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Sharp exponential Redheffer-type inequalities for Bessel functions

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Abstract. By using some known results on the zeros of Bessel functions of the first kind sharp exponential Redheffer-type inequalities are established for Bessel and modified Bessel functions of the first kind. The results presented in this paper extend and improve the other known results in the literature.

1. Redheffer-type inequalities for Bessel functions

The following inequality

$$\frac{\sin x}{x} \geq \frac{\pi^2 - x^2}{\pi^2 + x^2},$$

which holds for all $x \in \mathbb{R}$, is known in literature as Redheffer's inequality [24]. For an interesting proof in the case of $|x| < \pi$ of the above Redheffer inequality we refer to the paper [22]. Recently, motivated by this inequality, Chen et al. [11], by using mathematical induction and infinite product representation of $\cos x$, $\sinh x$ and $\cosh x$, established the following Redheffer-type inequalities

$$\cos x \geq \frac{\pi^2 - 4x^2}{\pi^2 + 4x^2} \quad \text{and} \quad \cosh x \leq \frac{\pi^2 + 4x^2}{\pi^2 - 4x^2} \quad \text{for all } |x| \leq \pi/2,$$

$$\frac{\sinh x}{x} \leq \frac{\pi^2 + x^2}{\pi^2 - x^2} \quad \text{for all } |x| < \pi.$$

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In order to sharpen and extend the above results, recently Zhu and Sun [26] presented six new Redheffer-type inequalities involving circular functions and hyperbolic functions. We note that Redheffer's inequality and the above Redheffer-type inequalities has been extended recently by the first author [4] to the Bessel and modified Bessel functions of the first kind. Moreover, it is worth mentioning that in the recent years many known trigonometric inequalities and their hyperbolic analogues were extended by the first author and coauthors. The interested reader is referred to the papers [3], [4], [5], [6], [7], [8], [9], [10] and to the references therein. In this paper our aim is to continue these investigations by extending all of the results of Zhu and Sun [26] and by improving the results from [4]. The paper is organized as follows: in the first two sections we present sharp exponential Redheffer-type inequalities for Bessel and modified Bessel functions of the first kind, while in the third section we offer some immediate applications of these results. Moreover, in this section we present alternative proofs for some inequalities for Bessel functions of the first kind established by Ifantis and Siafarikas [17]. The key tools in our proofs are some known results on the zeros of Bessel functions of the first kind, like the well-known Rayleigh bounds on the square of the first positive zero of Bessel functions, and some recent results of the first author on Bessel functions.

To achieve our goal first let us recall some basic facts. Suppose that $\nu > -1$ and consider the normalized Bessel function of the first kind $\mathcal{J}_\nu : \mathbb{R} \to (-\infty, 1]$, defined by

$$\mathcal{J}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} J_\nu(x) = \sum_{n\geq 0} \frac{(-1/4)^n}{(\nu+1)_n n!} x^{2n},$$

where, as usual, $(\nu+1)_n = \Gamma(\nu+n+1)/\Gamma(\nu+1)$ for each $n \geq 0$ is the so-called Pochhammer (or Appell) symbol, and J_ν, defined by

$$J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)},$$

stands for the Bessel function of the first kind of order ν (see [25, p. 40]).

The following preliminary result will be useful in the sequel.

Lemma 1. *Let $\nu > -1$ and let $j_{\nu,1}$ be the first positive zero of the Bessel function J_ν. Then the equation $j_{\nu,1}^2 = 8(\nu+1)$ has exactly one positive root $\nu_0 \in (1,2)$. Moreover, if $\nu \in (-1, \nu_0]$, then $j_{\nu,1}^2 \leq 8(\nu+1)$, and if $\nu \geq \nu_0$, then the above inequality is reversed.*

Proof. Due to Ismail and Muldoon [18, Theorem 2] it is known that the function $\nu \mapsto j_{\nu,1}^2/(\nu+1)$ is increasing on $(-1,\infty)$ (see also [19, p. 9], [14, p. 57]).

One the other hand, concerning the local behavior of $j_{\nu,1}$, in the neighborhood of $\nu = -1$ the following representation is valid [23]

$$j_{\nu,1} = 2(\nu+1)^{1/2}\left[1 + \frac{1}{4}(\nu+1) - \frac{7}{96}(\nu+1)^2 + \dots\right],$$

which implies that $j_{\nu,1}^2/(\nu+1) \to 4$ as $\nu \to -1$. It is also known (see for example [18, equation 6.8]) that for each $\nu > -1$ we have the following lower and upper bounds for the square of the first positive zero

$$4(\nu+1)(\nu+2)^{1/2} < j_{\nu,1}^2 < 2(\nu+1)(\nu+3).$$

Hence $j_{\nu,1}^2/(\nu+1) \to \infty$ as $\nu \to \infty$, and consequently indeed the equation $j_{\nu,1}^2 = 8(\nu+1)$ has an unique solution $\nu_0 > -1$. Now we prove that $\nu_0 \in (1,2)$. Using the above inequalities we get that $j_{\nu,1}^2 < 8(\nu+1)$ for each $\nu \in (-1,1)$ and $j_{\nu,1}^2 > 8(\nu+1)$ for each $\nu > 2$. Since $j_{1,1} = 3.83171\dots$ and $j_{2,1} = 5.13562\dots$ (see [1, p. 409]), it is easy to see that for $\nu \in \{1,2\}$ the equation $j_{\nu,1}^2 = 8(\nu+1)$ is not satisfied and thus ν_0 must lies in the interval $(1,2)$. With this the proof is complete. □

Our first main result of this paper, which improves and complements [4, equation 2.1], reads as follows.

Theorem 1. *Let $\nu > -1$ and let*

$$\lambda_\nu = [8(\nu+1) - j_{\nu,1}^2]^{1/2}, \qquad \omega_\nu = \begin{cases} j_{\nu,1}, & \text{if } \nu \in (-1,-1/2] \\ \lambda_\nu, & \text{if } \nu \in (-1/2,0) \\ j_{\nu,1}, & \text{if } \nu \in [0,1/2] \\ \lambda_\nu, & \text{if } \nu \in (1/2,\nu_0), \end{cases}$$

where $j_{\nu,1}$ is the first positive zero of the Bessel function of the first kind J_ν and ν_0 is the unique solution of the equation $j_{\nu,1}^2 = 8(\nu+1)$. Then the following sharp exponential Redheffer-type inequalities hold

$$\mathcal{J}_\nu(x) \geq \left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 + x^2}\right)^{\alpha_\nu} \qquad \text{for all } |x| < \omega_\nu \text{ and } \nu \in (-1,\nu_0), \tag{1.1}$$

$$\mathcal{J}_\nu(x) \leq \left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 + x^2}\right)^{\beta_\nu} \qquad \text{for all } |x| < j_{\nu,1} \text{ and } \nu \geq -7/8, \tag{1.2}$$

$$\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} \geq \left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)^{\gamma_\nu} \qquad \text{for all } |x| < j_{\nu,1} \text{ and } \nu \geq -7/8, \tag{1.3}$$

$$\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} \leq \left(\frac{j_{\nu,1}^2+x^2}{j_{\nu,1}^2-x^2}\right)^{\alpha_\nu} \qquad \text{for all } |x|<j_{\nu,1} \text{ and } \nu>-1, \tag{1.4}$$

with the best possible constants

$$\alpha_\nu = 1,\ \beta_\nu = \frac{j_{\nu,1}^2}{8(\nu+1)} \quad \text{and} \quad \gamma_\nu = \frac{j_{\nu,1}^2}{8(\nu+1)(\nu+2)}.$$

