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Matrix splittings and generalized inverses

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Abstract. In this paper we introduce a splitting of the class of square singular complex matrices induced by its inner inverses in two ways: using the Jordan normal form, and using the concept of the condiagonalizability. Then we use the introduced splitting to prove a special case of Harte's theorem [5] for complex matrices.

1. The idea

Let $A \in \mathbb{C}_r^{n\times n}$ be a square complex matrix whose rank is equal to $r < n$. A matrix $B \in \mathbb{C}_k^{n\times n}$ is called an inner, or $\{1\}_k$-generalized inverse of A, if $ABA = A$ holds. It is well known that there exists some k, $r \le k < n$, such that a matrix A has a $\{1\}_k$-inverse. Moreover, there exists a whole set of inner inverses, and that set consists of matrices whose rank is in range from r to n, inclusively

$$A\{1\} = \bigcup_{k=r}^{n} A\{1\}_k.$$

Let us now concentrate on matrices belonging to the class $A\{1\}_n$. They are invertible, and we denote with $A\{1\}_n^{-1}$ the set of their "ordinary" inverses. This paper deals with the question: How can a matrix A be "close" to the set $A\{1\}_n^{-1}$, in terms of the spectral norm?

If B satisfies $ABA=A$, $(AB)^*=AB$ and $(BA)^*=BA$, then B is an $\{1,3,4\}$ -inverse of A. Moreover, if B is an $\{1,3,4\}$-inverse of A satisfying $BAB = B$,

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then B is the Moore–Penrose inverse of A, denoted by $A^\dagger$. It is well-known that for any complex rectangular matrix A, the Moore–Penrose inverse $A^\dagger$ is unique and it always exists.

The paper is organized as follows: in Section 2 one can find some important definitions and lemmas related to the Jordan normal form, Harte's theorem and condiagonalizability of matrices, respectively. Section 3 consists of two main results that introduce (in two different ways) matrix splitting induced by its inner inverses, followed by some theorems which answer the question about distance between the matrix A and a family $A\{1\}_n^{-1}$. Theorem 3 is actually a proof of a special case of Harte's theorem. Finally, Section 4 presents some facts about the introduced matrix splitting.

2. Auxiliary results

We start with the Jordan form.

Lemma 1. *Let there be given a Jordan matrix of order k (whose dimensions are $k \times k$), $k \in \mathbb{N}$*

$$J_k(0) = \begin{pmatrix} 0 & I_{k-1} \\ 0 & 0 \end{pmatrix}. \tag{1}$$

Then we have

$$J_k(0)\{1,3,4\} = \begin{pmatrix} 0 & \mathbb{C} \\ I_{k-1} & 0 \end{pmatrix} = \left\{ \begin{pmatrix} 0 & \alpha \\ I_{k-1} & 0 \end{pmatrix} : \alpha \in \mathbb{C} \right\}. \tag{2}$$

Proof. We can reduce the matrix $J_k(0)$ to its Hermitian normal form (using a method described, for example, in [1], pp. 24), and obtain

$$T = EJ_k(0)P = \begin{pmatrix} I_{k-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where} \quad E = I_k, \ P = \begin{pmatrix} 0 & 1 \\ I_{k-1} & 0 \end{pmatrix}.$$

Since P is a permutation matrix, we have $\det P = (-1)^{k+1} \neq 0$. We look for a $\{1,3,4\}$-inverse of matrix T using the definition, in the form

$$T^{(1,3,4)} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where A, B, C and D are submatrices whose dimensions are $(k-1) \times (k-1)$, $(k-1) \times 1$, $1 \times (k-1)$ and 1×1, respectively. It is easy to find that $A = I_{k-1}$,

$B = 0$, $C = 0$, but the matrix D remains arbitrary, let it be denoted by a complex number α. Hence

$$T^{(1,3,4)} = \begin{pmatrix} I_{k-1} & 0 \\ 0 & \alpha \end{pmatrix}$$

Since E and P are nonsingular unitary matrices, and since we know that

$$T^{(1,3,4)} = P^{-1} J_k^{(1,3,4)}(0) E^{-1},$$

and we have

$$J_k^{(1,3,4)}(0) = PT^{(1,3,4)} = \begin{pmatrix} 0 & 1 \\ I_{k-1} & 0 \end{pmatrix} \begin{pmatrix} I_{k-1} & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ I_{k-1} & 0 \end{pmatrix}.$$

