Publ. Math. Debrecen
44 / 1–2 (1994), 123–135

On some generalized invariant means and almost approximately additive functions

By ROMAN BADORA (Katowice)

Abstract. We continue our investigations from [2]. We are going to prove the existence of generalized invariant means to some function spaces which are essentially larger than the space of all bounded functions. Our results are applied to the study of almost approximately additive functions.

1. Introduction

This paper is a continuation of our study which we carried out in [2].

Let $\mathcal{F}$ be a non-void subset of the space of all functions defined on a semigroup $(S,\cdot)$ with values in a set Y. We say that $\mathcal{F}$ is a *left (right) invariant* if and only if

$$f\in\mathcal{F} \text{ and } y\in S \text{ implies that } {}_yf\in\mathcal{F}\ (f_y\in\mathcal{F}), \tag{1}$$

where ${}_yf$ and f_y denote the *left* and *right translations* of $f\in\mathcal{F}$ defined by

$$ {}_yf(x)=f(yx),\ f_y(x)=f(xy),\ x,y\in S. \tag{2}$$

Let $\mathcal{F}$ be a left (right) invariant linear space of functions mapping a semigroup S into a real linear space Y. Let $\mathcal{C}$ be a family of subsets of Y and let $F:\mathcal{F}\to\mathcal{C}$. In [2] we have introduced a definition generalizing the concept of an invariant mean:

A linear operator $M:\mathcal{F}\to Y$ is termed a *left (right) invariant F-mean* if and only if it satisfies the following two conditions:

$$M(f)\in F(f),\ f\in\mathcal{F}; \tag{3}$$

$$M({}_yf)=M(f)\quad (M(f_y)=M(f)),\quad y\in S,\ f\in\mathcal{F}. \tag{4}$$

AMS 1980 Mathematics Subject Classification (1985 *Revision*): 43A07, 39B70.

In the case where $Y = \mathbb{R}$ and $\mathcal{F} = B(S, \mathbb{R})$, the space of all real bounded functions on a semigroup S, defining a map F from $B(S, \mathbb{R})$ into the family of all non-empty compact intervals of the real line $\mathbb{R}$ as follows:

$$F(f) = [\ \inf\{f(x) : x \in S\}, \sup\{f(x) : x \in S\}\ ],\ \ f \in B(S, \mathbb{R}),$$

we infer that our definition reduces to the classical definition of an invariant mean.

In agreement with the traditional terminology, if there exists at least one left (right) invariant mean on the space $B(S, \mathbb{R})$ then the underlying semigroup S is said to be *left (right) amenable*. For the theory of amenability of semigroups and groups see e.g. M. M. Day [5] and F. P. Greenleaf [16]. Here we only stress that every Abelian semigroup is (two-sided) amenable (see M. M. Day [4], and also, J. von Neuman [24], J. Dixmier [9] and E. Hewitt, K. A. Ross [18], Theorem 17.5), but amenability is a much weaker condition then commutativity.

Going back to our study from [2] we conssider some special collections $\mathcal{C}$ having the following property:

every subfamily of $\mathcal{C}$ any two members of which intersect has a non-empty intersection.

In this case, we say that the family $\mathcal{C}$ has the *binary intersection property*. Moreover, we recall that a *normed space has the binary intersection property* if and only if the family of all its closed balls has this property (see L. Nachbin [23] and M. M. Day [6]). For example, the space of all real bounded functions on a set has the binary intersection property.

In [2] we have shown that there exists a generalized invariant mean on the space $B(S, Y)$ of all bounded functions mapping a semigroup S into some real linear space Y. Precisely, we have proved the following

Theorem 1. *Let $(S, \cdot)$ be a left (right) amenable semigroup and let Y be a real locally convex linear topological space. Let $\mathcal{F}$ be a left (right) invariant linear subspace of the space $B(S, Y)$ of all bounded functions defined on S with values in Y. Let $\mathcal{C}$ be a subfamily of the family of all closed convex sets in Y having the binary intersection property and invariant with respect to translations by vectors of Y. Assume that the map $F : \mathcal{F} \to \mathcal{C}$ satisfies the following conditions:*

$$F(f + g) \subseteq F(f) + F(g), \quad f, g \in \mathcal{F}; \tag{5}$$

$$F(tf) = tF(f), \quad f \in \mathcal{F},\ t \in \mathbb{R} \setminus \{0\}; \tag{6}$$

$$f(S) \subseteq F(f), \quad f \in \mathcal{F}. \tag{7}$$

Then there exsists a left (right) invariant F-mean on the space $\mathcal{F}$.

