Publ. Math. Debrecen
74/3-4 (2009), 279–292

A degeneracy theorem for meromorphic mappings with few hyperplanes and low truncation level of multiplicities

By TRAN VAN TAN (Hanoi)

Abstract. The purpose of this article is to give a theorem of the linear degeneration for meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with $(2n+2)$ hyperplanes and multiplicities are truncated by $(n+1)$.

1. Introduction

In 1926, R. Nevanlinna [10] showed that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values, then $f = g$, and that g is a special type of a linear fractional transformation of f if they have the same inverse images, counted with multiplicities, for four distinct values. In 1975, H. Fujimoto [5] generalized Nevalinna's result to the case of meromorphic mappings of $\mathbb{C}$ into $\mathbb{C}P^n$. He showed that for two linearly nondegenerate meromorphic mappings f and g of $\mathbb{C}$ into $\mathbb{C}P^n$, if they have the same inverse images, counted with multiplicities for $(3n+2)$ hyperplanes in $\mathbb{C}P^n$ in general position, then $f \equiv g$, and there exists a projective linear transformation L of $\mathbb{C}P^n$ to itself such that $g = L \cdot f$ if they have the same inverse images counted with multiplicities for $(3n+1)$ hyperplanes in $\mathbb{C}P^n$ in general position. Since that time, this problem has been studied intensively by H. Fujimoto, W. Stoll, L. Smiley, S. Ji, Z. Tu, G. Dethloff, T. V. Tan, D. D. Thai, S. D. Quang and others.

Mathematics Subject Classification: 32H30, 30D35.
Key words and phrases: meromorphic mappings, truncated multiplicities, linearly nondegenerate.
The author was partially supported by the post-doctoral research program of the Abdus Salam International Centre for Theoretical Physics, Italy.

Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with reduced representation $(f_0 : \cdots : f_n)$. For each hyperplane $H : a_0w_0 + \cdots + a_nw_n = 0$ in $\mathbb{C}P^n$, we put $(f, H) = a_0f_0 + \cdots + a_nf_n$ and denote by $\nu_{(f,H)}$ the map of $\mathbb{C}^m$ into $\mathbb{N}_0$ such that $\nu_{(f,H)}(a)$ $(a \in \mathbb{C}^m$) is the intersection multiplicity of the image of f and H at $f(a)$.

Take q hyperplanes $H_1, \dots, H_q$ in $\mathbb{C}P^n$ in general position, a linearly nondegenerate meromorphic mapping f of $\mathbb{C}$ into $\mathbb{C}P^n$ such that

$$\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \le m - 2, \quad \text{for all } 1 \le i < j \le q.$$

Let p be a positive integer. We consider the family $\mathcal{F}(\{H_j\}_{j=1}^q, f, p)$ of all linearly nondegenerate meromorphic mappings $g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ satisfying the conditions:

(a) $\min\{\nu_{(g,H_j)}, p\} = \min\{\nu_{(f,H_j)}, p\}$ for all $j \in \{1, \dots, q\}$,

(b) $g = f$ on $\bigcup_{j=1}^q f^{-1}(H_j)$.

The uniqueness problem of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ means that we want to find conditions for q (the number of hyperplanes) and p (the value at which multiplicities are truncated) such that the set $\mathcal{F}\big(\{H_j\}_{j=1}^q, f, p)$ contains only one mapping (Uniqueness Theorem) or, more generally, we want to study the cardinality of the set $\mathcal{F}\big(\{H_j\}_{j=1}^q, f, p)$ and find the relations among the mappings of this set.

In 1983, L. Smiley [11] gave the following uniqueness theorem.

Theorem 1.1. *If $q \ge 3n + 2$ and $p = 1$, then $g = f$ for any $g \in \mathcal{F}(\{H_j\}_{j=1}^q, f, p)$.*

In 1988, S. Ji [9] showed that

Theorem 1.2. *Assume that $q = 3n + 1$ and $p = 1$. Then for three maps $g_1, g_2, g_3 \in \mathcal{F}(\{H_j\}_{j=1}^q, f, p)$, the map $g_1 \times g_2 \times g_3 : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$ is algebraically degenerate, namely, $\{(g_1(z), g_2(z), g_3(z)),\ z \in \mathbb{C}^m\}$ is included in a proper algebraic subset of $\mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$.*

In 2006, G. Dethloff and T. V. Tan [4] showed that the above result of S. Ji remains valid if $q \ge \big[\frac{5(n+1)}{2}\big]$, where we denote $[x] := \max\{k \in \mathbb{Z} : k \le x\}$ for a constant x.

For the case of fewer hyperplanes, in [3], [6]–[8], [13] the authors obtained some other degeneracy theorems. We would like to emphasize here that in all of them either multiplicities are not truncated or multiplicities are truncated by a big positive integer. This point plays an essential role in their proofs. We formulate the result of H. Fujimoto [7] in 1998 with the best truncation level available at present.

Theorem 1.3. *Suppose that $q \geq 2n+2, p = \frac{n(n+1)}{2} + n$ and take arbitrary $n+2$ mappings $f^0, \dots, f^{n+1}$ in $\mathcal{F}(\{H_j\}_{j=1}^q, f, p)$. Then, there are $n+1$ hyperplanes $H_{j_0}, \dots, H_{j_n}$ among H_j's such that for each pair (i,k) with $0 \leq i < k \leq n$, we have that*

$$\frac{(H_{j_i}, f^1)}{(H_{j_k}, f^1)} - \frac{(H_{j_i}, f^0)}{(H_{j_k}, f^0)}, \frac{(H_{j_i}, f^2)}{(H_{j_k}, f^2)} - \frac{(H_{j_i}, f^0)}{(H_{j_k}, f^0)}, \dots, \frac{(H_{j_i}, f^{n+1})}{(H_{j_k}, f^{n+1})} - \frac{(H_{j_i}, f^0)}{(H_{j_k}, f^0)}$$

are linearly dependent.