□

Lemma 2. *There exists an invertible element in the set $J_k(0)\{1,3,4\}$.*

Proof. If $\alpha \neq 0$, then $\det J_k^{(1,3,4)}(0) = (-1)^{k+1}\alpha \neq 0$. This means that in the set $J_k(0)\{1,3,4\}$ there exists an invertible element, whose inverse we denote with $\widetilde{J_k}(0)$. Now it is obvious that

$$\widetilde{J_k}(0) = \begin{pmatrix} 0 & I_{k-1} \\ \frac{1}{\alpha} & 0 \end{pmatrix}. \tag{3}$$

□

We use $\|\cdot\|$ to denote the spectral norm of elements in $\mathbb{C}^{n\times n}$. The spectral norm of a matrix A is the square root of the spectral radius of A^*A. The following result is well-known.

Lemma 3. *Let A and B be two square complex matrices. Then we have*

a) *If $W = \left(\begin{smallmatrix} 0 & 0 \\ 0 & A \end{smallmatrix}\right)$, then $\|W\| = \|A\|$;*

b) *If $W = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, then $\|W\| = \max\{\|A\|, \|B\|\}$.*

Lemma 4. *A spectral norm of the matrix $J_k(0) - \widetilde{J_k}(0)$ is given by*

$$\|J_k(0) - \widetilde{J_k}(0)\| = \frac{1}{|\alpha|}. \tag{4}$$

Proof. Let W be the difference $J_k(0) - \widetilde{J_k}(0)$. We know that

$$W = \begin{pmatrix} 0 & 0_{(k-1)\times(k-1)} \\ -\frac{1}{\alpha} & 0 \end{pmatrix},$$

which means that

$$W^*W = \begin{pmatrix} \frac{1}{\alpha\bar{\alpha}} & 0 \\ 0 & 0_{(k-1)\times(k-1)} \end{pmatrix},$$

so $\sigma(W^*W) = \{0, \frac{1}{|\alpha|^2}\}$, which implies $\|W\| = \frac{1}{\alpha}$. □

For any complex matrix A, let $A^\dagger$ denote its unique Moore–Penrose inverse (which always exists). The Moore–Penrose inverse of matrix $J_k(0)$ will be $J_k^\dagger(0) = \left(\begin{smallmatrix} 0 & 0 \\ I_{k-1} & 0 \end{smallmatrix}\right)$.

Lemma 5. *The spectral norm of a matrix* ${J_k}^{(1,3,4)}(0) - {J_k}^\dagger(0)$ *is given by*

$$\|{J_k}^{(1,3,4)}(0) - {J_k}^\dagger(0)\| = |\alpha|. \tag{5}$$

Proof. Let W denote difference ${J_k}^{(1,3,4)}(0) - {J_k}^\dagger(0)$. We know that

$$W = \begin{pmatrix} 0 & \alpha \\ 0_{(k-1)\times(k-1)} & 0 \end{pmatrix},$$

which means

$$W^*W = \begin{pmatrix} 0_{(k-1)\times(k-1)} & 0 \\ 0 & \alpha\bar{\alpha} \end{pmatrix},$$

so $\sigma(W^*W) = \{0, |\alpha|^2\}$, which implies $\|W\| = |\alpha|$. □

From Lemma 4 and Lemma 5 we can infer interesting conclusion

$$\|J_k(0) - \widetilde{J_k}(0)\| \cdot \|{J_k}^{(1,3,4)}(0) - {J_k}^\dagger(0)\| = 1,$$

which means that requests for simultaneous approximating both $J_k(0)$ with $\widetilde{J_k}(0)$ and $J_k^\dagger(0)$ with $J_k^{(1,3,4)}(0)$ are mutually opposed.

Lemma 6. *The following holds*

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m \end{pmatrix}^\dagger = \begin{pmatrix} A_1^\dagger & 0 & \cdots & 0 \\ 0 & A_2^\dagger & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m^\dagger \end{pmatrix}, \tag{6}$$

where A_i, $i = \overline{1,m}$, *are square complex matrices.*

Now we present some facts related to Harte's Theorem.

