $\mathrm{Ker}(\rho)$ elements. Thus, $|P| = |\mathrm{Ker}(\rho)|$ divides $|\Phi|^s$.

If the normal subgroups of a p-group G are characteristic, then the elements of $\mathrm{Aut}(G)$ induce power automorphisms on G/Φ. Since G/Φ is abelian these induced automorphisms are universal power automorphisms on G/Φ (see [2]). Therefore the group B is either trivial or it is a cyclic group of order $p-1$. So we have

Corollary 1. *Let G be a finite p-group having a cyclic characteristic series of subgroups. If each automorphism induces a power automorphism on G/Φ, then the automorphism group of G is either a p-group, or it is a semidirect product of a normal p-group with an abelian group of order $p-1$.*

Theorem 2. *If a finite p-group G has a cyclic characteristic series then it has a cyclic characteristic series containing Φ.*

Proof. Let

$$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_n = \{1\}$$

be the characteristic series of G with $|N_{i-1}/N_i| = p$. Let $N_i/N_{i+1} = \langle x_i N_{i+1}\rangle$. Each element of $g \in G$ may be written as a product of n factors, i.e.

$$g = \prod_{i=1}^{n} x_i^{h_i}, \quad \text{where} \quad 0 \le h_i \le p-1.$$

Let $I = \{i \mid (\Phi \cap N_i)N_{i+1} \subset N_i\}$. It is easy to see that

$$|(\Phi \cap N_i) : (\Phi \cap N_{i+1})| = \begin{cases} 1 & \text{if } i \in I \\ p & \text{if } i \notin I. \end{cases}$$

The set $T = \{h_i \mid h_i = 0 \text{ whenever } i \notin I\}$, is identical to the set of coset representatives of G/Φ, therefore $|T| = |G/\Phi| = s = |I|$ and $|G/\Phi| = p^s$.

Considering the series

$$\Phi = N_0 \cap \Phi \supseteq N_1 \cap \Phi \supseteq N_2 \cap \Phi \supseteq \cdots \supseteq N_n \cap \Phi = 1 \tag{3}$$

we get the c.c.s. of G restricted to Φ. Ignoring those terms $N_i \cap \Phi$, where $i \notin I$ we have an increasing characteristic series in G.

With the notation $X_i = \{x_i^m \mid 0 \leq m < p\}$ we have

$$T \cap N_i = X_i(T \cap N_{i+1}) \quad \text{whenever } i \in I.$$

Since for $i_j \in I$ we have $T \cap N_{i_j} \supset T \cap N_{i_{j+1}}$, we obtain that $X_{i_j} = \{x_{i_j}^m \mid 0 \leq m < p\} \subseteq T \cap N_{i_j}$ and

$$T \cap N_{i_j} = X_{i_j}(T \cap N_{i_{j+1}}).$$

Clearly for $1 \leq i_j \leq n$ and $i_j \in I$ we have

$$0 < i_1 < i_2 < \dots i_{s-1} < i_s < n$$

and $x_{i_j} \in T \cap N_{i_j}$ while $x_{i_j} \notin T \cap N_{i_{j+1}}$.

Since G/Φ is a faithful B-module, the image of $\alpha \in B$ at the map

$$\bar{\rho} : B \mapsto \mathrm{Aut}(G/\Phi)$$

is $\bar{\rho}(\alpha) = A$, where A is a diagonal matrix

$$\begin{pmatrix} \alpha_{i_1} & 0 & 0 & 0 \\ 0 & \alpha_{i_1} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \alpha_{i_s} \end{pmatrix} \in GL(s, \mathrm{Z}_p).$$

Consequently, there exist a characteristic series

$$G = \langle x_{i_1}, x_{i_2}, \dots x_{i_s}, \Phi \rangle \supset \langle x_{i_2}, x_{i_3}, \dots x_{i_s}, \Phi \rangle \supset \dots \supset \langle x_{i_s}, \Phi \rangle \supset \Phi. \qquad \square$$

3. Examples

We give examples for p-groups having c.c.s. which are not in the above classes (a), (b), (c).