In this paper we will prove the following degeneracy theorem for the case where multiplicities are truncated by a smaller number, namely $(n+1)$.

Theorem 1.4. *Suppose that $q \geq 2n+2$ and $p = n+1$, then for each mapping g in $\mathcal{F}(\{H_j\}_{j=1}^q, f, p)$, there exist a constant $\alpha \in \mathbb{C}$ and a pair (i,j) with $1 \leq i < j \leq q$, such that*

$$\frac{(H_i, f)}{(H_j, f)} \equiv \alpha \frac{(H_i, g)}{(H_j, g)}.$$

Remark. a) As a corollary of Theorem 1.4, we get that the mapping $f \times g : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ is linearly degenerate (with the algebraic structure in $\mathbb{C}P^n \times \mathbb{C}P^n$ given by the Segre embedding into $\mathbb{C}P^{n^2+2n}$).

b) By the Second Main Theorem and since $g = f$ on $\bigcup_{j=1}^q f^{-1}(H_j)$, it is easy to see that in Theorem 1.4 if $n \geq 2$ then $\alpha = 1$. But this dose not hold if $n = 1$, in fact, consider two holomorphic mappings $f = (e^z : 1)$, $g = (1 : e^z)$ of $\mathbb{C}$ into $\mathbb{C}P^1$ and four points $a_1 = (1:0)$, $a_2 = (0:1)$, $a_3 = (1:1), a_4 = (1:-1)$.

The idea of our proof is completely different from Fujimoto's. The proof of Fujimoto is based on using the Cartan auxiliary function. Our proof consists two main ideas: First of all, we use the Second Main Theorem for estimating the counting function of the set $A := \cup_{i=1}^{2n+2}\{z : \nu_{(f,H_j)}(z) \neq n \text{ or } \nu_{(g,H_j)}(z) \neq n\}$, after that, we use Borel's method. We would like to note that so far the versions of Borel's lemma are only for nowhere vanishing holomorphic functions (classical version) and for meromorphic functions with very small zero and pole sets (the version in [3] which is best available at present). When studying uniqueness problem, so far, Borel's method was only used for the case where multiplicities are not truncated or truncated by a big constant. This comes from the fact that if multiplicities are not truncated, we get immediately that the function $\frac{(f,Hj)}{(g,H_j)}$ is holomorphic nowhere vanishing and if multiplicities are truncated by a big number, we estimate easily that the counting function of the set of all zero and pole points of the meromorphic function $\frac{(f,Hj)}{(g,H_j)}$ is small. Our new idea in this paper is to use Borel's method for the case where multiplicities are truncated by a small number.

2. Preliminaries

We set $\|z\| = \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and define

$$B(r) := \{z \in \mathbb{C}^m : \|z\| < r\}, \quad S(r) := \{z \in \mathbb{C}^m : \|z\| = r\} \text{ for all } 0 < r < \infty.$$

Define $d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$, $\upsilon := (dd^c\|z\|^2)^{m-1}$ and $\sigma := d^c \log \|z\|^2 \wedge (dd^c \log \|z\|^2)^{m-1}$.

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, expanding F as $F = \sum P_i(z-a)$ with homogeneous polynomials P_i of degree i around a, we define

$$\nu_F(a) := \min\{i : P_i \not\equiv 0\}.$$

Let φ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the divisor ν_φ as follows: For each $z \in \mathbb{C}^m$, we choose nonzero holomorphic functions F and G on a neighborhood U of z such that $\varphi = \frac{F}{G}$ on U and $\dim\big(F^{-1}(0) \cap G^{-1}(0)\big) \le m-2$ and then we put $\nu_\varphi(z) := \nu_F(z)$.

Let ν be a divisor in $\mathbb{C}^m$ and k, M be positive integers or $+\infty$. Set $|\nu| := \overline{\{z : \nu(z) \ne 0\}}$ and

$$^{\le M}\nu^{[k]}(z) = 0 \quad \text{if } \nu(z) > M \quad \text{and} \quad {}^{\le M}\nu^{[k]}(z) = \min\{\nu(z), k\} \quad \text{if } \nu(z) \le M$$

$$^{> M}\nu^{[k]}(z) = 0 \quad \text{if } \nu(z) \le M \quad \text{and} \quad {}^{> M}\nu^{[k]}(z) = \min\{\nu(z), k\} \quad \text{if } \nu(z) > M.$$

The counting function is defined by

$$^{\le M}N^{[k]}(r, \nu) := \int_1^r \frac{^{\le M}n(t)}{t^{2m-1}} dt$$

and

$$^{> M}N^{[k]}(r, \nu) := \int_1^r \frac{^{> M}n(t)}{t^{2m-1}} dt \quad (1 \le r < +\infty)$$

where

$$^{\le M}n(t) := \int_{|\nu| \cap B(r)} {}^{\le M}\nu^{[k]}.\upsilon \text{ for } m \ge 2, \quad {}^{\le M}n(t) := \sum_{|z| \le t} {}^{\le M}\nu^{[k]}(z) \text{ for } m = 1$$

$$^{> M}n(t) := \int_{|\nu| \cap B(r)} {}^{> M}\nu^{[k]}.\upsilon \text{ for } m \ge 2, \quad {}^{> M}n(t) := \sum_{|z| \le t} {}^{> M}\nu^{[k]}(z) \text{ for } m = 1.$$