Proof. Observe that due to Lemma 1 λ_ν is well defined. On the other hand, since the function $\nu \mapsto j_{\nu,1}^2/(\nu+1)$ is increasing, clearly for all $\nu>-1$ one has $j_{\nu,1}^2 > 4(\nu+1)$, i.e. for all $\nu \in (-1,\nu_0)$ we have $\lambda_\nu < j_{\nu,1}$. Since all functions which appear in inequalities (1.1), (1.2), (1.3) and (1.4) are even, without loss of generality, in what follows we assume that $x\in(0,\lambda_\nu)$ or $x\in(0,j_{\nu,1})$, depending on the inequality in the question. First we show that, assuming that the inequalities (1.1), (1.2), (1.3) and (1.4) hold, then the constants α_ν, β_ν and γ_ν are the best possible. For this consider the functions $f_\nu, g_\nu : (0,j_{\nu,1}) \to \mathbb{R}$, defined by

$$f_\nu(x) = \frac{\log \mathcal{J}_\nu(x)}{\log\left(\frac{j_{\nu,1}^2-x^2}{j_{\nu,1}^2+x^2}\right)} \quad \text{and} \quad g_\nu(x) = \frac{\log \frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)}}{\log\left(\frac{j_{\nu,1}^2+x^2}{j_{\nu,1}^2-x^2}\right)}.$$

Since for all $|x|<j_{\nu,1}$ and $\nu>-1$ we have $\mathcal{J}_\nu(x)>0$ (see for example [7, Theorem 3]), clearly the function f_ν is well defined. Similarly, because for all $|x|<j_{\nu+1,1}$ and $\nu>-1$ we have $\mathcal{J}_{\nu+1}(x)>0$ and $(-j_{\nu,1},j_{\nu,1}) \subset (-j_{\nu+1,1},j_{\nu+1,1})$, it follows that the function g_ν is well defined too. Using the l'Hospital rule it is easy to verify that we have

$$\lim_{x\to 0} f_\nu(x) = \lim_{x\to 0} \frac{\mathcal{J}_\nu'(x)}{\mathcal{J}_\nu(x)} \cdot \frac{x^4-j_{\nu,1}^4}{4xj_{\nu,1}^2} = \lim_{x\to 0} \frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} \cdot \frac{j_{\nu,1}^4-x^4}{8(\nu+1)j_{\nu,1}^2} = \beta_\nu,$$

$$\lim_{x\to j_{\nu,1}} f_\nu(x) = \lim_{x\to j_{\nu,1}} \frac{\mathcal{J}_\nu'(x)}{\mathcal{J}_\nu(x)} \cdot \frac{x^4-j_{\nu,1}^4}{4xj_{\nu,1}^2} = \lim_{x\to j_{\nu,1}} \frac{\mathcal{J}_{\nu+1}(j_{\nu,1})}{8(\nu+1)j_{\nu,1}^2} \cdot \frac{j_{\nu,1}^4-x^4}{\mathcal{J}_\nu(x)} = \alpha_\nu,$$

$$\lim_{x\to 0} g_\nu(x) = \lim_{x\to 0} \frac{(\nu+2)[\mathcal{J}_{\nu+1}(x)]^2-(\nu+1)\mathcal{J}_\nu(x)\mathcal{J}_{\nu+2}(x)}{\mathcal{J}_\nu(x)\mathcal{J}_{\nu+1}(x)} \cdot \frac{j_{\nu,1}^4-x^4}{8(\nu+1)(\nu+2)j_{\nu,1}^2} = \gamma_\nu$$

and then

$$\lim_{x\to j_{\nu,1}} g_\nu(x) = -\lim_{x\to j_{\nu,1}} \frac{\log \mathcal{J}_\nu(x)}{\log\left(\frac{j_{\nu,1}^2+x^2}{j_{\nu,1}^2-x^2}\right)} = \lim_{x\to j_{\nu,1}} \frac{\log \mathcal{J}_\nu(x)}{\log\left(\frac{j_{\nu,1}^2-x^2}{j_{\nu,1}^2+x^2}\right)} = \lim_{x\to j_{\nu,1}} f_\nu(x) = \alpha_\nu,$$

where we have used the differentiation formula

$$\mathcal{J}_\nu'(x) = -\frac{x}{2(\nu+1)}\mathcal{J}_{\nu+1}(x), \tag{1.5}$$

which can be verified easily by using the series representation of the function $\mathcal{J}_\nu$. With other words, we have $f_\nu(0^+)=\beta_\nu$, $g_\nu(0^+)=\gamma_\nu$ and $f_\nu(j_{\nu,1}^-)=g_\nu(j_{\nu,1}^-)=\alpha_\nu$, which show that the constants α_ν, β_ν and γ_ν are the best possible.

Now let us focus on the inequality (1.1). Recall that this inequality was recently proved by the first author [4, equation 2.1], but under the assumption that if $\Delta_\nu(n) = j_{\nu,n+1}^2 - j_{\nu,1}j_{\nu,n} - j_{\nu,n}j_{\nu,n+1} \geq 0$ for all $n \in \{1,2,3,\dots\}$, then $|x| < \xi_\nu$, where

$$\xi_\nu = \min_{n\geq 1, \nu > -1} \left\{ j_{\nu,1}, \sqrt{\Delta_\nu(n)} \right\}.$$

Here $j_{\nu,n}$ stands for the nth positive zero of the Bessel function J_ν. Here we show that the above condition can be relaxed for $\nu \in [0,1/2]$ and $\nu \in (-1,-1/2]$. For this consider the (unique) solution $j = j_{\nu,\kappa}$ of the differential equation

$$\frac{\mathrm{dj}}{\mathrm{d}\nu} = 2j \int_0^\infty K_0(2j \sinh t) e^{-2\nu t}\,\mathrm{dt},$$

which satisfies the condition $j \to 0$ as $\nu \to -\kappa^+$. Here K_0 stands for the modified Bessel function of the second kind of zero order and for $\kappa \in \{1,2,3,\dots\}$ the $j_{\nu,\kappa}$ becomes exactly the κth positive zero of the Bessel function J_ν of the first kind. Due to Elbert and Laforgia [13] it is known that if $\nu \in [0,1/2]$, then $j_{\nu,\kappa}$ is convex with respect to κ. With other words, in particular we have that the sequence $\{j_{\nu,n}\}_{n\geq 0}$, where $j_{\nu,0} = 0$, is convex when $\nu \in [0,1/2]$. This implies that for each $\nu \in [0,1/2]$ and $n \in \{1,2,3,\dots\}$ we have

$$\Delta_\nu(n) - j_{\nu,1}^2 = (j_{\nu,n+1} + j_{\nu,1})(j_{\nu,n+1} - j_{\nu,n} - j_{\nu,1}) \geq 0,$$

since

$$j_{\nu,n+1} - j_{\nu,n} \geq j_{\nu,n} - j_{\nu,n-1} \geq \dots \geq j_{\nu,2} - j_{\nu,1} \geq j_{\nu,1}.$$

Consequently, if $\nu \in [0,1/2]$, then $\xi_\nu = j_{\nu,1}$.

On the other hand it is known (see for example [12, Theorem 21]) that if $|\nu| \geq 1/2$, then for all $n \in \{1,2,3,\dots\}$ we have $j_{\nu,n+1} - j_{\nu,n} \geq \pi$, which implies that for all $\nu \in (-1,-1/2]$ and all $n \in \{1,2,3,\dots\}$ one has

$$j_{\nu,n+1} - j_{\nu,n} \geq \pi > \pi/2 = j_{-1/2,1} \geq j_{\nu,1},$$

i.e. $\xi_\nu = j_{\nu,1}$. Here we used that every positive zero $j_{\nu,n}$ of J_ν satisfies the inequality $\mathrm{dj_{\nu,n}}/\mathrm{d}\nu > 1$ for all $\nu > -1$ (see [15, Corollary 3.1]) and in particular the function $\nu \mapsto j_{\nu,1}$ is increasing on $(-1,\infty)$.