Let $\mathcal{A}$ be Banach algebra with the unit 1. An element $a \in \mathcal{A}$ is regular (regular in a von Neumann sense) in $\mathcal{A}$ if there exists some $x \in \mathcal{A}$ such that $axa = a$. We denote by $\mathcal{A}^\sqcap$ the set consisting of all regular elements from $\mathcal{A}$. The next notation is also correct: $\mathcal{A}^\sqcap = \{a \in \mathcal{A} : a \in a\mathcal{A}a\}$. We use $\mathcal{A}^\bullet$ to denote set which consists of all idempotents from $\mathcal{A}$. Then we have $\{a \in \mathcal{A} : a \in a\mathcal{A}^{-1}a\} = \mathcal{A}^\bullet\mathcal{A}^{-1} = \mathcal{A}^{-1}\mathcal{A}^\bullet$ [4].

Proposition 1 (Harte [5]). *Let $\mathcal{A}$ be a Banach algebra with the unit 1. Then we have*

$$\mathcal{A}^{\sqcap} \cap \mathrm{cl}(\mathcal{A}^{-1}) = \mathcal{A}^{-1}\mathcal{A}^{\bullet}.$$

The proof of this proposition can be found in [5], alternatively in [8], p. 181. More general results, concerning Fredholm operators can be found in [7], and for Fredholm theory related to Banach algebra homomorphisms in [3]. In this paper Theorem 3 presents a constructive proof for Harte's theorem for a class of singular square complex matrices, based on a matrix splitting induced by inner inverses.

We will list three definitions, one comment, and the main theorem from [6], which deals with condiagonalizability concept.

For a matrix $A = [a_{ij}]_{n\times n}$, its component-wise conjugate is the matrix $\overline{A} = [\overline{a_{ij}}]_{n\times n}$. The component-wise conjugate is related to the adjoint and the transpose: $\overline{A} = (A^*)^T = (A^T)^*$, so that also $A^* = (\overline{A})^T$.

Definition 1. A matrix $A \in \mathbb{C}^{n\times n}$ is condiagonalizable if $A_R = A\overline{A}$ (or, which is the same, $A_L = \overline{A}A$) is diagonalizable by a similarity transformation.

Definition 2. Matrices $A, B \in \mathbb{C}^{n\times n}$ are said to be consimilar if $A = SB\overline{S}^{-1}$ for a nonsingular matrix $S \in \mathbb{C}^{n\times n}$.

Definition 3. Let $\sigma(A_L) = \{\lambda_1, \ldots, \lambda_m\}$ be the spectrum of A_L. The coneigenvalues of A are the m scalars $\mu_1, \ldots, \mu_m$, defined as follows:
If $\lambda_i \notin (-\infty, 0)$, then the corresponding coneigenvalue μ_i is defined as: $\mu_i = \sqrt{\lambda_i}$, $Re(\mu_i) \geq 0$; the multiplicity of μ_i is set to that of λ_i.
If $\lambda_i \in (-\infty, 0)$, then we associate two conjugate purely imaginary coneigenvalues $\mu_i = \pm\sqrt{\lambda_i}$. The multiplicity of each is set to a half of that of λ_i.

If $A \in \mathbb{R}^{n\times n}$, then each eigenvalue of A with a nonnegative real part is at the same time a coneigenvalue of this matrix. If an eigenvalue λ has a negative real part, then $\mu = -\lambda$ is a coneigenvalue of A.

We use $\ker A$ to denote the null-space of A.

Proposition 2 (Ikramov [6]). *Let $A \in \mathbb{C}^{n\times n}$ be a condiagonalizable matrix. Then A can be brought by a consimilarity transformation to its canonical form which is a direct sum of 1×1 and 2×2 blocks. The 1×1 blocks are the real nonnegative coneigenvalues of A, while each 2×2 block corresponds to a pair of complex conjugate coneigenvalues μ, $\overline{\mu}$, and has the form*

$$\begin{pmatrix} 0 & \overline{\mu} \\ \mu & 0 \end{pmatrix}.$$

If A is singular and $k = \dim\ker A_L - \dim\ker A > 0$, then the canonical form of A also contains k blocks of the form

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

3. Main results

We start with the following result

Theorem 1. *Any complex matrix $A \in \mathbb{C}_r^{n\times n}$, which Jordan normal form does not include a $J_1(0)$ block, can be split as a sum*

$$A = \tilde{A} + N, \tag{7}$$

where $\tilde{A}$ is invertible, $\tilde{A}^{-1} \in A\{1\}$, and N is a nilpotent matrix of the nilpotency order equal to 2 (which means that $N^2 = 0$).