For a nonzero meromorphic function φ on $\mathbb{C}^m$, we set

$$^{\le M}N^{[k]}_\varphi(r) := {}^{\le M}N^{[k]}(r, \nu_\varphi) \quad \text{and} \quad {}^{> M}N^{[k]}_\varphi(r) := {}^{> M}N^{[k]}(r, \nu_\varphi).$$

For brevity we will omit the character $^{[k]}$ (respectively $^{\leqslant M}$) in the counting function and in the divisor if $k = +\infty$ (respectively $M = +\infty$).

We have the following Jensen's formula:

$$N_{\varphi}(r) - N_{\frac{1}{\varphi}}(r) = \int_{S(r)} \log|\varphi|\sigma - \int_{S(1)} \log|\varphi|\sigma.$$

Let $f : \mathbb{C}^m \longrightarrow \mathbb{C}P^n$ be a meromorphic mapping. For an arbitrary fixed homogeneous coordinate system $(w_0 : \cdots : w_n)$ in $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension $\geqslant 2$. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_n|^2\big)^{1/2}$. The characteristic function $T_f(r)$ of f is defined by

$$T_f(r) := \int_{S(r)} \log\|f\|\sigma - \int_{S(1)} \log\|f\|\sigma, \quad 1 < r < +\infty.$$

For a meromorphic function φ on $\mathbb{C}^m$, the characteristic function $T_{\varphi}(r)$ of φ is defined by considering φ as a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

The proximity function $m(r,\varphi)$ is defined by

$$m(r,\varphi) = \int_{S(r)} \log^+|\varphi|\sigma,$$

where $\log^+ x = \max\big\{\log x, 0\big\}$ for $x \geqslant 0$.

We state the First and Second Main Theorems in Value Distribution Theory:

First Main Theorem.

1) *For a nonzero meromorphic function φ, on $\mathbb{C}^m$ we have*

$$T_{\varphi}(r) = N_{\frac{1}{\varphi}}(r) + m(r,\varphi) + O(1).$$

2) *Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$, and H be a hyperplane in $\mathbb{C}P^n$ such that $(f,H) \not\equiv 0$. Then*

$$N_{(f,H)}(r) \leqslant T_f(r) + O(1) \quad \text{for all } r > 1.$$

Second Main Theorem. *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $H_1, \ldots, H_q$ $(q \geqslant n+1)$ hyperplanes in $\mathbb{C}P^n$ in general position. Then*

$$(q-n-1)T_f(r) \leqslant \sum_{j=1}^{q} N^{[n]}_{(f,H_j)}(r) + o\big(T_f(r)\big)$$

for all r except for a subset E of $(1,+\infty)$ of finite Lebesgue measure.

3. Proof of Theorem 1.4

In order to prove Theorem 1.4 we need the following lemma.

Let $\mathcal{G}$ be a torsion free abelian group and $A = (x_1, \dots, x_q)$ be a q-tuple of elements x_i in $\mathcal{G}$. Let $1 < s < r \le q$. We say that A has the property $P_{r,s}$ if any r elements $x_{p_1}, \dots, x_{p_r}$ in A satisfy the condition that for any subset $I \subset \{p_1, \dots, p_r\}$ with $\#I = s$, there exists a subset $J \subset \{p_1, \dots, p_r\}$, $J \ne I, \#J = s$ such that $\prod_{i \in I} x_i = \prod j \in J x_j$.

Lemma 3.1. *If A has the property $P_{r,s}$, then there exists a subset $\{i_1, \dots, i_{q-r+2}\}$ of $\{1, \dots, q\}$ such that $x_{i_1} = \dots = x_{i_{q-r+2}}$.*

Proof. We refer to [6], Lemma 2.6. □

We now begin to prove Theorem 1.4.
We introduce an equivalence relation on $\mathcal{A} := \{1, \dots, q\}$ as follows: $i \sim j$ if and only if

$$\det \begin{pmatrix} (f, H_i) & (f, H_j) \\ \\ (g, H_i) & (g, H_j) \end{pmatrix} \equiv 0.$$

Set $\{\mathcal{A}_1, \dots, \mathcal{A}_s\} = \mathcal{A}/\sim$. Since $f \not\equiv g$ and $\{H_j\}_{j=1}^q$ are in general position, we have that $\sharp \mathcal{A}_k \le n$ for all $k \in \{1, \dots, s\}$. Without loss of generality, we may assume that $\mathcal{A}_k := \{i_{k-1}+1, \dots, i_k\}$ $(k \in \{1, \dots, s\})$ where $0 = i_0 < \dots < i_s = q$. We define the map $\sigma : \{1, \dots, q\} \to \{1, \dots, q\}$ by

$$\sigma(i) = \begin{cases} i + n & \text{if } i + n \le q, \\ \\ i + n - q & \text{if } i + n > q. \end{cases}$$

