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On continuous solutions of functional equations

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To the memory of Professor András Rapcsák

Abstract. In this work it is proved that under certain conditions the real continuous solutions f of the functional equation

$$f(t) = h(t, y, f(g_1(t, y)), \dots, f(g_n(t, y))),$$

are locally Lipschitz functions.

As it is treated in ACZÉL's classical book [1961], regularity is very important in the theory and practice of functional equations. The regularity problem of functional equations with two variables can be formulated as follows (see ACZÉL [1984] and JÁRAI [1986]):

Problem. Let T and Z be open subsets of $\mathbb{R}^s$ and $\mathbb{R}^m$, respectively, and let D be an open subset of $T \times T$. Let $f : T \to Z$, $g_i : D \to T$ $(i = 1, 2, \dots, n)$ and $h : D \times Z^{n+1} \to Z$ be functions. Suppose that

(1) $\quad f(t) = h(t, y, f(y), f(g_1(t, y)), \dots, f(g_n(t, y)))$ whenever $(t, y) \in D$;

(2) h is analytic;

(3) g_i is analytic and for each $t \in T$ there exists a y for which $(t, y) \in D$ and $\dfrac{\partial g_i}{\partial y}(t, y)$ has rank s $(i = 1, 2, \dots, n)$.

Is it true that every f, which is measurable or has the Baire property is analytic?

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The following steps may be used:

(I) Measurability implies continuity.
(II) Almost open solutions are continuous.
(III) Continuous solutions are locally Lipschitz.
(IV) Locally Lipschitz solutions are continuously differentiable.
(V) All p times continuously differentiable solutions are $p+1$ times continuously differentiable.
(VI) Infinitely many times differentiable solutions are analytic.

The complete answer to this problem is unknown. The problems corresponding to (I), (II), (IV) and (V) are solved in Járai [1986]. In the same paper, partial results in connection with (III) are treated. A partial result in connection with (VI) is treated in Járai [1988] (in Hungarian). Papers Járai [1992b] and [1992c] deal with locally Hölder continuous solutions, proving that locally Hölder continuous solutions with exponent $0<\alpha<1$ are also locally Hölder continuous with exponent $2\alpha/(1+\alpha)$. The paper Járai [1992a] deals with the local Lipschitz property of real solutions having locally bounded variation.

In this paper we deal with continuous real solution, and under certain additional conditions on the given functions g_i we prove that continuous solutions of the above functional equation are locally Lipschitz functions. The following lemma is similar to that proved in Járai [1992c]:

Lemma. *Let V, W and U be open real intervals, $R>0$, $[y_0-R, y_0+R]\subset W$, $g: V\times W\to U$ a continuously differentiable function, and $f: U\to\mathbb{R}$ a continuous function. Suppose that all partial functions $y\mapsto g(t,y)$ are monotonic with inverse denoted by $x\mapsto G_t(x)$. If there exist constants B, B', L and L' such that $|f(x)|\le B$, $|G_t'(x)|\le B'$, $|g(t,y)-g(t',y')|\le L(|t-t'|+|y-y'|)$ and $|G_t'(x)-G_{t'}'(x)|\le L'|t-t'|$ whenever $t,t'\in V$ and the left hand sides are defined, then the absolute value of the integral*

$$\int_{y_0-R}^{y_0+R} f(g(t,y))-f(g(t',y))\,dy$$

is bounded by $2LBB'|t-t'|+LBL'|t-t'|(|t-t'|+2R)$ whenever $t,t'\in V$.

Proof. The integral above can be written as the difference of two integrals. Using the substitution $x=g(t,y)$ in the first, and the substitution $x=g(t',y)$ in the second integral, we get

$$\int_{g(t,y_0-R)}^{g(t,y_0+R)} f(x)G_t'(x)dx-\int_{g(t',y_0-R)}^{g(t',y_0+R)} f(x)G_{t'}'(x)\,dx=$$

$$= \int_{g(t,y_0-R)}^{g(t',y_0-R)} f(x)G_t'(x)dx + \int_{g(t',y_0-R)}^{g(t,y_0+R)} f(x)(G_t'(x) - G_{t'}'(x))\,dx + \\ + \int_{g(t',y_0+R)}^{g(t,y_0+R)} f(x)G_{t'}'(x)\,dx.$$

The first and the last term can be estimated by $L|t-t'|BB'$, and the middle term by $L(|t-t'|+2R)BL'|t-t'|$.