Proof. Any square complex matrix can be reduced to the Jordan normal form

$$A = X \begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix} X^{-1},$$

where X is nonsingular, $J_1 \in \mathbb{C}_r^{r\times r}$ is invertible, and J_0 is nilpotent and consists from blocks $J_k(0)$, $k > 1$, each of them is a Jordan matrix. We look for $A^{(1)}$ in the form

$$A^{(1)} = X \begin{pmatrix} P & Q \\ R & S \end{pmatrix} X^{-1}.$$

The equation $AA^{(1)}A = A$ must be satisfied by any $\{1\}$-inverse, so we conclude that

$$J_1PJ_1 = J_1 \Rightarrow P = J_1^{-1}$$

$$J_1QJ_0 = 0 \Rightarrow QJ_0 = 0$$

$$J_0RJ_1 = 0 \Rightarrow J_0R = 0$$

$$J_0SJ_0 = J_0 \Rightarrow S = J_0^{(1)}$$

For the sake of clarity we choose $Q = 0$, $R = 0$. Hence

$$X \begin{pmatrix} J_1^{-1} & 0 \\ 0 & J_0^{(1)} \end{pmatrix} X^{-1} \in A\{1\}. \tag{8}$$

We now find a $\{1,3,4\}$-inverse of a block

$$J_0 = \begin{pmatrix} J_{k_1}(0) & 0 & \cdots & 0 \\ 0 & J_{k_2}(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{k_m}(0) \end{pmatrix}. \tag{9}$$

We can assume that this inverse can be in the form

$$J_0^{(1,3,4)} = \begin{pmatrix} J_{k_1}^{(1,3,4)}(0) & 0 & \cdots & 0 \\ 0 & J_{k_2}^{(1,3,4)}(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{k_m}^{(1,3,4)}(0) \end{pmatrix}. \tag{10}$$

This special inner inverse for each block $J_{k_i}(0), i = \overline{1,m}$, can be found as it is described in Lemma 1. Arbitrary complex element which participate in the i-th block we denote as α_i. If we enforce the natural condition $\alpha_i \neq 0,\ i = \overline{1,m}$, then Lemma 2 implies the existence of the inverse of each submatrix $J_{k_i}^{(1,3,4)}$, and further implies the existence of the inverse for the matrix ${J_0}^{(1,3,4)}$

$$\widetilde{J_0} = \begin{pmatrix} \widetilde{J_{k_1}}(0) & 0 & \cdots & 0 \\ 0 & \widetilde{J_{k_2}}(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{J_{k_m}}(0) \end{pmatrix}. \tag{11}$$

Since the inverse of the i-th submatrix depends on a parameter α_i, $\tilde{J}_0$ depends on complex nonzero parameters $\alpha_1, \ldots, \alpha_m$.

If we use $\tilde{A}$ for the inverse of $A^{(1)}$, it is obvious that

$$\tilde{A} = X \begin{pmatrix} J_1 & 0 \\ 0 & \widetilde{J_0} \end{pmatrix} X^{-1}. \tag{12}$$

Let $N = A - \tilde{A}$. We have

$$N = X \begin{pmatrix} 0 & 0 \\ 0 & J_0 - \widetilde{J_0} \end{pmatrix} X^{-1}. \tag{13}$$

The block-diagonal matrix $J_0 - \widetilde{J_0}$ consists of blocks

$$J_{k_i}(0) - \widetilde{J_{k_i}}(0) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ -\alpha_i^{-1} & \cdots & 0 \end{pmatrix}, \tag{14}$$

and now it is easy to obtain that

$$N^2 = 0. \qquad \square$$

Remark 1. The condition from the statement of a previous theorem, which relates to block $J_1(0)$, significantly decreases the class of matrices for which the theorem is applicable. For example, the class of all matrices of the index equal to 1 (there belong non-invertible hermitian, normal and range-hermitian matrices) is an excellent example for a class to which previous theorem is unapplicable.