It is easy to see that σ is bijective and $|\sigma(i) - i| \ge n$ (note that $q \ge 2n + 2$). This implies that i and $\sigma(i)$ belong two distinct sets among $\mathcal{A}'_k$ s. This implies that

$$\det \begin{pmatrix} (f, H_i) & (f, H_{\sigma(i)}) \\ \\ (g, H_i) & (g, H_{\sigma(i)}) \end{pmatrix} \not\equiv 0.$$

Let i_0 be an arbitrary fixed index, $i_0 \in \{1, \dots, q\}$. Set

$$\phi := \frac{(f, H_{i_0})}{(f, H_{\sigma(i_0)})} - \frac{(g, H_{i_0})}{(g, H_{\sigma(i_0)})} \not\equiv 0.$$

Let z_0 be an arbitrary zero point of (f, H_{i_0}) (if there exist any), then we have that z_0 is also a zero point of ϕ with multiplicity $\geq \min\{\nu_{(f,H_{i_0})}(z_0), n+1\}$ (outside an analytic set of codimension ≥ 2).

For any $j \in \{1, \ldots, q\} \backslash \{i_0, \sigma(i_0)\}$, since $f = g$ on $f^{-1}(H_j)$ we have that a zero point of (f, H_j) is also a zero point of ϕ (outside an analytic set of codimension ≥ 2).

On the other hand $\dim \big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leq m-2$ for all $1 \leq i < j \leq q$. Hence, we have

$$N_\phi(r) \geq N^{[n+1]}_{(f,H_{i_0})}(r) + \sum_{j=1, j\neq i_0, \sigma(i_0)}^{q} N^{[1]}_{(f,H_j)}(r). \tag{3.1}$$

By the First Main Theorem we have

$$\begin{aligned} m\left(r, \frac{(f, H_{i_0})}{(f, H_{\sigma(i_0)})}\right) &= T_{\frac{(f,H_{i_0})}{(f,H_{\sigma(i_0)})}}(r) - N_{\frac{(f,H_{\sigma(i_0)})}{(f,H_{i_0})}}(r) + O(1) \\ &\leq T_f(r) - N_{(f,H_{\sigma(i_0)})}(r) + O(1). \end{aligned}$$

Similarly,

$$m\left(r, \frac{(g, H_{i_0})}{(g, H_{\sigma(i_0)})}\right) \leq T_g(r) - N_{(g,H_{\sigma(i_0)})}(r) + O(1).$$

Hence, we have

$$\begin{aligned} m(r, \phi) &\leq m\left(r, \frac{(f, H_{i_0})}{(f, H_{\sigma(i_0)})}\right) + m\left(r, \frac{(g, H_{i_0})}{(g, H_{\sigma(i_0)})}\right) + O(1) \\ &\leq T_f(r) + T_g(r) - N_{(f,H_{\sigma(i_0)})}(r) - N_{(g,H_{\sigma(i_0)})}(r) + O(1). \end{aligned} \tag{3.2}$$

Set $\nu = \max\{\nu_{(f,H_{\sigma(i_0)})}, \nu_{(g,H_{\sigma(i_0)})}\}$.

Since $\min\{\nu_{(f,H_{\sigma(i_0)})}, n+1\} = \min\{\nu_{(g,H_{\sigma(i_0)})}, n+1\}$, we have

$$\nu + \nu^{[n+1]}_{(f,H_{\sigma(i_0)})} - \nu_{(f,H_{\sigma(i_0)})} - \nu_{(g,H_{\sigma(i_0)})} \leq 0. \tag{3.3}$$

This implies that

$$N_{(f,H_{\sigma(i_0)})}(r) + N_{(g,H_{\sigma(0)})}(r) \geq N(r, \nu) + N^{[n+1]}_{(f,H_{\sigma(i_0)})}(r).$$

Combining with (3.2) we have

$$m(r, \phi) \leq T_f(r) + T_g(r) - N(r, \nu) - N^{[n+1]}_{(f,H_{\sigma(i_0)})}(r) + O(1).$$

On the other hand, it is clear that

$$N(r,\nu) \geq N_{\frac{1}{\phi}}(r).$$

Hence, we get

$$m(r,\phi) \leq T_f(r) + T_g(r) - N_{\frac{1}{\phi}}(r) - N^{[n+1]}_{(f,H_{\sigma(i_0)})}(r) + O(1).$$

Then, by the First Main Theorem we have

$$\begin{aligned} N_{\phi}(r) &\leq T_{\phi}(r) + O(1) = m(r,\phi) + N_{\frac{1}{\phi}}(r) + O(1) \\ &\leq T_f(r) + T_g(r) - N^{[n+1]}_{(f,H_{\sigma(i_0)})}(r) + O(1). \end{aligned} \tag{3.4}$$

By (3.1) and (3.4) we have

$$N^{[n+1]}_{(f,H_{i_0})}(r) + \sum_{j=1, j\neq i_0,\sigma(i_0)}^{q} N^{[1]}_{(f,H_j)}(r) \leq T_f(r) + T_g(r) - N^{[n+1]}_{(f,H_{\sigma(i_0)})}(r) + O(1).$$

This gives

$$\sum_{j=1, j\neq i_0,\sigma(i_0)}^{q} N^{[1]}_{(f,H_j)}(r) + N^{[n+1]}_{(f,H_{i_0})}(r) + N^{[n+1]}_{(f,H_{\sigma(i_0)})}(r) \leq T_f(r) + T_g(r) + O(1). \tag{3.5}$$

for all $i_0 \in \{1,\ldots,q\}$.
Taking the sum of both sides of the inequality (3.5) over all $i_0 \in \{1,\ldots,q\}$, we have

$$(q-2)\sum_{j=1}^{q} N^{[1]}_{(f,H_j)}(r) + \sum_{i=1}^{q} \big(N^{[n+1]}_{(f,H_i)}(r) + N^{[n+1]}_{(f,H_{\sigma(i)})}(r)\big) \leq q\big(T_f(r) + T_g(r)\big) + O(1).$$