For the remained part, i.e. when $\nu \in (-1/2, 0)$ or $\nu \in (1/2, \nu_0)$, consider the function $h_\nu : [0, j_{\nu,1}) \to \mathbb{R}$, defined by

$$h_\nu(x) = \log \mathcal{J}_\nu(x) - \log\left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 + x^2}\right).$$

Taking into account the inequality [17, equation 2.17] (for an alternative proof of this inequality see the Concluding remarks below (part **5**))

$$\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} = \frac{2(\nu+1)}{x}\frac{J_{\nu+1}(x)}{J_\nu(x)} < \frac{j_{\nu,1}^2}{j_{\nu,1}^2 - x^2}, \tag{1.6}$$

which holds for all $\nu > -1$ and $x \in (0, j_{\nu,1})$, and using (1.5), clearly we have

$$\begin{aligned} h'_\nu(x) = \frac{\mathcal{J}'_\nu(x)}{\mathcal{J}_\nu(x)} + \frac{4xj_{\nu,1}^2}{j_{\nu,1}^4 - x^4} &= \frac{4xj_{\nu,1}^2}{j_{\nu,1}^4 - x^4} - \frac{x}{2(\nu+1)}\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} \\ &\geq \frac{4xj_{\nu,1}^2}{j_{\nu,1}^2 - x^2}\left[\frac{1}{j_{\nu,1}^2 + x^2} - \frac{1}{8(\nu+1)}\right] \geq 0, \end{aligned}$$

where $x \in [0, \lambda_\nu)$ and $\nu \in (-1, \nu_0)$, i.e. the function h_ν is increasing on $[0, \lambda_\nu)$. This in turn implies that $h_\nu(x) \geq h_\nu(0) = 0$ for all $x \in [0, \lambda_\nu)$ and $\nu \in (-1, \nu_0)$. With this the proof of (1.1) is complete.

Now we are going to prove (1.2). Let us consider the function $\varphi_\nu : [0, j_{\nu,1}) \to \mathbb{R}$, defined by

$$\varphi_\nu(x) = \frac{j_{\nu,1}^2}{8(\nu+1)} \log\left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 + x^2}\right) - \log \mathcal{J}_\nu(x).$$

In what follows we show that for each $\nu \geq -7/8$ the function φ_ν is increasing, and consequently $\varphi_\nu(x) \geq \varphi_\nu(0) = 0$, i.e. (1.2) holds. For this recall the Rayleigh inequalities [25, p. 502]

$$\left[\sigma_\nu^{(2m)}\right]^{-1/m} < j_{\nu,1}^2 < \sigma_\nu^{(2m)}/\sigma_\nu^{(2m+2)}, \tag{1.7}$$

which hold for all $m \in \{1, 2, 3, \dots\}$ and $\nu > -1$, where

$$\sigma_\nu^{(2m)} = \sum_{n \geq 1} j_{\nu,n}^{-2m}$$

is the Rayleigh function of order $2m$, and the Kishore's formula [20]

$$\frac{x}{2}\frac{J_{\nu+1}(x)}{J_\nu(x)} = \sum_{m \geq 1} \sigma_\nu^{(2m)} x^{2m}, \tag{1.8}$$

where $|x| < j_{\nu,1}$ and $\nu > -1$. Using (1.5), (1.7) and (1.8) we obtain

$$\varphi_\nu'(x) = -\frac{x}{2(\nu+1)}\frac{j_{\nu,1}^4}{j_{\nu,1}^4 - x^4} - \frac{\mathcal{J}_\nu'(x)}{\mathcal{J}_\nu(x)} = \frac{x}{2(\nu+1)}\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} - \frac{x}{2(\nu+1)}\frac{j_{\nu,1}^4}{j_{\nu,1}^4 - x^4}$$

$$= \frac{1}{2(\nu+1)x}\frac{1}{j_{\nu,1}^2 + x^2}\left[4(\nu+1)(j_{\nu,1}^2 + x^2)\frac{x}{2}\frac{J_{\nu+1}(x)}{J_\nu(x)} - \frac{j_{\nu,1}^4 x^2}{j_{\nu,1}^2 - x^2}\right]$$

$$= \frac{1}{2(\nu+1)x}\frac{1}{j_{\nu,1}^2+x^2}\left[4(\nu+1)(j_{\nu,1}^2+x^2)\sum_{m\geq 1}\sigma_\nu^{(2m)}x^{2m} - j_{\nu,1}^2 x^2\sum_{m\geq 0}\left(\frac{x}{j_{\nu,1}}\right)^{2m}\right]$$

$$= \frac{1}{2(\nu+1)x}\frac{1}{j_{\nu,1}^2+x^2}\left[4(\nu+1)j_{\nu,1}^2\sum_{m\geq 2}\sigma_\nu^{(2m)}x^{2m} +, 4(\nu+1)\sum_{m\geq 2}\sigma_\nu^{(2m-2)}x^{2m}\right.$$

$$\left. - \sum_{m\geq 2}\frac{1}{j_{\nu,1}^{2m-2}}x^{2m}\right]$$

$$= \frac{1}{2(\nu+1)x}\frac{1}{j_{\nu,1}^2 + x^2}\sum_{m\geq 2}\left[4(\nu+1)j_{\nu,1}^2\sigma_\nu^{(2m)} + 4(\nu+1)\sigma_\nu^{(2m-2)} - \frac{1}{j_{\nu,1}^{2m-2}}\right]x^{2m}$$

$$\geq \frac{1}{2(\nu+1)x}\frac{1}{j_{\nu,1}^2 + x^2}\sum_{m\geq 2}\left[8(\nu+1)j_{\nu,1}^2\sigma_\nu^{(2m)} - \frac{1}{j_{\nu,1}^{2m-2}}\right]x^{2m}$$

$$\geq \frac{1}{2(\nu+1)x}\frac{1}{j_{\nu,1}^2 + x^2}\sum_{m\geq 2}\frac{8\nu+7}{j_{\nu,1}^{2m-2}}x^{2m} \geq 0,$$

where we have used that due to the Rayleigh formula [25, p. 502]

$$\sigma_\nu^{(2)} = \sum_{n\geq 1}\frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}$$

one has

$$4(\nu+1)j_{\nu,1}^2\sigma_\nu^{(2)}x^2 = j_{\nu,1}^2 x^2.$$

Finally, we prove the inequalities (1.3) and (1.4). For this first we show that the inequality (1.3) is in fact an immediate consequence of the inequality (1.2). Recall that the function $\nu \mapsto [\mathcal{J}_\nu(x)]^{\nu+1}$ is increasing on $(-1,\infty)$ for each fixed $x \in (0, j_{\nu,1})$ (see [7, Theorem 3]), and thus we have

$$\mathcal{J}_{\nu+1}(x) \geq [\mathcal{J}_\nu(x)]^{(\nu+1)/(\nu+2)}$$

for all $\nu > -1$ and $x \in (0, j_{\nu,1})$. This in turn together with (1.2) implies that

$$\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_\nu(x)} \geq [\mathcal{J}_\nu(x)]^{(\nu+1)/(\nu+2)-1} = \frac{1}{[\mathcal{J}_\nu(x)]^{1/(\nu+2)}} \geq \left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)^{\beta_\nu/(\nu+2)}$$

$$= \left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)^{\gamma_\nu},$$

as we required. Secondly, observe that inequality (1.4) follows easily from inequality (1.6). □

Concluding remarks and particular cases

1. First we note that using again the inequality [18, equation 6.8] $j_{\nu,1}^2 < 2(\nu+1)(\nu+3)$ we obtain that $\gamma_\nu < 1$ for all $\nu > -1$. Moreover, using the inequality (1.7) for $m = 1$, i.e. $4(\nu+1) < j_{\nu,1}^2 < 4(\nu+1)(\nu+2)$ (see for example [18, equation 6.7]) we obtain that $\gamma_\nu < 1/2 < \beta_\nu$ for all $\nu > -1$. Here we used that [25, p. 502]

$$\sigma_\nu^{(2)} = \frac{1}{4(\nu+1)} \quad \text{and} \quad \sigma_\nu^{(4)} = \frac{1}{16(\nu+1)^2(\nu+2)}.$$