Counterexample. Let us deal with the matrix

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Theorem 1 can not be applied to our matrix A, and A can not be split using a method described in Theorem 1 as a sum

$$A = \tilde{A} + N, \quad \det \tilde{A} \neq 0, \quad N^2 = 0.$$

But, we try to find that splitting on some other way. We use the next easy-to-prove observation

$$(\forall N \in \mathbb{C}^{2\times 2})\, N^2 = 0 \neq N \Leftrightarrow N = \begin{pmatrix} t & s \\ -\frac{t^2}{s} & -t \end{pmatrix}, \quad t \in \mathbb{C},\ s \in \mathbb{C} \setminus \{0\}.$$

Now we have

$$A = \tilde{A} + N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} t & s \\ -\frac{t^2}{s} & -t \end{pmatrix},$$

where we assume that $\det \tilde{A} = ad - bc \neq 0$. This imply $a = 1 - t$, $b = -s$, $c = \frac{t^2}{s}$ i $d = t$, and further

$$A = \tilde{A} + N = \begin{pmatrix} 1-t & -s \\ \frac{t^2}{s} & t \end{pmatrix} + \begin{pmatrix} t & s \\ -\frac{t^2}{s} & -t \end{pmatrix}, \quad \det \tilde{A} = t,$$

and we now conclude that must be $t \neq 0$. Indeed, we find the desired splitting!

Theorem 2. *Any singular condiagonalizable complex matrix $A \in \mathbb{C}_r^{n\times n}$ such that $k = \dim\ker A_L - \dim\ker A > 0$ can be split into the sum*

$$A = \tilde{A} + N, \tag{15}$$

where $\tilde{A}$ is invertible, $\tilde{A}^{-1} \in A\{1\}$, and N satisfy $N\overline{N} = \overline{N}N = 0$.

PROOF. Let the matrix A satisfy conditions from this theorem. Then, it can be, because of Proposition 2, expressed as

$$A = S \begin{pmatrix} D_1 & 0 \\ 0 & D_0 \end{pmatrix} \overline{S}^{-1} = SB\overline{S}^{-1}, \tag{16}$$

where

$$D_1 = \operatorname{diag}\left\{ \lambda_1, \dots, \lambda_p, \begin{pmatrix} 0 & \overline{\mu}_1 \\ \mu_1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & \overline{\mu}_q \\ \mu_q & 0 \end{pmatrix} \right\}$$

is the invertible matrix, and

$$D_0 = \operatorname{diag}\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

consists of exactly k blocks, where $k = \dim\ker(A_L) - \dim\ker(A) > 0$. We use $\lambda_i, i = \overline{1,p}$, to denote nonnegative coneigenvalues of A, while μ_j and $\overline{\mu}_j, j = \overline{1,q}$, are pairs of complex conjugate coneigenvalues.

We look for $A\{1\}$ in the form

$$A^{(1)} = \overline{S} \begin{pmatrix} M & N \\ P & Q \end{pmatrix} S^{-1}.$$

Because of $AA^{(1)}A = A$, we conclude that

$$D_1 M D_1 = D_1 \Rightarrow M = D_1^{-1}$$

$$D_1 N D_0 = 0 \Rightarrow N D_0 = 0$$

$$D_0 P D_1 = 0 \Rightarrow D_0 P = 0$$

$$D_0 Q D_0 = D_0 \Rightarrow Q = D_0^{(1)}$$

We put $N = 0$, $P = 0$. Hence,

$$\overline{S} \begin{pmatrix} D_1^{-1} & 0 \\ 0 & D_0^{(1)} \end{pmatrix} S^{-1} \in A\{1\}. \tag{17}$$

Now, we look for a $\{1,3,4\}$-inverse of a block-matrix

$$D_0 = \begin{pmatrix} 0 & 0 & & & & & \\ 1 & 0 & & & & & \\ & & 0 & 0 & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 0 \\ & & & & & 1 & 0 \end{pmatrix}. \tag{18}$$

in the form

$$D_0^{(1,3,4)} = \mathrm{diag}\left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{(1,3,4)}, \dots, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}^{(1,3,4)} \right\}. \tag{19}$$

It is easy to find that

$$D_0^{(1,3,4)} = \begin{pmatrix} 0 & 1 & & & & & \\ \beta_1 & 0 & & & & & \\ & & 0 & 1 & & & \\ & & \beta_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & 1 \\ & & & & & \beta_k & 0 \end{pmatrix}. \tag{20}$$