This gives

$$(q-2)\sum_{j=1}^{q} N^{[1]}_{(f,H_j)}(r) + 2\sum_{i=1}^{q} N^{[n+1]}_{(f,H_i)}(r) \leq q\big(T_f(r) + T_g(r)\big) + O(1),$$

(note that σ is bijective).
Similarly,

$$(q-2)\sum_{j=1}^{q} N^{[1]}_{(g,H_j)}(r) + 2\sum_{i=1}^{q} N^{[n+1]}_{(g,H_i)}(r) \leq q\big(T_f(r) + T_g(r)\big) + O(1).$$

Therefore, we get

$$(q-2)\sum_{j=1}^{q}\big(N^{[1]}_{(f,H_j)}(r)+N^{[1]}_{(g,H_j)}(r)\big)+2\sum_{i=1}^{q}\big(N^{[n+1]}_{(f,H_i)}(r)+N^{[n+1]}_{(g,H_i)}(r)\big)$$
$$\leq 2q\big(T_f(r)+T_g(r)\big)+O(1). \tag{3.6}$$

By the Second Main Theorem, we have

$$(q-n-1)(T_f(r)+T_g(r))\leq \sum_{j=1}^{q}\big(N^{[n]}_{(f,H_j)}(r)+N^{[n]}_{(g,H_j)}(r)\big)+o(T_f(r)+T_g(r)).$$

Hence, by (3.6) we get

$$\begin{aligned}
&(q-2)\sum_{j=1}^{q}\big(N^{[1]}_{(f,H_j)}(r)-\frac{1}{n}N^{[n]}_{(f,H_j)}(r)\big)+(q-2)\sum_{j=1}^{q}\big(N^{[1]}_{(g,H_j)}(r)-\frac{1}{n}N^{[n]}_{(g,H_j)}(r)\big)g\\
&\quad+2\sum_{j=1}^{q}\big(N^{[n+1]}_{(f,H_j)}(r)-N^{[n]}_{(f,H_j)}(r)\big)+2\sum_{j=1}^{q}\big(N^{[n+1]}_{(g,H_j)}(r)-N^{[n]}_{(g,H_j)}(r)\big)\\
&\quad\leq (q-2)\sum_{j=1}^{q}\big(N^{[1]}_{(f,H_j)}(r)+N^{[1]}_{(g,H_j)}(r)\big)+2\sum_{j=1}^{q}\big(N^{[n+1]}_{(f,H_j)}(r)+N^{[n+1]}_{(g,H_j)}(r)\big)\\
&\quad\quad-\frac{(q-2+2n)(q-n-1)}{n}(T_f(r)+T_g(r))+o(T_f(r)+T_g(r))\\
&\quad\overset{(3.6)}{\leq}\left(2q-\frac{(q-2+2n)(q-n-1)}{n}\right)(T_f(r)+T_g(r))+o(T_f(r)+T_g(r))\\
&\quad\leq o(T_f(r)+T_g(r))\quad \text{(note that } q\geq 2n+2).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sum_{j=1}^{q}\big(N^{[1]}_{(f,H_j)}(r)-\frac{1}{n}N^{[n]}_{(f,H_j)}(r)\big)\leq o(T_f(r)+T_g(r))\quad\text{and}\\
&\sum_{j=1}^{q}\big(N^{[n+1]}_{(f,H_j)}(r)-N^{[n]}_{(f,H_j)}(r)\big)\leq o(T_f(r)+T_g(r)),\\
&\sum_{j=1}^{q}\big(N^{[1]}_{(g,H_j)}(r)-\frac{1}{n}N^{[n]}_{(g,H_j)}(r)\big)\leq o(T_f(r)+T_g(r))\quad\text{and}\\
&\sum_{j=1}^{q}\big(N^{[n+1]}_{(g,H_j)}(r)-N^{[n]}_{(g,H_j)}(r)\big)\leq o(T_f(r)+T_g(r)).
\end{aligned}$$

This implies that for all $j \in \{1,\dots,q\}$, we have

$$
{}^{\le n-1}N^{[1]}_{(f,H_j)}(r) \le o(T_f(r)+T_g(r)) \text{ and } {}^{>n}N^{[1]}_{(f,H_j)}(r) \le o(T_f(r)+T_g(r)),
$$
$$
{}^{\le n-1}N^{[1]}_{(g,H_j)}(r) \le o(T_f(r)+T_g(r)) \text{ and } {}^{>n}N^{[1]}_{(g,H_j)}(r) \le o(T_f(r)+T_g(r)). \quad (3.7)
$$

Define functions

$$
h_j = \frac{(f,H_j)}{(g,,H_j)}, \quad j \in \{1,\dots,q\}.
$$

We choose an arbitrary subset $\mathcal{Q} = \{j_1,\dots,j_{2n+2}\}$ of the index set $\mathcal{A} := \{1,\dots,q\}$. Assume that $H_j : a_{j0}w_0+\dots+a_{jn}w_n = 0$ ($j \in \{1,\dots,q\}$).
We have,

$$
\begin{cases} a_{j0}f_0+\dots+a_{jn}f_n = h_j(a_{j0}g_0+\dots+a_{jn}g_n) \\ j \in \mathcal{Q} \end{cases}
$$
$$
\Rightarrow \begin{cases} a_{j_s0}f_0+\dots+a_{j_sn}f_n - h_{j_s}a_{j_s0}g_0 - \dots - h_{j_s}a_{j_sn}g_n = 0 \\ 1 \le s \le 2n+2 \end{cases}
$$