2. Secondly, observe that in (1.1) the condition $\nu < \nu_0$ is not only sufficient, but also necessary. More precisely, since for all $\nu > -1$ and $x \in (-j_{\nu,1}, j_{\nu,1})$ we have

$$\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 + x^2} \leq 1,$$

form (1.1) and (1.2) it follows that β_ν must be less than $\alpha_\nu = 1$, i.e. ν must lies in $(-1, \nu_0)$. Moreover, when $\nu \geq \nu_0$, then the inequality (1.1) is reversed and for $\nu > \nu_0$ is weaker than (1.2). This observation is in the agreement with the fact that for example

$$\Delta_2(2) = j_{2,3}^2 - j_{2,1}j_{2,2} - j_{2,2}j_{2,3} = -6.01404 < 0,$$

where we have used that $j_{2,1} = 5.13562$, $j_{2,2} = 8.41724$ and $j_{2,3} = 11.61984$ (see [1, p. 409]). With other words, for $\nu > \nu_0$ the expression $\Delta_\nu(n) = j_{\nu,n+1}^2 - j_{\nu,1}j_{\nu,n} - j_{\nu,n}j_{\nu,n+1}$ is not necessarily positive for all $n \in \{1, 2, 3, \dots\}$.

3. It is worth mentioning that in particular the function $\mathcal{J}_\nu$ reduces to some elementary functions, like sine and cosine. More precisely, in particular we have

$$\mathcal{J}_{-1/2}(x) = \sqrt{\pi/2} \cdot x^{1/2} J_{-1/2}(x) = \cos x, \tag{1.9}$$

$$\mathcal{J}_{1/2}(x) = \sqrt{\pi/2} \cdot x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}, \tag{1.10}$$

$$\mathcal{J}_{3/2}(x) = 3\sqrt{\pi/2} \cdot x^{-3/2} J_{3/2}(x) = 3\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right), \tag{1.11}$$

respectively, which can verified easily by using the series representation of the function $\mathcal{J}_\nu$ and of the cosine and sine functions, respectively. Now, choosing in (1.1) and (1.2) $\nu = -1/2$, in view of (1.9) we obtain the following sharp Redheffer-type inequalities [26, Theorem 2]

$$\left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^{\alpha_{-1/2}} \leq \cos x \leq \left(\frac{\pi^2 - 4x^2}{\pi^2 + 4x^2}\right)^{\beta_{-1/2}} \quad \text{for all} \quad |x| < \pi/2,$$

with the best possible constants $\alpha_{-1/2} = 1$ and $\beta_{-1/2} = \pi^2/16$ (see Figure 1).

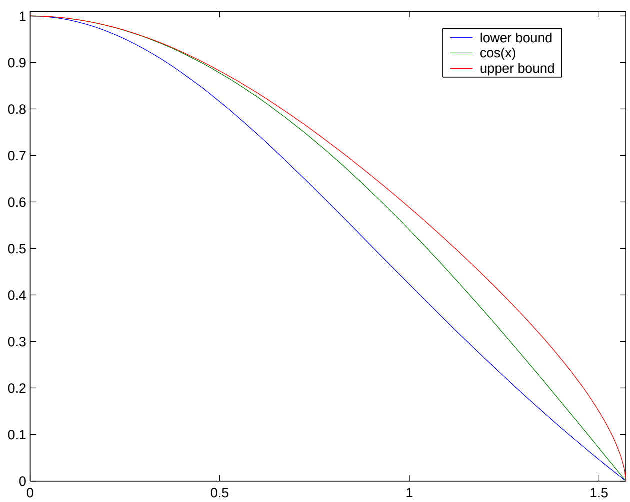

Figure 1. The graph of the functions $\frac{\pi^2-4x^2}{\pi^2+4x^2}$, $\cos x$ and $\left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\pi^2/16}$ on $(0, \pi/2)$.

Similarly, taking $\nu = 1/2$ in (1.1) and (1.2), in view of (1.10), we reobtain the following sharp inequalities [26, Theorem 1]

$$\left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{\alpha_{1/2}} \leq \frac{\sin x}{x} \leq \left(\frac{\pi^2 - x^2}{\pi^2 + x^2}\right)^{\beta_{1/2}} \quad \text{for all } |x| < \pi,$$

with the best possible constants $\alpha_{1/2} = 1$ and $\beta_{1/2} = \pi^2/12$ (see Figure 2).

Analogously, if we take $\nu = -1/2$ in (1.3) and (1.4), then in view of (1.9) and (1.10) we get the following sharp Redheffer-type inequalities [26, Theorem 3]

$$\left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\gamma_{-1/2}} \leq \frac{\tan x}{x} \leq \left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\alpha_{-1/2}} \quad \text{for all } |x| < \pi/2,$$

with the best possible constants $\alpha_{-1/2} = 1$ and $\gamma_{-1/2} = \pi^2/24$ (see Figure 3).

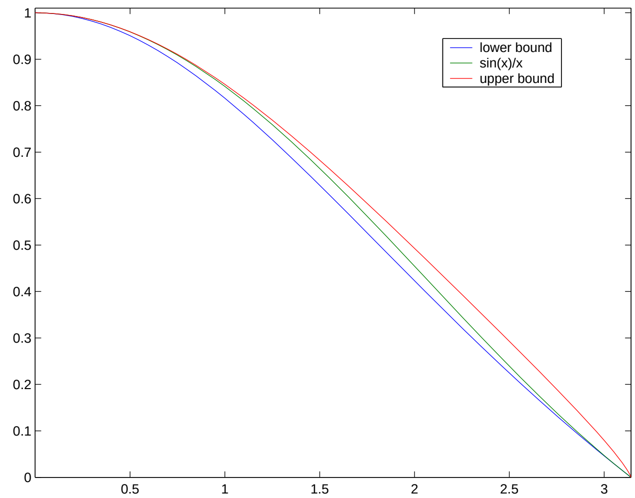

Figure 2. The graph of the functions $\frac{\pi^2-x^2}{\pi^2+x^2}$, $\frac{\sin x}{x}$ and $\left(\frac{\pi^2-x^2}{\pi^2+x^2}\right)^{\pi^2/12}$ on $(0,\pi)$.

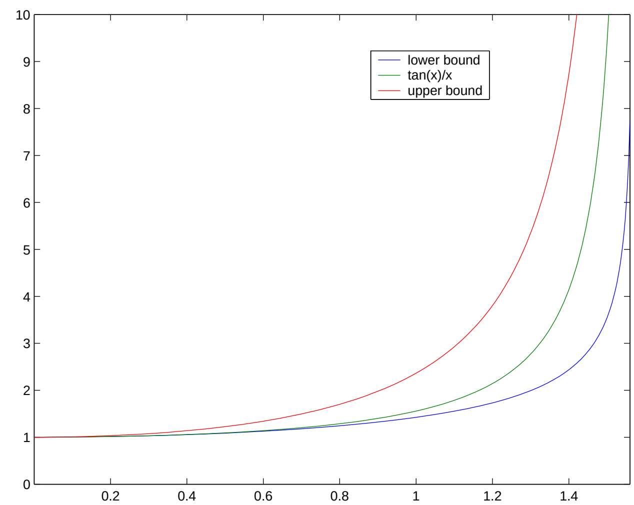

Figure 3. The graph of the functions $\left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\pi^2/24}$, $\frac{\tan x}{x}$ and $\frac{\pi^2+4x^2}{\pi^2-4x^2}$ on $(0,\pi/2)$.

Here we used that $j_{-1/2,1}=\pi/2$ and $j_{1/2,1}=\pi$, which can be verified easily by using the infinite product representation of the cosine and sine functions [1, p. 75], and of the function [1, p. 370] J_ν, respectively.