Theorem. *Let T, Y, and Z be open subsets of $\mathbb{R}$, let D be an open subset of $T \times Y$, and let C be a compact subset of T. Consider the functions $f : T \to Z$, $g_i : D \to T$ $(i = 1, \dots, n)$, $h : D \times Z^n \to Z$. Suppose, that*

(1) *for each $(t, y) \in D$,*

$$f(t) = h(t, y, f(g_1(t, y)), \dots, f(g_n(t, y)));$$

(2) *h is twice continuously differentiable;*

(3) *g_i is twice continuously differentiable on D and for each $t \in T$ there exists a y such that $(t, y) \in D$, $g_i(t, y) \in C$ and $\dfrac{\partial g_i}{\partial y}(t, y) \neq 0$ for $i = 1, \dots, n$;*

(4) *the function f is continuous.*

Then f is locally Lipschitz function on T.

Proof. For an $\varepsilon > 0$ let $C_\varepsilon = \{x : \operatorname{dist}(x, C) \le \varepsilon\}$ denote the (closed) ε neighbourhood of C. Let us fix an $\varepsilon > 0$ such that $C_\varepsilon \subset T$. Then $|f|$ is bounded by B on C_ε. For each $0 \le r \le \varepsilon$ let

$$\omega(r) = \sup\{|f(t) - f(t')| : t \in C_\varepsilon,\ |t - t'| \le r, t' \in C_{\varepsilon - |t-t'|}\}.$$

Clearly ω is increasing, $\omega(0) = 0$, ω is continuous in 0 because f is uniformly continuous on the compact set C_ε, and $\omega(r_1 + r_2) \le \omega(r_1) + \omega(r_2)$ whenever $0 \le r_1, r_2, r_1 + r_2 \le \varepsilon$. To prove the last assertion, suppose that this inequality does not hold. Then there exist t, t' such that $|t-t'| \le r_1+r_2$, $t \in C_\varepsilon$ and $t' \in C_{\varepsilon-|t-t'|}$, but $|f(t)-f(t')| > \omega(r_1)+\omega(r_2)$. Choosing t'' between t and t' such that $|t - t''| \le r_1$ and $|t'' - t'| \le r_2$, we have $t'' \in C_{\varepsilon-|t-t''|}$, hence $|f(t)-f(t'')| \le \omega(r_1)$ and $|f(t'')-f(t')| \le \omega(r_2)$, which is a contradiction.

For an arbitrary $t_0 \in C_\varepsilon$, let us choose a y_0 (depending on t_0) by (3). Let us choose $\delta_{t_0} > 0$ and $R_{t_0} > 0$ such that for the open interval V_{t_0} with center t_0 and length $4\delta_{t_0}$ and for the closed interval W_{t_0} with center y_0 and length $2R_{t_0}$ the closure of $V_{t_0} \times W_{t_0}$ be contained in D, and $\dfrac{\partial g_i}{\partial y}$ does not vanish on $V_{t_0} \times W_{t_0}$. Hence the partial functions $y \mapsto g_i(t, y)$ have inverse on W_{t_0} for all $t \in V_{t_0}$ and $i = 1, 2, \dots, n$. Decreasing δ_{t_0} and R_{t_0} if necessary we may suppose that these inverses have derivatives bounded (in absolute value) by B'_{t_0} and are Lipschitz continuous with Lipschitz constant L'_{t_0} for $i = 1, 2, \ldots, n$. Similarly, we may suppose that g_i is a Lipschitz function with Lipschitz constant L_{t_0} on $V_{t_0} \times W_{t_0}$. We may suppose that L_{t_0} is an integer. Let us choose a positive constant η_{t_0} such that the closed $(n+2)\eta_{t_0}$ neighbourhood of $\{t_0\}\times\{y_0\}\times f(C)^n$ be contained in $D\times Z^n$. Let us choose an $0 < \varepsilon_{t_0} \le \varepsilon$ such that $\omega(\varepsilon_{t_0}) \le \eta_{t_0}$. Decreasing δ_{t_0} and R_{t_0} if necessary we may suppose that $2\delta_{t_0} \le \eta_{t_0}$, $R_{t_0} \le \eta_{t_0}$, and $g_i(V_{t_0}\times W_{t_0})$ is contained in the open interval with midpoint $g_i(t_0, y_0)$ and length $2\varepsilon_{t_0}$.