Therefore

$$
\det(a_{j_s0},\dots,a_{j_sn},h_{j_s}a_{j_s0},\dots,h_{j_s}a_{j_sn},\ 1 \le s \le 2n+2) \equiv 0.
$$

For each $I = \{j_{s_0},\dots,j_{s_n}\} \subset \mathcal{Q}$, $1 \le s_0 < \dots < s_n \le 2n+2$, we define

$$
A_I = (-1)^{\frac{n(n+1)}{2}+s_0+\dots+s_n} \cdot \det(a_{j_{s_k}i}, 0 \le k, i \le n) \cdot \det(a_{j_{s'_k}i}, \quad 0 \le k,\ i \le n)
$$

where $\{s'_0,\dots,s'_n\} = \{1,\dots,2n+2\} \setminus \{s_0,\dots,s_n\}$, $s'_0 < \dots < s'_n$. We have $A_I \in \mathbb{C}^*$. Set $\mathcal{L} := \{I \subset \mathcal{Q}, \#I = n+1\}$, then $\#\mathcal{L} = \binom{2n+2}{n+1}$. By the Laplace expansion Theorem, we have

$$
\sum_{I \in \mathcal{L}} A_I h_I \equiv 0. \quad (3.8)
$$

where $h_I := \prod_{i \in I} h_i$.
We introduce an equivalence relation $I \sim J$ on $\mathcal{L}$ as follows: $I \sim J$ if and only if $\frac{h_I}{h_J} \in \mathbb{C}$ (then $\frac{h_I}{h_J} \in \mathbb{C}^*$ since $h_I \not\equiv 0$ for all $I \in \mathcal{L}$). Set $\{L_1,\dots,L_s\} = \mathcal{L}/\sim$ $\left(s \le \binom{2n+2}{n+1}\right)$.
For each $k \in \{1,\dots,s\}$ we choose $I_k \in L_k$ and define $\alpha_k \in \mathbb{C}$ by

$$
\sum_{I \in L_k} A_I h_I = \alpha_k h_{I_k}.
$$

Then (3.8) can be written as

$$\sum_{k=1}^{s} \alpha_k h_{I_k} \equiv 0. \tag{3.9}$$

Case 1. There exists some $\alpha_k \neq 0$.

Without loss of generality, we may assume that $\alpha_i \neq 0$ for all $i \in \{1, \dots, \ell\}$ and $\alpha_k = 0$ for all $i \in \{\ell + 1, \dots, s\}$ $(1 \leq \ell \leq s)$.

Denote by $\mathcal{P}$ the set of all positive integers $k \leq \ell$ such that there exist a subset $P_k \subset \{1, \dots, \ell\}, \# P_k = k$ and nonzero constants c_i $(i \in P_k)$ with

$$\sum_{i \in P_k} c_i h_{I_i} \equiv 0.$$

It follows from (3.9) that $\mathcal{P} \neq \varnothing$. Let t be the smallest integer in $\mathcal{P}$. Without loss of generality, we may assume that $P_t = \{1, \dots, t\}$. Then there exist nonzero constants c_i $(1 \leq i \leq t)$ such that

$$\sum_{i=1}^{t} c_i h_{I_i} \equiv 0. \tag{3.10}$$

Since $\frac{h_{I_i}}{h_{I_j}} \notin \mathbb{C}$ and $h_{I_i} \not\equiv 0$ for all $1 \leq i \neq j \leq t$, we have $t \geq 3$.

Without loss of generality, we may assume that

$$T_{\frac{c_1 h_{I_1}}{c_2 h_{I_2}}}(r) = \max\{T_{\frac{c_1 h_{I_1}}{c_2 h_{I_2}}}(r), T_{\frac{c_2 h_{I_2}}{c_3 h_{I_3}}}(r), T_{\frac{c_3 h_{I_3}}{c_1 h_{I_1}}}(r)\} \quad \text{for all } r \in \mathcal{R}, \tag{3.11}$$

where $\mathcal{R}$ is a subset of $[1, +\infty)$ with infinite Lebesgue measure.

We define a meromorphic mapping φ of $\mathbb{C}^m$ into $\mathbb{C}P^{t-2}$ by $\varphi := (c_1 h_{I_1} : \cdots : c_{t-1} h_{I_{t-1}})$. Since $t = \min \mathcal{P}$, we have that φ is linearly nondegenerate.

Since $t \geq 3$ and by (3.11) we have

$$T_\varphi(r) \geq T_{\frac{c_1 h_{I_1}}{c_2 h_{I_2}}}(r) \geq \frac{1}{3}\big(T_{\frac{c_1 h_{I_1}}{c_2 h_{I_2}}}(r) + T_{\frac{c_2 h_{I_2}}{c_3 h_{I_3}}}(r) + T_{\frac{c_3 h_{I_3}}{c_1 h_{I_1}}}(r)\big) \quad \text{for all } r \in \mathcal{R}. \tag{3.12}$$

It is easy to see that

$$((I_1 \cup I_2) \setminus (I_1 \cap I_2)) \cap ((I_2 \cup I_3) \setminus (I_2 \cap I_3)) \cap ((I_3 \cup I_1) \setminus (I_3 \cap I_1)) = \varnothing.$$

So, $\mathcal{X}_{12} \cup \mathcal{X}_{23} \cup \mathcal{X}_{31} = \{1, \dots, q\}$, where $\mathcal{X}_{uv} = \{1, \dots, q\} \setminus ((I_u \cup J_v) \setminus (I_u \cap J_v))$.