4. We note that the proof of inequality (1.2) is similar to those given by Zhu and Sun [26] for Theorems 1 and 2. However, our approach in the proof of inequalities (1.3) and (1.4) is much simpler than the method given in [26]. It is worth mentioning that Kishore [20] pointed out that, if B_m denotes the mth Bernoulli number in the even suffix notation and $G_m = 2(1-2^m)B_m$, then

$$\sigma_{-1/2}^{(2m)} = (-1)^m \frac{2^{2m-2}}{(2m)!} G_{2m} \quad \text{and} \quad \sigma_{1/2}^{(2m)} = (-1)^{m-1} \frac{2^{2m-1}}{(2m)!} B_{2m}.$$

Using (1.9), (1.10), (1.11) and Kishore's expansion (1.8) for $\nu = -1/2$ and ν=1/2 these formulas leads to the well known power series expansions [1, p. 75]

$$\begin{aligned}\tan x &= \sum_{m\geq 1} (-1)^m \frac{2^{2m}(1-2^{2m})}{(2m)!} B_{2m} x^{2m-1} \\ &= \sum_{m\geq 1} \frac{2^{2m}(2^{2m}-1)}{(2m)!} \left|B_{2m}\right| x^{2m-1}, \quad |x| < \pi/2,\end{aligned}$$

$$x \cot x = 1 - \sum_{m\geq 1} (-1)^{m-1} \frac{2^{2m}}{(2m)!} B_{2m} x^{2m} = 1 - \sum_{m\geq 1} \frac{2^{2m}}{(2m)!} \left|B_{2m}\right| x^{2m}, \quad |x| < \pi,$$

which were the chief tools in the proof of the main results in [26].

5. We note that Ifantis and Siafarikas [17] in order to prove (1.6) have used the power series representation [16, p. 95]

$$\frac{\mathcal{J}_{\nu+1}(x)}{\mathcal{J}_{\nu}(x)} = \frac{2(\nu+1)}{x} \frac{J_{\nu+1}(x)}{J_{\nu}(x)} = 1 + \sum_{m\geq 1} \frac{\|S_{\nu}^m e_1\|^2}{2^{2m}} x^{2m},$$

where $\nu > -1$ and $x \in (0, j_{\nu,1})$. Here $S_{\nu} = L_{\nu}^{1/2}(V+V^*)L_{\nu}^{1/2}$ is a compact and self-adjoint operator, V is an unilateral shift operator with respect to the orthonormal basis e_m, $m \in \{1,2,3,\dots\}$ in an abstract Hilbert space, L_{ν} is the diagonal operator such that $L_{\nu} e_m = (1/(\nu+m)) e_m$ and V^* is the adjoint of V. We show here that inequality (1.6) in fact can be deduced easily without of the above power series representation of the quotient $\mathcal{J}_{\nu+1}(x)/\mathcal{J}_{\nu}(x)$. For this we prove first that for all $\nu > -1$ and $m \in \{0,1,2,\dots\}$ the following inequality holds

$$4(\nu+1)\sigma_{\nu}^{(2m+2)} j_{\nu,1}^{2m} \leq 1, \tag{1.12}$$

and for $m \geq 1$ is strict. For $m = 0$ clearly we have equality, while for $m = 1$ the above inequality becomes exactly the right hand side of (1.7) for the case

$m = 1$, i.e. $j_{\nu,1}^2 < 4(\nu+1)(\nu+2)$. Now suppose that (1.12) holds for some $m = k-1 \geq 2$, i.e. we have

$$4(\nu+1)\sigma_\nu^{(2k)} j_{\nu,1}^{2k-2} < 1.$$

Then by using the right hand side of (1.7) we have

$$4(\nu+1)\sigma_\nu^{(2k+2)} j_{\nu,1}^{2k} < 4(\nu+1)\frac{\sigma_\nu^{(2k)}}{j_{\nu,1}^2} j_{\nu,1}^{2k} = 4(\nu+1)\sigma_\nu^{(2k)} j_{\nu,1}^{2k-2} < 1,$$

and thus by mathematical induction we have that indeed the inequality (1.12) is true. Consequently, by using the Kishore expansion (1.8) and inequality (1.12) it follows that

$$\begin{aligned}\frac{2(\nu+1)}{x}\frac{J_{\nu+1}(x)}{J_\nu(x)} - \frac{j_{\nu,1}^2}{j_{\nu,1}^2 - x^2} &= \frac{4(\nu+1)}{x^2}\frac{x}{2}\frac{J_{\nu+1}(x)}{J_\nu(x)} - \frac{j_{\nu,1}^2}{j_{\nu,1}^2 - x^2} \\ &= \frac{4(\nu+1)}{x^2}\sum_{m\geq 1}\sigma_\nu^{(2m)} x^{2m} - \sum_{m\geq 0}\frac{1}{j_{\nu,1}^{2m}} x^{2m} \\ &= \sum_{m\geq 1}\left[4(\nu+1)\sigma_\nu^{(2m+2)} - \frac{1}{j_{\nu,1}^{2m}}\right] x^{2m} < 0\end{aligned}$$

for all $x \in (0, j_{\nu,1})$ and $\nu > -1$, i.e. the proof of (1.6) is complete. Moreover, using the above argument it can be proved that

$$\begin{aligned}&\frac{2(\nu+1)}{x}\frac{J_{\nu+1}(x)}{J_\nu(x)} - \left[1 + \frac{x^2 j_{\nu,1}^2}{4(\nu+1)(\nu+2)(j_{\nu,1}^2 - x^2)}\right] \\ &\quad = \frac{4(\nu+1)}{x^2}\sum_{m\geq 1}\sigma_\nu^{(2m)} x^{2m} - \left[1 + \frac{x^2}{4(\nu+1)(\nu+2)}\sum_{m\geq 0}\frac{1}{j_{\nu,1}^{2m}} x^{2m}\right] \\ &\quad = \sum_{m\geq 2}\left[4(\nu+1)\sigma_\nu^{(2m+2)} - \frac{1}{4(\nu+1)(\nu+2) j_{\nu,1}^{2m-2}}\right] x^{2m} < 0\end{aligned}$$

for all $x \in (0, j_{\nu,1})$ and $\nu > -1$. Here we used that from the right hand side of (1.7) by using again mathematical induction it follows that

$$\sigma_\nu^{(2m+2)} \leq \sigma_\nu^{(4)} / j_{\nu,1}^{2m-2}$$

for all $m \in \{1, 2, 3, \dots\}$, and when $m \geq 2$ the above inequality is strict. This leads to the known inequality [17, equation 2.18]

$$\frac{2(\nu+1)}{x}\frac{J_{\nu+1}(x)}{J_\nu(x)} < 1 + \frac{x^2 j_{\nu,1}^2}{4(\nu+1)(\nu+2)(j_{\nu,1}^2 - x^2)},$$

which is better than (1.6).

It is also worth mentioning here that [15] the eigenvalues of S_ν are precisely the values $\pm 2/j_{\nu,m}$ and $\|S_\nu\| = 2/j_{\nu,1}$. However, comparing (1.8) with the above expansion of Ifantis and Siafarikas, it follows that for all $m \in \{0, 1, 2, \dots\}$ we have

$$\|S_\nu^m e_1\|^2 = (\nu+1)2^{2m+2}\sigma_\nu^{(2m+2)}.$$

This relation complements the results of Ifantis and Siafarikas [15], [16], [17].

6. Finally, we note that based on numerical experiments, we conjecture that the inequality (1.1) holds true for all $\nu \in (-1, \nu_0)$ and $|x| < j_{\nu,1}$, while the inequalities (1.2) and (1.3) hold true for all $\nu > -1$ and $|x| < j_{\nu,1}$.