Under the assumption that arbitrary complex elements in i-th block is subjected by condition $\beta_i \neq 0, i = \overline{1,k}$, we can find also

$$\widetilde{D_0} = (D_0^{(1,3,4)})^{-1} = \begin{pmatrix} 0 & \beta_1^{-1} & & & & & \\ 1 & 0 & & & & & \\ & & 0 & \beta_2^{-1} & & & \\ & & 1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & \beta_k^{-1} \\ & & & & & 1 & 0 \end{pmatrix}. \tag{21}$$

We use $\tilde{A}$ for the ordinary inverse of $A^{(1)}$. It is obvious that

$$\tilde{A} = S \begin{pmatrix} D_1 & 0 \\ 0 & \widetilde{D_0} \end{pmatrix} \overline{S}^{-1}. \tag{22}$$

We have

$$N = A - \tilde{A} = S \begin{pmatrix} 0 & 0 \\ 0 & D_0 - \widetilde{D_0} \end{pmatrix} \overline{S}^{-1}. \tag{23}$$

The block-diagonable matrix $D_0 - \widetilde{D_0}$ contains the blocks

$$\begin{pmatrix} 0 & -\beta_i{}^{-1} \\ 0 & 0 \end{pmatrix}. \tag{24}$$

Now it is easy to obtain that

$$N\overline{N} = \overline{N}N = 0. \qquad \square$$

The next result is a constructive proof of a special case of HARTE's theorem from [5].

Theorem 3. *Under the assumptions from Theorem 1, we have*

$$\inf_{\substack{(\alpha_1,\dots,\alpha_m)\in\mathbb{C}^m \\ \alpha_i\neq 0, i=\overline{1,m}}} \|A-\tilde{A}\| = 0, \tag{25}$$

where α_i are parameters which $\tilde{A}$ is depended from.

PROOF. Writing "inf" for the infimum over the parameters listed in formula (25), we have

$$\inf \|A-\tilde{A}\| = \inf \|N\| = \inf \|X \begin{pmatrix} 0 & 0 \\ 0 & J_0-\widetilde{J_0} \end{pmatrix} X^{-1}\|$$
$$\leq \|X\|\,\|X^{-1}\| \inf \left\| \begin{pmatrix} 0 & 0 \\ 0 & J_0-\widetilde{J_0} \end{pmatrix} \right\| = \|X\|\,\|X^{-1}\| \inf \left(\max_{i=\overline{1,m}} \frac{1}{|\alpha_i|} \right) = 0,$$

because, using Lemmas 3 and 4, we have

$$\inf \|J_0-\widetilde{J_0}\| = \inf \left(\max_{i=\overline{1,m}} \|J_{k_i}(0)-\widetilde{J_{k_i}}(0)\| \right) = \inf \left(\max_{i=\overline{1,m}} \frac{1}{|\alpha_i|} \right) = \inf_{\alpha_s\in\mathbb{C}\setminus\{0\}} \frac{1}{|\alpha_s|},$$

and it is equal to 0 when $\alpha_s\to\infty$ in a complex plane. □

Theorem 4. *Under the assumptions from Theorem 1, we have*

$$\inf_{\substack{c(\alpha_1,\dots,\alpha_m)\in\mathbb{C}^m \\ \alpha_i\neq 0, i=\overline{1,m}}} \|\widetilde{A}^{-1}-A^\dagger\| = 0, \tag{26}$$

where α_i are parameters which $\tilde{A}$ is depended from.

PROOF. Again, writing "inf" for the infimum over the parameters listed in formula (26), we have

$$\inf \|\widetilde{A}^{-1}-A^\dagger\| = \inf \|X \begin{pmatrix} 0 & 0 \\ 0 & J_0^{(1,3,4)}-J_0^\dagger \end{pmatrix} X^{-1}\|$$
$$\leq \|X\|\,\|X^{-1}\| \inf \left\| \begin{pmatrix} 0 & 0 \\ 0 & J_0^{(1,3,4)}-J_0^\dagger \end{pmatrix} \right\| = \|X\|\,\|X^{-1}\| \inf \left(\max_{i=\overline{1,m}} |\alpha_i| \right) = 0,$$

because, using Lemmas 5 and 6:

$$\begin{aligned}\inf \|J_0^{(1,3,4)} - J_0^\dagger\| &= \inf \left(\max_{i=\overline{1,m}} \|J_{k_i}^{(1,3,4)}(0) - J_{k_i}^\dagger(0)\| \right) \\ &= \inf \left(\max_{i=\overline{1,m}} |\alpha_i| \right) = \inf_{\alpha_s \in \mathbb{C}\setminus\{0\}} |\alpha_s|,\end{aligned}$$

and it is equal to 0 when $\alpha_s \to 0$ in a complex plane. □

We actually concluded that we deal with good approximation, but a parameter in one case tends to zero, and in other to the infinity. It raises a question whether it is possible to simultaneously approximate both A with $\tilde{A}$, and $A^\dagger$ with $\tilde{A}^{-1}$. Two next theorems answer to this question negatively.

Theorem 5. *Under the assumptions from Theorem 1, we have*

$$\inf_{\substack{(\alpha_1,\dots,\alpha_m)\in\mathbb{C}^m \\ \alpha_i \neq 0, i=\overline{1,m}}} \|A - \tilde{A}\| \cdot \|\tilde{A}^{-1} - A^\dagger\| \leq \left(\|X\| \cdot \|X^{-1}\|\right)^2. \tag{27}$$

Proof. Writing "inf" for the infimum over the parameters listed in formula (27), and h for $\|X\| \cdot \|X^{-1}\|$, we have

$$\begin{aligned}&\inf \|A - \tilde{A}\| \cdot \|\tilde{A}^{-1} - A^\dagger\| \\ &= \inf \left\| X \begin{pmatrix} 0 & 0 \\ 0 & J_0 - \widetilde{J_0} \end{pmatrix} X^{-1} \right\| \cdot \left\| X \begin{pmatrix} 0 & 0 \\ 0 & J_0^{(1,3,4)} - J_0^\dagger \end{pmatrix} X^{-1} \right\| \\ &\leq h^2 \cdot \inf \|J_0 - \widetilde{J_0}\| \cdot \|J_0^{(1,3,4)} - J_0^\dagger\| \\ &= h^2 \inf \left(\max_{i=\overline{1,m}} \|J_{k_i}(0) - \widetilde{J_{k_i}}(0)\| \cdot \max_{i=\overline{1,m}} \|J_{k_i}^{(1,3,4)}(0) - J_{k_i}^\dagger(0)\| \right) \\ &= h^2 \inf \left(\max_{i=\overline{1,m}} 1/|\alpha_i| \cdot \max_{j=\overline{1,m}} |\alpha_j| \right) = h^2 \inf \frac{|\alpha_t|}{|\alpha_s|} = h^2,\end{aligned}$$

since $|\alpha_t| \geq |\alpha_i| \geq |\alpha_s|, i = \overline{1,m}$ implies $\frac{|\alpha_t|}{|\alpha_s|} \geq 1$; the required infimum is equal to 1, and it can be reached for a good choice of $\alpha = (z, z, \dots, z), z \in \mathbb{C}\setminus\{0\}$. □

Theorem 6. *Under the assumptions from Theorem 1, we have*

$$\inf_{\substack{(\alpha_1,\dots,\alpha_m)\in\mathbb{C}^m \\ \alpha_i \neq 0, i=\overline{1,m}}} (\|A - \tilde{A}\| + \|\tilde{A}^{-1} - A^\dagger\|) \leq 2\|X\| \cdot \|X^{-1}\|. \tag{28}$$

Proof. Writing "inf" for the infimum over the parameters listed in formula (28), and h for $\|X\|\cdot\|X^{-1}\|$, we have

$$\begin{aligned}
&\inf(\|A-\tilde{A}\|+\|\tilde{A}^{-1}-A^{\dagger}\|)\\
&\quad=\inf\left(\left\|X\begin{pmatrix}0&0\\0&J_0-\widetilde{J_0}\end{pmatrix}X^{-1}\right\|+\left\|X\begin{pmatrix}0&0\\0&J_0^{(1,3,4)}-J_0^{\dagger}\end{pmatrix}X^{-1}\right\|\right)\\
&\quad\le h^2\cdot\inf\|J_0-\widetilde{J_0}\|\cdot\|J_0^{(1,3,4)}-J_0^{\dagger}\|\\
&\quad=h^2\inf\left(\max_{i=\overline{1,m}}\|J_{k_i}(0)-\widetilde{J_{k_i}}(0)\|+\max_{i=\overline{1,m}}\|J_{k_i}^{(1,3,4)}(0)-J_{k_i}^{\dagger}(0)\|\right)\\
&\quad=h^2\inf\left(\max_{i=\overline{1,m}}1/|\alpha_i|+\max_{j=\overline{1,m}}|\alpha_j|\right)=h^2\inf\left(\frac{1}{|\alpha_t|}+|\alpha_s|\right)=2h^2,
\end{aligned}$$