On the other hand, since $f = g$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$ and $\dim\big(f^{-1}(H_i)\cap f^{-1}(H_j)\big) \leq m-2$ for all $0 \leq i \neq j \leq q$ we have

$$N_{\frac{h_{I_u}}{h_{I_v}}-1}(r) \geq \sum_{j\in\mathcal{X}_{uv}} N^{[1]}_{(f,H_j)}(r).$$

Hence, by the Second Main Theorem we have

$$\begin{aligned} N_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N_{\frac{h_{I_2}}{h_{I_3}}-1}(r) + N_{\frac{h_{I_3}}{h_{I_1}}-1}(r) &\geq \sum_{j=1}^{q} N^{[1]}_{(f,H_j)}(r) \\ &\geq \frac{q-n-1}{n} T_f(r) - o(T_f(r)). \end{aligned} \tag{3.13}$$

Similarly,

$$N_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N_{\frac{h_{I_2}}{h_{I_3}}-1}(r) + N_{\frac{h_{I_3}}{h_{I_1}}-1}(r) \geq \frac{q-n-1}{n} T_g(r) - o(T_g(r)). \tag{3.14}$$

By (3.12), (3.13), (3.14) and by the First Main Theorem, we have

$$\begin{aligned} T_\varphi(r) &\geq \frac{1}{3}\Big(T_{\frac{c_1h_{I_1}}{c_2h_{I_2}}}(r) + T_{\frac{c_2h_{I_2}}{c_3h_{I_3}}}(r) + T_{\frac{c_3h_{I_3}}{c_1h_{I_1}}}(r)\Big) \\ &= \frac{1}{3}\Big(T_{\frac{h_{I_1}}{h_{I_2}}}(r) + T_{\frac{h_{I_2}}{h_{I_3}}}(r) + T_{\frac{h_{I_3}}{h_{I_1}}}(r)\Big) + O(1) \\ &\geq \frac{1}{3}\Big(N_{\frac{h_{I_1}}{h_{I_2}}-1}(r) + N_{\frac{h_{I_2}}{h_{I_3}}-1}(r) + N_{\frac{h_{I_3}}{h_{I_1}}-1}(r)\Big) \\ &\geq \frac{q-n-1}{6n}\big(T_f(r) + T_g(r)\big) - o(T_f(r)+T_g(r)) - O(1),\ r \in \mathcal{R}. \end{aligned} \tag{3.15}$$

Since $\dim\big(f^{-1}(H_i) \cap f^{-1}(H_j)\big) \leq m-2$, for all $1 \leq i < j \leq q$, it is easy to see that for each $I \in \mathcal{L}$ there exists an analytic subset S of $\mathbb{C}^m$ with codimension at least 2 such that $\nu_{h_I} = 0 = \nu_{\frac{1}{h_I}}$ on $\bigcup_{j=1}^{q}\{z : \nu_{(f,H_j)} = n = \nu_{(g,H_j)}\}\diagdown S$.
Then by (3.7) we get

$$\begin{aligned} \sum_{I\in\mathcal{L}} N^{[1]}_{h_I}(r) + \sum_{I\in\mathcal{L}} N^{[1]}_{\frac{1}{h_I}}(r) &\leq O\Big(\sum_{j=1}^{q}\big({}^{\leq n-1}N^{[1]}_{(f,H_j)}(r) + {}^{>n}N^{[1]}_{(f,H_j)}(r)\big) \\ &\quad + O\Big(\sum_{j=1}^{q}\big({}^{\leq n-1}N^{[1]}_{(g,H_j)}(r) + {}^{>n}N^{[1]}_{(g,H_j)}(r)\big) \leq o(T_f(r) + T_g(r)). \end{aligned}$$

Combining with (3.15) we have

$$\sum_{I\in\mathcal{L}} N_{h_I}(r) + \sum_{I\in\mathcal{L}} N_{\frac{1}{h_I}}(r) \le o\big(T_\varphi(r)\big),\ r\in\mathcal{R}. \tag{3.16}$$

Let $(u_1 : \cdots : u_{t-1})$ be a reduced representation of φ. Set $u_t = \frac{c_t h_{I_t} u_1}{c_1 h_{I_1}}$.
By (3.10) we have

$$\sum_{i=1}^{t} u_i \equiv 0. \tag{3.17}$$

It is easy to see that a zero of $u_i (i = 1, \dots, t)$ is a zero or a pole of some h_{I_j} $(j \in \{1, \dots, t\})$. Thus

$$\sum_{i=1}^{t} N^{[1]}_{u_i}(r) \le t \sum_{i=1}^{t} (N^{[1]}_{h_{I_i}}(r) + N^{[1]}_{\frac{1}{h_{I_i}}}(r)). \tag{3.18}$$