2. Redheffer-type inequalities for modified Bessel functions

In this section we are going to present the hyperbolic counterpart of the results from the previous section. For $\nu > -1$ let us consider the function $\mathcal{I}_\nu : \mathbb{R} \to [1, \infty)$, defined by

$$\mathcal{I}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} I_\nu(x) = \sum_{n\geq 0} \frac{(1/4)^n}{(\nu+1)_n n!} x^{2n},$$

where I_ν is the modified Bessel function of the first kind, defined by [25, p. 77]

$$I_\nu(x) = \sum_{n\geq 0} \frac{(x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}.$$

Corresponding to Theorem 1 we have the following results for the function $\mathcal{I}_\nu$. We note that this theorem improves and complements the earlier result of the first author [4, equation 2.2].

Theorem 2. *Let $\nu > -1$ and let $|x| < j_{\nu,1}$, where $j_{\nu,1}$ is the first positive zero of the Bessel function of the first kind J_ν. Then the following sharp exponential Redheffer-type inequalities hold*

$$\left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)^{\alpha_\nu} \leq \mathcal{I}_\nu(x) \leq \left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)^{\beta_\nu}, \tag{2.1}$$

$$\left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 + x^2}\right)^{\gamma_\nu} \leq \frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)} \leq \left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 + x^2}\right)^{\alpha_\nu}, \tag{2.2}$$

with the best possible constants

$$\alpha_\nu = 0,\ \beta_\nu = \frac{j_{\nu,1}^2}{8(\nu+1)} \quad \textit{and} \quad \gamma_\nu = \frac{j_{\nu,1}^2}{8(\nu+1)(\nu+2)}.$$

Proof. As in the proof of Theorem 1, since all functions which appear in inequalities (2.1) and (2.2) are even, without loss of generality, in what follows we assume that $x \in (0, j_{\nu,1})$. First we show that the constants α_ν, β_ν and γ_ν are the best possible. For this consider the functions $f_\nu, g_\nu : (0, j_{\nu,1}) \to \mathbb{R}$, defined by

$$f_\nu(x) = \frac{\log \mathcal{I}_\nu(x)}{\log\left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)} \quad \text{and} \quad g_\nu(x) = \frac{\log \frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)}}{\log\left(\frac{j_{\nu,1}^2 - x^2}{j_{\nu,1}^2 + x^2}\right)}.$$

Using the l'Hospital rule it is easy to verify that we have

$$\lim_{x\to 0} f_\nu(x) = \lim_{x\to 0} \frac{\mathcal{I}_\nu'(x)}{\mathcal{I}_\nu(x)} \cdot \frac{j_{\nu,1}^4 - x^4}{4x j_{\nu,1}^2} = \lim_{x\to 0} \frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)} \cdot \frac{j_{\nu,1}^4 - x^4}{8(\nu+1) j_{\nu,1}^2} = \beta_\nu$$

and

$$\lim_{x\to 0} g_\nu(x) = \lim_{x\to 0} \frac{(\nu+2)[\mathcal{I}_{\nu+1}(x)]^2 - (\nu+1)\mathcal{I}_\nu(x)\mathcal{I}_{\nu+2}(x)}{\mathcal{I}_\nu(x)\mathcal{I}_{\nu+1}(x)} \cdot \frac{j_{\nu,1}^4 - x^4}{8(\nu+1)(\nu+2) j_{\nu,1}^2} = \gamma_\nu,$$

where we have used the differentiation formula

$$\mathcal{I}_\nu'(x) = \frac{x}{2(\nu+1)} \mathcal{I}_{\nu+1}(x), \tag{2.3}$$

which can be verified easily by using the series representation of the function $\mathcal{I}_\nu$. On the other hand we have

$$\lim_{x\to j_{\nu,1}} f_\nu(x) = \lim_{x\to j_{\nu,1}} g_\nu(x) = \alpha_\nu.$$

With other words, we have $f_\nu(0^+) = \beta_\nu$, $g_\nu(0^+) = \gamma_\nu$ and $f_\nu(j_{\nu,1}^-) = g_\nu(j_{\nu,1}^-) = \alpha_\nu$, which show that the constants α_ν, β_ν and γ_ν are the best possible.

Now let us focus on inequalities (2.1) and (2.2). Since the function $x \mapsto \mathcal{I}_\nu(x)$ is increasing on $(0,\infty)$ for each $\nu > -1$, we have $\mathcal{I}_\nu(x) \geq 1$, and thus the left hand side of (2.1) is obvious. Similarly, since the function $\nu \mapsto \mathcal{I}_\nu(x)$ is decreasing on

$(-1, \infty)$ for each fixed $x \in \mathbb{R}$ (see [7, Theorem 1]), one has $\mathcal{I}_{\nu+1}(x) \leq \mathcal{I}_\nu(x)$, and thus the right hand side of (2.2) is true. For the right hand side of (2.1) consider the function $h_\nu : [0, j_{\nu,1}) \to \mathbb{R}$, defined by

$$h_\nu(x) = \frac{j_{\nu,1}^2}{8(\nu+1)} \log\left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right) - \log \mathcal{I}_\nu(x).$$

Then by using again the Rayleigh formula [25, p. 502]

$$\sigma_\nu^{(2)} = \sum_{n\geq 1} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}$$

and the factorization

$$\mathcal{I}_\nu(x) = \prod_{n\geq 1}\left(1 + \frac{x^2}{j_{\nu,n}^2}\right), \tag{2.4}$$

which can be easily derived from [25, p. 498]

$$\mathcal{J}_\nu(x) = \prod_{n\geq 1}\left(1 - \frac{x^2}{j_{\nu,n}^2}\right), \tag{2.5}$$

we have

$$\begin{aligned} h_\nu'(x) &= \frac{1}{4(\nu+1)} \frac{2x j_{\nu,1}^4}{(j_{\nu,1}^4 - x^4)} - \sum_{n\geq 1} \frac{2x}{j_{\nu,n}^2 + x^2} = \frac{2x j_{\nu,1}^4}{(j_{\nu,1}^4 - x^4)} \sum_{n\geq 1} \frac{1}{j_{\nu,n}^2} - \sum_{n\geq 1} \frac{2x}{j_{\nu,n}^2 + x^2} \\ &= 2x \sum_{n\geq 1} \left[\frac{j_{\nu,1}^4}{(j_{\nu,1}^4 - x^4) j_{\nu,n}^2} - \frac{1}{j_{\nu,n}^2 + x^2}\right] \geq 0, \end{aligned}$$

for all $\nu > -1$ and $x \in [0, j_{\nu,1})$, i.e. the function h_ν is increasing. Hence $h_\nu(x) \geq h_\nu(0) = 0$, and thus the proof of the right hand side of (2.1) is done. However, there is another way to deduce (2.1). Namely, by the well-known monotone form of l'Hospital's rule (see [2, Lemma 2.2]) to prove that the function f_ν, defined above, is decreasing, it is enough to show that the function

$$x \mapsto \frac{\frac{\mathrm{d}}{\mathrm{dx}} \log \mathcal{I}_\nu(x)}{\frac{\mathrm{d}}{\mathrm{dx}} \log\left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 - x^2}\right)} = \frac{1}{2 j_{\nu,1}^2} \sum_{n\geq 1} \frac{j_{\nu,1}^4 - x^4}{j_{\nu,n}^2 + x^2}$$

is decreasing too on $(0, j_{\nu,1})$, which is clearly true, since each terms in the above series are decreasing as functions of x. Consequently for each $x \in (0, j_{\nu,1})$ the inequalities

$$\alpha_\nu = f_\nu(j_{\nu,1}^-) < f_\nu(x) < f_\nu(0^+) = \beta_\nu$$

hold, as we required. Here we used on the one hand that the numerator and denominator of $f_\nu(x)$ vanishes at zero and on the other hand that from the infinite product formula (2.4) one has

$$\frac{\mathcal{I}'_\nu(x)}{\mathcal{I}_\nu(x)} = \sum_{n\geq 1} \frac{2x}{j^2_{\nu,n}+x^2}.$$