The open intervals with center $t_0 \in C_\varepsilon$ and length $2\delta_{t_0}$ give an open covering of the compact set C_ε. Hence there exists a finite set $T_0 \subset C_\varepsilon$ such that the open intervals corresponding to all $t_0 \in T_0$ give a finite open covering of C_ε. Let $L = \sup\{L_{t_0} : t_0 \in T_0\}$ and let $0 < \delta \le \inf\{\delta_{t_0} : t_0 \in T_0\}$, $0 < R_0 \le \inf\{R_{t_0} : t_0 \in T_0\}$ be such that $L(\delta + R_0) \le \varepsilon$. Let K denote the closure of $\bigcup_{t_0\in T_0} V_{t_0} \times W_{t_0}$. Clearly, K is a compact subset of D. Similarly, let K' denote the union of the closed $(n + 2)\eta_{t_0}$ neighbourhoods of $\{t_0\} \times \{y_0\} \times f(C)^n$ for $t_0 \in T_0$. Then K' is a compact subset of $D \times Z^n$, and hence the functions $\dfrac{\partial h}{\partial z_i}$ are Lipschitz continuous with Lipschitz constant L'_i, and the functions $\left|\dfrac{\partial h}{\partial t}\right|$ and $\left|\dfrac{\partial h}{\partial z_i}\right|$ are bounded by B'_0 and B'_i, respectively on K' $(i = 1, 2, \ldots, n)$. Moreover, let $B' = \sup\{B'_{t_0} : t_0 \in T_0\}$ and $L' = \sup\{L'_{t_0} : t_0 \in T_0\}$. Let t be an arbitrary element of C_ε, and let t' be an element of $C_{\varepsilon-|t-t'|}$ for which $|t - t'| < \delta$. There exists a $t_0 \in T_0$ such that $|t - t_0| < \delta_{t_0}$. In what follows, let us fix this t_0, the corresponding y_0, $V = V_{t_0}$ and $W = W_{t_0}$. Clearly $t, t' \in V$. Let R be an arbitrary real number for which $0 < R < R_0$. Let us integrate the two sides of the functional equation over the interval $[y_0 - R,\, y_0 + R]$ with respect to y. We have

$$2Rf(t) = \int_{y_0-R}^{y_0+R} h(t, y, f(g_1(t, y)), \ldots, f(g_n(t, y)))\, dy.$$

Hence

$$|f(t) - f(t')| = \frac{1}{2R}\left| \int_{y_0-R}^{y_0+R} h(t, y, f(g_1(t, y)), \ldots, f(g_n(t, y))) - \right.$$
$$\left. - h(t', y, f(g_1(t', y)), \ldots, f(g_n(t', y)))\, dy \right|.$$

To get a good upper estimate for the left hand side we need an upper estimate for the difference

$$h(t,y,f(g_1(t,y)),\dots,f(g_n(t,y)))-h(t',y,f(g_1(t',y)),\dots \\ \dots,f(g_n(t',y))).$$

We may apply the Taylor theorem for the function h with points

$$z=(t,y,z_1,\dots,z_n) \quad \text{and} \quad z'=(t',y,z_1',\dots,z_n'),$$

where $t',t\in V$, $y\in W$, $z_i=f(g_i(t,y))$ and $z_i'=f(g_i(t',y))$ for $i=1,\dots,n$. The points z and z', and hence the segment connecting them are contained in the ball with center $(t_0,y_0,f(g_1(t_0,y_0)),\dots,f(g_n(t_0,y_0)))$ and radius $(n+2)\eta_{t_0}$ contained in K'. We have

$$h(z)-h(z')=\int_0^1\frac{\partial h}{\partial t}(\tau z+(1-\tau)z')(t-t')d\tau+ \\ +\sum_{i=1}^n\int_0^1\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')(z_i-z_i')\,d\tau.$$

Using this and omitting variables we have

$$2R|f(t')-f(t)|=\left|\int_{y_0-R}^{y_0+R}\left(\int_0^1\frac{\partial h}{\partial t}(\tau z+(1-\tau)z')(t-t')\,d\tau+\right.\right. \\ \left.\left.+\sum_{i=1}^n\int_0^1\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')(z_i-z_i')d\tau\right)dy\right|.$$

Using the triangle inequality, we get $n+1$ terms on the right hand side. For the first term we get the trivial upper bound $2RB_0'|t'-t|$, where B_0' is an upper bound of $\left|\frac{\partial h}{\partial t}\right|$. Let $z_i^0=f(g_i(t,y_0))$ $(i=1,2,\dots,n)$, and let $z^0=(t,y_0,z_1^0,\dots,z_n^0)$. If h_i' denotes the value of the partial derivative $\frac{\partial h}{\partial z_i}$ at the point z^0, then the other terms can be rewritten as the absolute value of

$$\int_{y_0-R}^{y_0+R}\int_0^1\left(\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')-h_i'\right)(z_i-z_i')\,d\tau\,dy+ \\ +h_i'\int_{y_0-R}^{y_0+R}(z_i-z_i')\,dy.$$

First we give an upper estimate for the absolute value of the first term of this sum. An upper estimate of $|z_i-z_i'|$ is $\omega(L|t-t'|)$, because L is a

Lipschitz-constant for g_i on $V \times W$. Hence

$$\left|\int_{y_0-R}^{y_0+R}\int_0^1 \left(\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')-h_i'\right)(z_i-z_i')d\tau\, dy\right| \le$$

$$\le \omega(L|t-t'|)\int_{y_0-R}^{y_0+R}\int_0^1 \left|\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')-h_i'\right| d\tau\, dy.$$