1. *Metacyclic p-group with cyclic maximal subgroup.*

Let

$$G = \langle a, b \mid a^{p^n} = 1, b^p = 1, ba = a^{p^{n-1}+1} \rangle.$$

Then G is a finite p-group of order p^{n+1} which is finite extension over a cyclic maximal subgroup of order p^n.

We have $\Phi = [G, G] = \langle a^p \rangle$ and

$$G \supset \langle b, a^p \rangle \supset \langle a^p \rangle \supset \langle a^{p^2} \rangle \supset \dots \langle a^{p^{n-1}} \rangle \supset \langle 1 \rangle$$

is a cyclic characteristic series. The subgroup $\langle a \rangle$ is normal but not characteristic subgroup in G. With some calculation it is easy to see that the elements $\varphi \in \mathrm{Aut}(G)$ are

$$\varphi : \begin{cases} a \mapsto a^i b^j & \text{where } i \in U(\mathrm{Z}_{p^n}),\ j \in \mathrm{Z}_p \\ b \mapsto a^k b & \text{where } k \in \mathrm{Z}_{p^n}/\mathrm{Z}_{p^{n-1}} \end{cases}.$$

The p'-automorphisms on G/Φ are of the form

$$\beta : \begin{cases} a\Phi \mapsto a^i \Phi & \text{where } 1 \le i < p \\ b\Phi \mapsto b\Phi \end{cases}.$$

The order of $\mathrm{Aut}(G)$ is $p^{n+1}(p-1)$. This group is the semidirect product of a p-group by C_{p-1}.

If G is in the above classes (a), (b) and (c) then it has unique central series. Next we give an example for a group with cyclic characteristic series and with not trivial p'-automorphism having different upper and lower central series.

2. $C_p \wr C_{p^m}$.

Let G be a standard wreath product of a cyclic group $C_p = \langle a \rangle$ of order p (p is prime) with a cyclic group $C_{p^m} = \langle b \rangle$ of order p^m ($m \ge 1$), i.e. $G = C_p \wr C_{p^m}$. The group G is generated by 2 elements and its order is p^{p^m+m} and it is a semidirect product of $K = \langle a_1 \rangle \times \langle a_2 \rangle \cdots \times \langle a_{p^m} \rangle$ by C_{p^m}. Here K is an elementary abelian group of exponent p and of rank p^m. Denote by

$$G = \Gamma_0 \supset \Gamma_1 \supset \cdots \supset \Gamma_c = \{1\}$$

the lower central series of G. Here $\Gamma_1 = [G, G] = [K, B]$ and the nilpotency index of G is p^m.

Let $\alpha_1 = a_1$ and for $j = 2, 3, \dots, p^m$ let $\alpha_j = [b, \alpha_{j-1}]$. Then $K = \langle \alpha_1, \alpha_2, \dots, \alpha_m \rangle$ and $\Gamma_k(G) = \langle \alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_{p^m} \rangle$, where $k = 1, 2, \dots p^m - 1$ and $c = p^m$. By some calculation for the upper central series

$$G = Z_{p^m} \supset Z_{p^m-1} \supset \cdots \supset Z_1 \supset Z_0 = \{1\}$$

we have

$$Z_{p^m-p^k+l} = \langle b^{p^k}, \alpha_{p^k+l+1}, \alpha_{p^k+l+2}, \dots, \alpha_{p^m}\rangle = \langle p^{p^k}, \Gamma_{p^k+l}(G)\rangle,$$
$$\text{for } 0 < k \leq m,\ 1 \leq l \leq p^k - p^{k-1}.$$

The group G has a series of characteristic subgroups with factor group of order p, for example

$$G \supset \langle b^p, K\rangle \supset \langle b^{p^2}, K\rangle \supset \dots K \supset \Gamma_1 \supset \Gamma_2 \supset \Gamma_{p^m}.$$

It is known (see [8]) that $\mathrm{Aut}(G)$ contains subgroups isomorphic to $\mathrm{Aut}(A)$ and $\mathrm{Aut}(B)$, thus in this case $|\,\mathrm{Aut}(G)| = p^t(p-1)^2$ for some $t > 1$.

Question. In all mentioned cases (a)–(c) and Examples 1. and 2. the c.c.s. are refinements of the lower central series. Is it always so?

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