Finally, we prove the left hand side of (2.2). It is known that the function $\nu \mapsto [\mathcal{I}_\nu(x)]^{\nu+1}$ is increasing on $(-1,\infty)$ for each $x \in \mathbb{R}$ (see [7, Theorem 1]), and thus we have

$$\mathcal{I}_{\nu+1}(x) \geq \big[\mathcal{I}_\nu(x)\big]^{(\nu+1)/(\nu+2)}$$

for all $\nu > -1$ and $x \in \mathbb{R}$. This in turn together with the right hand side of (2.1) implies that

$$\begin{aligned}\frac{\mathcal{I}_{\nu+1}(x)}{\mathcal{I}_\nu(x)} \geq \big[\mathcal{I}_\nu(x)\big]^{(\nu+1)/(\nu+2)-1} &= \frac{1}{[\mathcal{I}_\nu(x)]^{1/(\nu+2)}} \geq \left(\frac{j^2_{\nu,1}-x^2}{j^2_{\nu,1}+x^2}\right)^{\beta_\nu/(\nu+2)} \\ &= \left(\frac{j^2_{\nu,1}-x^2}{j^2_{\nu,1}+x^2}\right)^{\gamma_\nu},\end{aligned}$$

and with this the proof is complete. □

Concluding remarks and particular cases

1. First note that in [4], by using mathematical induction and the infinite product representation (2.4), we proved that if $\Delta_\nu(n) \geq 0$ for each $n = \{1,2,3,\dots\}$ and $\nu > -1$, then for all $|x| < j_{\nu,1}$ the following Redheffer-type inequality

$$\mathcal{I}_\nu(x) \leq \frac{j^2_{\nu,1}+x^2}{j^2_{\nu,1}-x^2} \tag{2.6}$$

holds. Due to Lemma 1 if $\nu \in (-1,\nu_0)$, then $\beta_\nu < 1$, and thus the right hand side of (2.1) is better than (2.6). When $\nu = \nu_0$, then $\beta_\nu = 1$, and thus the right hand side of (2.1) and (2.6) are the same. However, when $\nu > \nu_0$ the inequality (2.6) does not hold necessarily, since for all $\nu > \nu_0$ and $n \in \{1,2,3,\dots\}$ the inequality $\Delta_\nu(n) \geq 0$ is not true. For example, as we have pointed out above, $\Delta_2(2) = -6.01404 < 0$.

2. It is worth mentioning that in particular the function $\mathcal{I}_\nu$ reduces to some elementary functions, like hyperbolic sine and hyperbolic cosine. More precisely, in particular we have

$$\mathcal{I}_{-1/2}(x) = \sqrt{\pi/2}\cdot x^{1/2} I_{-1/2}(x) = \cosh x, \tag{2.7}$$

$$\mathcal{I}_{1/2}(x) = \sqrt{\pi/2} \cdot x^{-1/2} I_{1/2}(x) = \frac{\sinh x}{x}, \tag{2.8}$$

$$\mathcal{I}_{3/2}(x) = 3\sqrt{\pi/2} \cdot x^{-3/2} I_{3/2}(x) = -3\left(\frac{\sinh x}{x^3} - \frac{\cosh x}{x^2}\right), \tag{2.9}$$

respectively, which can verified easily by using the series representation of the function $\mathcal{I}_\nu$ and of the hyperbolic cosine and hyperbolic sine functions, respectively. Now, choosing in (2.1) the value $\nu = -1/2$, in view of (2.7) we obtain the following sharp Redheffer-type inequalities (see [26, Theorem 5] for $r = \pi/2$)

$$\left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\alpha_{-1/2}} \leq \cosh x \leq \left(\frac{\pi^2 + 4x^2}{\pi^2 - 4x^2}\right)^{\beta_{-1/2}} \quad \text{for all } |x| < \pi/2,$$

with the best possible constants $\alpha_{-1/2} = 0$ and $\beta_{-1/2} = \pi^2/16$ (see Figure 4).

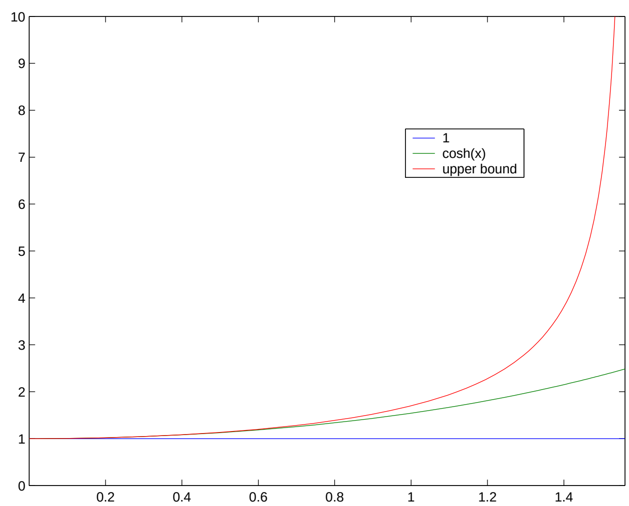

Figure 4. The graph of the functions $1, \cosh x$ and $\left(\frac{\pi^2+4x^2}{\pi^2-4x^2}\right)^{\pi^2/16}$ on $(0, \pi/2)$.

Similarly, taking $\nu = 1/2$ in (2.1), in view of (2.8), we reobtain the following sharp inequalities (see [26, Theorem 4] for $r = \pi$)

$$\left(\frac{\pi^2 + x^2}{\pi^2 - x^2}\right)^{\alpha_{1/2}} \leq \frac{\sinh x}{x} \leq \left(\frac{\pi^2 + x^2}{\pi^2 - x^2}\right)^{\beta_{1/2}} \quad \text{for all } |x| < \pi,$$

with the best possible constants $\alpha_{1/2} = 0$ and $\beta_{1/2} = \pi^2/12$ (see Figure 5).

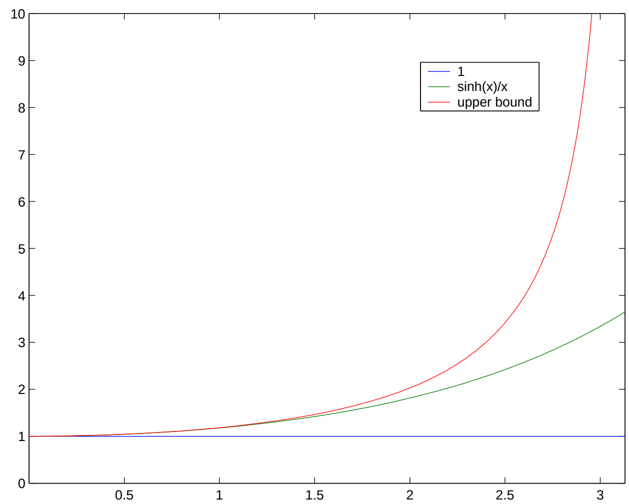

Figure 5. The graph of the functions 1, $\frac{\sinh x}{x}$ and $\left(\frac{\pi^2+x^2}{\pi^2-x^2}\right)^{\pi^2/12}$ on $(0,\pi)$.

Analogously, if we take $\nu=-1/2$ in (2.2), then in view of (2.7) and (2.8) we get the following sharp Redheffer-type inequalities (see [26, Theorem 6] for $r=\pi/2$)

$$\left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\gamma_{-1/2}} \leq \frac{\tanh x}{x} \leq \left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\alpha_{-1/2}} \quad \text{for all } |x|<\pi/2,$$

with the best possible constants $\alpha_{-1/2}=0$ and $\gamma_{-1/2}=\pi^2/24$ (see Figure 6).