since $|\alpha_t|\ge|\alpha_s|$ imply $|\alpha_s|+\frac{1}{|\alpha_t|}\ge|\alpha_t|+\frac{1}{|\alpha_t|}\ge 2$; the required infimum can be reached for a good choice of $\alpha=(1,1,\dots,1)$. □

The results analogous to Theorems 3–6 are valid, with slightly changed proofs, under the assumptions of Theorem 2, instead of Theorem 1.

4. Another results

We can prove the following results.

Proposition 3 (The spectrum of matrix $\tilde{A}$)**.** *If $\sigma(A)=\sigma(J_1)\cup\{0\}$, then $\sigma(N)=\{0\}$ and*

$$\sigma(\tilde{A})=\sigma(J_1)\cup\bigcup_{k_i=1}^{m}\left\{\frac{1}{\sqrt[k_i]{\alpha_{k_i}}}\right\},\tag{29}$$

where we take exactly k_i values of a root of complex number α_{k_i}.

Proof. If $\sigma(A)=\sigma(J_1)\cup\{0\}$, then $\sigma(\tilde{A})=\sigma(J_1)\cup\sigma(\widetilde{J_0})$. Clearly, $\sigma(\widetilde{J_0})=$

$\bigcup_{k_i=1}^{m} \sigma(\widetilde{J_{k_i}}(0))$, and eigenvalues of $\widetilde{J_{k_i}}(0)$ are $\frac{1}{\sqrt[k_i]{\alpha_{k_i}}}$, because

$$0 = \begin{vmatrix} -\sigma & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\sigma & 1 & \cdots & 0 & 0 \\ 0 & 0 & -\sigma & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\sigma & 1 \\ \frac{1}{\alpha_i} & 0 & 0 & \cdots & 0 & -\sigma \end{vmatrix}$$

$$= (-1)^{k_i}\sigma^{k_i} + (-1)^{k_i+1}\frac{1}{\alpha_i} = (-1)^{k_i}(\sigma^{k_i} - \frac{1}{\alpha_i}),$$

and now the conclusion can easily be obtained. □

Proposition 4 (Properties of the splitting)**.**

1. *If we multiply by N from the left (right) side the formula $A = \tilde{A} + N$, we get that $NA = N\tilde{A}$, $(AN = \tilde{A}N)$;*
2. *Since $\tilde{A}^{-1} \in A\{1\}$, it have to be $A\tilde{A}^{-1}A = A$; if we multiply this formula from the left (right) side by $\tilde{A}^{-1}$, we obtain that $A\tilde{A}^{-1}$ and $\tilde{A}^{-1}A$ are projectors $P_{R(A),S}$, $(P_{T,N(A)})$ (under the condition $R(A) \oplus S = \mathbb{C}^n$,$(T \oplus N(A) = \mathbb{C}^m)$); this condition is, using Corollary 10, pp. 73, from* [1] *equivalent to the existence of a matrix $X \in A\{1,2\}$, where $R(X) = T,\ N(X) = S$).*
3. *If we replace $A = \tilde{A} + N$ instead of the first (second) A in $A\tilde{A}^{-1}A = A$, then we obtain: $N\tilde{A}^{-1}A = NP_{T,N(A)} = 0$, $(A\tilde{A}^{-1}N = P_{R(A),S}N = 0)$.*
4. *We can use a result from* [2]*, pp. 9 (If $S \in T\{1\}$, then $STS \in T\{1,2\}$), and then conclude*

$$\tilde{A}^{-1}A\tilde{A}^{-1} = \tilde{A}^{-1}(\tilde{A} + N)\tilde{A}^{-1} = \tilde{A}^{-1} + \tilde{A}^{-1}N\tilde{A}^{-1} \in A\{1,2\},$$

because $\tilde{A}^{-1} \in A\{1\}$.

If a matrix splitting of A is obtained using the method described in Theorem 2, previous proposition still remains true, with some minor changes.

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