By (3.17), (3.18) and by the Second Main Theorem we have

$$\begin{aligned} T_\varphi(r) &\le \sum_{i=1}^{t-1} N^{[t-2]}_{u_i}(r) + N^{[t-2]}_{u_1+\cdots+u_{t-1}}(r) + o(T_\varphi(r)) \\ &\overset{(3.17)}{\le} \sum_{i=1}^{t} N^{[t-2]}_{u_i}(r) \le (t-2)\sum_{i=1}^{t} N^{[1]}_{u_i}(r) + o(T_\varphi(r)) \\ &\le t(t-2)\sum_{i=1}^{t} (N^{[1]}_{h_{I_i}}(r) + N^{[1]}_{\frac{1}{h_{I_i}}}(r)) + o(T_\varphi(r)) \\ &\le t(t-2)\sum_{I\in\mathcal{L}} \big(N_{h_I}(r) + N_{\frac{1}{h_I}}(r)\big) + o(T_\varphi(r)). \end{aligned}$$

Combining with (3.16) we have $T_\varphi(r) \le o(T_\varphi(r)), r \in \mathcal{R}$. This is a contradiction.

Case 2. $\alpha_k \ne 0$ for all $k \in \{1, \dots, s\}$. Then $\sum_{I\in L_k} A_I h_I \equiv 0$ for all $k \in \{1, \dots, s\}$. On the other hand $A_I h_I \not\equiv 0$. Hence, $\#L_k \ge 2$ for all $k \in \{1, \dots, s\}$. So, for each $I \in \mathcal{L}$ there exists $J \in \mathcal{L}$, $J \ne I$ such that

$$\frac{h_I}{h_J} \in \mathbb{C}^*. \tag{3.19}$$

Let $\mathcal{M}^*$ be the abelian multiplication group of all nonzero meromorphic functions on $\mathbb{C}^m$. It is clear that the multiplication group $\mathcal{G} := \mathcal{M}^*\big/\mathbb{C}^*$ is a torsion free abelian group. We denote by $[h]$ the class in $\mathcal{G}$ containing $h \in \mathcal{M}^*$.
By (3.19) we get that $A := ([h_1], \dots, [h_q])$ has the property $P_{2n+2,n+1}$. Then by Lemma 3.1 there exist $i, j \in \{1, \dots, q\}$, $i \ne j$ such that $[h_i] = [h_j]$. This means that there exists constant $\alpha \ne 0$ such that $\frac{(H_i,f)}{(H_j,f)} \equiv \alpha\frac{(H_i,g)}{(H_j,g)}$. This completes the proof of Theorem 1.4. $\square$

References

[1] H. Cartan, Sur la croissance des fonctions méromorphes d'une ou plusiers variables complexes, *C. R. Acad. Sci. Paris* **188** (1929), 1374–1376.

[2] G. Dethloff and T. V. Tan, Uniqueness problem for meromorphic mappings with truncated multiplicities and moving targets, *Nagoya Math. J.* **181** (2006), 75–101.

[3] G. Dethloff and T. V. Tan, Uniqueness problem for meromorphic mappings with truncated multiplicities and few targets, *Annales de la Faculté des Sciences de Toulous* **15** (2006), 217–242.

[4] G. Dethloff and T. V. Tan, An extension of uniqueness theorems for meromorphic mappings, *Vietnam J. Math.* **34** (2006), 71–94.

[5] H. Fujimoto, The uniqueness problem of meromorphic maps into the complex projective space, *Nayoya Math. J.* **58** (1975), 1–23.

[6] H. Fujimoto, A uniqueness theorem of algebraically nondegenerate meromorphic maps into $\mathbf{C}P^n$, *Nayoya Math. J.* **64** (1976), 117–147.

[7] H. Fujimoto, Uniqueness problem with truncated multiplicities in value distribution theory, *Nayoya Math. J.* **152** (1998), 131–152.

[8] H. Fujimoto, Uniqueness problem with truncated multiplicities in value distribution theory, II, *Nayoya Math. J.* **155** (1999), 161–188.

[9] S. Ji, Uniqueness problem without multiplicities in value distribution theory, *Pacific J. Math.* **135** (1988), 323–348.

[10] R. Nevanlinna, Einige Eideutigkeitssätze in der Theorie der meromorphen Funktionen, *Acta. Math.* **48** (1926), 367–391.

[11] L. Smiley, Geometric conditions for unicity of holomorphic curves, *Contemp. Math.* **25** (1983), 149–154.

[12] W. Stoll, On the propagation of dependences, *Pacific J. Math.* **139** (1989), 311–337.

[13] T. V. Tan, A degeneracy theorem for meromorphic mappings with moving targets, *Inter. J. Math.* **18** (2007), 235–244.

[14] D. D. Thai and S. D. Quang, Uniqueness problem with truncated multiplicities of meromorphic mappings in several complex variables for moving targets, *Inter. J. Math.* **16** (2005), 903–939.

[15] D. D. Thai and S. D. Quang, Uniqueness problem with truncated multiplicities of meromorphic mappings in several complex variables, *Inter. J. Math.* **17** (2006), 1223-1257.

[16] Z-H. Tu, Uniqueness problem of meromorphic mappings in several complex variables for moving targets, *Tohoku Math. J.* **54** (2002), 567–579.

TRAN VAN TAN
DEPARTMENT OF MATHEMATICS
HANOI NATIONAL UNIVERSITY OF EDUCATION
136 XUAN THUY STREET, CAU GIAY, HANOI
VIETNAM

E-mail: vtran@ictp.it; tranvantanhn@yahoo.com

(Received April 29, 2008)