3. Further results on Bessel and modified Bessel functions

Observe that combining (1.2) with the right hand side of (2.1) we easily obtain that $\mathcal{J}_\nu(x)\mathcal{I}_\nu(x)\leq 1$ for all $|x|<j_{\nu,1}$ and $\nu\geq -7/8$. Moreover, combining (1.3) with the left hand side of (2.2) we obtain that $\mathcal{J}_\nu(x)\mathcal{I}_\nu(x)\leq \mathcal{J}_{\nu+1}(x)\mathcal{I}_{\nu+1}(x)$ for all $|x|<j_{\nu,1}$ and $\nu\geq -7/8$. The next result shows that the above properties hold true for all $\nu>-1$.

Theorem 3. *The following assertions are true:*

a. *the function* $x\mapsto \mathcal{J}_\nu(x)\mathcal{I}_\nu(x)$ *is increasing on* $(-j_{\nu,1},0]$ *and decreasing on* $[0,j_{\nu,1})$ *for all* $\nu>-1$;

b. *the function* $\nu\mapsto \mathcal{J}_\nu(x)\mathcal{I}_\nu(x)$ *is increasing on* $(-1,\infty)$ *for all* $|x|<j_{\nu,1}$ *fixed;*

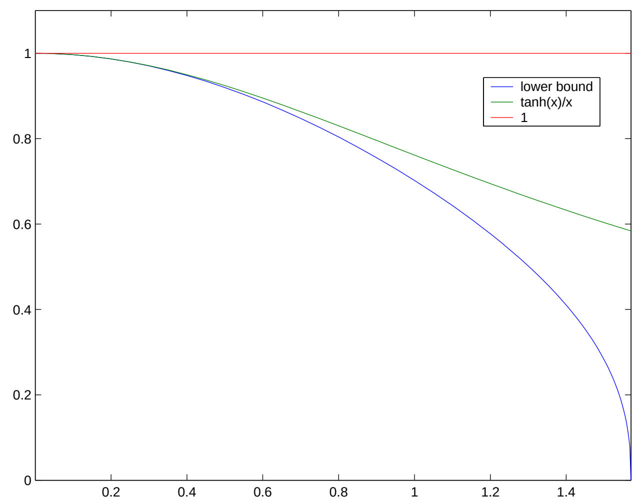

Figure 6. The graph of the functions $\left(\frac{\pi^2-4x^2}{\pi^2+4x^2}\right)^{\pi^2/24}$, $\frac{\tanh x}{x}$ and 1 on $(0,\pi/2)$.

c. *the following inequalities hold*

$$0 < \mathcal{J}_\nu(x)\mathcal{I}_\nu(x) \le \mathcal{J}_{\nu+1}(x)\mathcal{I}_{\nu+1}(x) \le 1 \tag{3.1}$$

for all $|x| < j_{\nu,1}$ *and* $\nu > -1$.

Proof. **a.** By using the differentiation formulas (1.5) and (2.3) we obtain

$$\frac{\mathrm{d}}{\mathrm{dx}}\left[\mathcal{J}_\nu(x)\mathcal{I}_\nu(x)\right] = \frac{x}{2(\nu+1)}\left[\mathcal{I}_{\nu+1}(x)\mathcal{J}_\nu(x) - \mathcal{I}_\nu(x)\mathcal{J}_{\nu+1}(x)\right].$$

Since the function $x \mapsto \mathcal{J}_\nu(x)\mathcal{I}_\nu(x)$ is even, it is enough to show that the above expression is negative for all $[0, j_{\nu,1})$ and all $\nu > -1$. For this recall that [7] the function $\nu \mapsto \mathcal{J}_\nu(x)$ is increasing on $(-1,\infty)$ for each fixed $|x| < j_{\nu,1}$, while the function $\nu \mapsto \mathcal{I}_\nu(x)$ is decreasing on $(-1,\infty)$ for all $x \in \mathbb{R}$ fixed. These properties in particular imply that for all $|x| < j_{\nu,1}$ and $\nu > -1$ we have

$$\mathcal{I}_{\nu+1}(x)/\mathcal{I}_\nu(x) \le 1 \le \mathcal{J}_{\nu+1}(x)/\mathcal{J}_\nu(x),$$

and thus the proof of this part is complete. Another proof can be obtained if we consider the factorizations (2.4) and (2.5). Namely, in view of these formulas, it is enough to show that

$$\prod_{n\ge1}\left(1+\frac{x^2}{j_{\nu+1,n}^2}\right)\left(1-\frac{x^2}{j_{\nu,n}^2}\right) \le \prod_{n\ge1}\left(1-\frac{x^2}{j_{\nu+1,n}^2}\right)\left(1+\frac{x^2}{j_{\nu,n}^2}\right),$$

which clearly holds because each terms in the above products are positive and

$$\left(1+\frac{x^2}{j_{\nu+1,n}^2}\right)\left(1-\frac{x^2}{j_{\nu,n}^2}\right)\leq\left(1-\frac{x^2}{j_{\nu+1,n}^2}\right)\left(1+\frac{x^2}{j_{\nu,n}^2}\right)$$

holds for all $\nu>-1$, $n\in\{1,2,3,\dots\}$ and $x\in[0,j_{\nu,1})$. Here we have used that

$$1/j_{\nu+1,n}^2-1/j_{\nu,n}^2\leq 1/j_{\nu,n}^2-1/j_{\nu+1,n}^2,$$

that is, $j_{\nu,n}<j_{\nu+1,n}$ holds for all $\nu>-1$ and $n\in\{1,2,3,\dots\}$.

b. Recall that the function $\nu\mapsto j_{\nu,n}$ is increasing on $(-1,\infty)$ for each $n\in\{1,2,3,\dots\}$. From this we deduce that the function $\nu\mapsto\log(1-x^4/j_{\nu,n}^4)$ is increasing too on $(-1,\infty)$ for each $n\in\{1,2,3,\dots\}$ and $|x|<j_{\nu,1}$ fixed. Consequently, by using the infinite product formulas (2.4) and (2.5), the function

$$\nu\mapsto\log\left[\mathcal{J}_\nu(x)\mathcal{I}_\nu(x)\right]=\sum_{n\geq 1}\log\left(1-\frac{x^4}{j_{\nu,n}^4}\right)$$

is increasing on $(-1,\infty)$ for each $|x|<j_{\nu,1}$ fixed.

c. This follows from part **a** and **b**. □

Particular cases

We note that if we choose $\nu\in\{-1/2,1/2,3/2\}$ in part **a** of Theorem 3, then in view of (1.9), (1.10), (1.11), (2.7), (2.8) and (2.9) we obtain the following inequalities:

$$0<(\cos x)(\cosh x)\leq 1\quad\text{for all } |x|<\pi/2,$$

$$0<\left(\frac{\sin x}{x}\right)\left(\frac{\sinh x}{x}\right)\leq 1\quad\text{for all } |x|<\pi,$$

$$0<9\left(\frac{\sin x}{x^3}-\frac{\cos x}{x^2}\right)\left(\frac{\cosh x}{x^2}-\frac{\sinh x}{x^3}\right)\leq 1\quad\text{for all } |x|<j_{3/2,1},$$

where $j_{3/2,1}=4.493409$ in view of (1.11) is in fact the first positive zero of the equation $\tan x=x$. We note that the first two inequalities presented above were communicated to the first author by Professor Matti Vuorinen. Thanks are due to him for this information. It is also worth mentioning here that after we have finished the first draft of this manuscript we have found the paper [21], which the first inequality appears with interval of validity $(0,\pi/4)$ and the second with

$(0, \pi/2)$. Finally, observe that using (3.1) for all $|x| < \pi/2$ the following chain of inequalities holds true

$$0 < (\cos x)(\cosh x) \leq \left(\frac{\sin x}{x}\right)\left(\frac{\sinh x}{x}\right)$$
$$\leq 9\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right)\left(\frac{\cosh x}{x^2} - \frac{\sinh x}{x^3}\right) \leq 1.$$

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