We need to estimate the difference $\left|\frac{\partial h}{\partial z_i}(\tau z+(1-\tau)z')-\frac{\partial h}{\partial z_i}(z^0)\right|$. This is not greater than L_i' multiplied by the norm of $\tau z+(1-\tau)z'-z^0$, that is, L_i' times the maximal distance between the vectors z' and $z^0 = (t, y_0, z_1^0, \dots, z_n^0)$, where L_i' is a Lipschitz-constant for $\frac{\partial h}{\partial z_i}$. The maximal distance between z' and z^0 can be estimated by $|t-t'|+R+n\omega(L(|t-t'|+R))$. Hence we have the upper bound

$$2R\omega(|t-t'|L)L_i'(|t-t'|+R+n\omega(L(|t-t'|+R)))$$

for the first term.

To get an upper bound for the second term, we need an upper bound for the absolute value of

$$\int_{y_0-R}^{y_0+R}(z_i-z_i')dy=\int_{y_0-R}^{y_0+R} f(g_i(t,y))-f(g_i(t',y))dy,$$

because $|h_i'|$ is trivially bounded by the upper bound B_i' of $\left|\frac{\partial h}{\partial z_i}\right|$. By our lemma we get the upper bound $2L|t-t'|BB'+L(|t-t'|+2R)BL'|t-t'|$ for this integral.

Summing up all these estimates, we get

$$|f(t)-f(t')| \le$$

$$\le B_0'|t-t'|+\omega(L|t-t'|)\sum_{i=1}^{n} L_i'(|t-t'|+R+n\omega(L(|t-t'|+R)))+$$

$$+\sum_{i=1}^{n} B_i'(2L|t-t'|BB'+L(|t-t'|+2R)BL'|t-t'|))\,/\,R.$$

If $|t-t'|\le R$, this can be rewritten as

$$|f(t)-f(t')| \le C_0|t-t'|+C_1\omega(|t-t'|)R+C_2\omega(|t-t'|)\omega(R)+C_3|t-t'|/R,$$

where C_0, C_1 C_2 and C_3 do not depend on t, t' and R. Taking supremum first on the right, and then on the left hand side for $t\in C_\varepsilon$, $t' \in C_{\varepsilon-|t-t'|}$,

$|t - t'| \le r$, we have

$$\omega(r) \le C_0 r + C_1 \omega(r) R + C_2 \omega(r)\omega(R) + C_3 r/R$$

whenever $0 \le r \le \delta$. If we choose R such that it satisfies the condition $C_1 R + C_2 \omega(R) \le 1/2$ — which can always be done by decreasing δ if necessary — we have

$$\omega(r) \le 2(C_0 + C_3/R) r$$

whenever $0 \le r \le \delta$. This proves that f is a locally Lipschitz function on C. For an arbitrary $t \in T$, we may suppose without restricting generality, that t is an inner point of C, because otherwise we may replace C by the union of C and of a compact neighbourhood of t. Hence the theorem follows.

Corollary. *Suppose that the conditions of the above problem are satisfied with $s = m = 1$, and the additional condition (3) from the theorem is satisfied, too. Then f is infinitely many times differentiable.*

Proof. From theorem 1.4 in Járai [1986] it follows that f is continuous. The above theorem implies that f is locally Lipschitz. From theorem 1.5 in Járai [1986] it follows that f is infinitely many times differentiable.

Example. Usually the above theorem can be used with some additional argument. We illustrate this by a simple example. More complicated cases can be treated similarly. Suppose that f satisfies the functional equation

$$f(x+y) = h(x, y, f(x), f(y), f(x-y)), \quad x, y \in \mathbb{R}.$$

Here $f : \mathbb{R} \to \mathbb{R}$ is the unknown function and h is a given $\mathcal{C}^\infty$ function. Then all continuous solutions are locally Lipschitz (hence every measurable or almost open solution is $\mathcal{C}^\infty$ by results in Járai [1986]). To prove this let us substitute $t = x + y$ in the equation above. We get

$$f(t) = h(t-y, y, f(t-y), f(y), f(t-2y)), \quad t, y \in \mathbb{R}.$$

Now let $N > 0$ be a natural number, T the open interval $]-2N, 2N[$, $D = \{(t, y) : |y| < N,\ t = x + y,\ |x| < N,\ |x - y| < N\}$, and $C = [-N, N]$. Using the above theorem we have that f is locally Lipschitz on T, hence everywhere, because N is arbitrary. We remark that although there is a method to solve this functional equation (see Aczél [1961], pp. 62, 110) it is not clear from the method whether the solutions are smooth.

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