In the present paper we are going to extend the concept of an invariant mean to some function spaces that are larger that the space $B(S, Y)$ (see [1], Z. Gajda [11] and [12]).

We shall consider some general situation (Theorem 3) and its special case with the space of all essentially bounded functions defined on a semigroup S with values in Y (Theorem 4).

Next, we present an application of these results to the study of almost approximately additive functions (see M. Kuczma [21]).

2. Existence theorem

We shall need the following results which were proved in [2]:

Lemma 1. *If the family $\mathcal{C}$ has the binary intersection property, then the family $\tilde{\mathcal{C}}$ of all non-empty intersections of subfamilies of the family $\mathcal{C}$ has also the binary intersection property.*

Lemma 2. *Let $\mathcal{C}$ be a family of subsets of the real linear space Y having the binary intersection property, invariant with respect to translations by vectors of Y and symmetry to zero. If $\{A_i : i \in \mathbb{I}\}$ and $\{B_q : q \in \mathbb{Q}\}$ are two subfamilies of $\mathcal{C}$ such that*

$$\bigcap\{A_i : i \in \mathbb{I}\} \neq \emptyset \text{ and } \bigcap\{B_q : q \in \mathbb{Q}\} \neq \emptyset,$$

then

$$\bigcap\{A_i + B_q : i \in \mathbb{I}, \quad q \in \mathbb{Q}\} = \bigcap\{A_i : i \in \mathbb{I}\} + \bigcap\{B_q : q \in \mathbb{Q}\}.$$

Theorem 2. *Let X and Y be two real linear spaces and let $\mathcal{C}$ be a family of subsets of Y having the binary intersection property and invariant with respect to translations by vectors from Y. Assume that the map $F : X \to \mathcal{C}$ satisfies the following two conditions:*

$$F(x+y) \subseteq F(x) + F(y), \quad x, y \in X;$$
$$F(tx) = tF(x), \quad x \in X,\ t \in \mathbb{R} \setminus \{0\}.$$

Next, let X_0 be a linear subspace of the space X and let $L_0 : X_0 \to Y$ be a linear operator on X_0 such that $L_0(x) \in F(x)$ for all $x \in X_0$. Then there exists a linear operator $L : X \to Y$ which is an extension of L_0 and $L(x) \in F(x)$ for all $x \in X$.

Now, we shall prove the existence theorem.

Theorem 3. *Let $(S, \cdot)$ be a left (right) amenable semigroup and let Y be a real locally convex linear topological space. Let $\mathcal{F}$ be a left (right) invariant linear space of functions defined on S with values in Y and let $\mathcal{C}$ be a subfamily of the family of all bounded closed convex subsets of Y having the binary intersection property and invariant with respect to translations by vectors of Y. Assume that the map $F : \mathcal{F} \to \mathcal{C}$ satisfies conditions (5), (6) and the following condition:*

$$F({}_yf) \subseteq F(f) \quad (F(f_y) \subseteq F(f)), \quad f \in \mathcal{F},\ y \in S. \tag{8}$$

Then there exists a left (right) invariant F-mean on the space $\mathcal{F}$.

Proof. We shall restrict ourselves to the proof of the "left-hand side version" of this theorem.

To start with, note that $0 \in F(0_S)$, where 0_S denotes the function equal zero on the whole semigroup S. Indeed, by our assumptions, the non-empty set $F(0_S)$ is convex and putting $t = -1$ in (6) we get that the set $F(0_S)$ is symmetric; therefore, $0 \in F(0_S)$.

The generalized Hahn–Banach theorem (Theorem 2), for the space $X = \mathcal{F}$ and the subspace X_0 degenerated to zero, implies that there exists a linear operator $L : \mathcal{F} \to Y$ such that

$$L(f) \in F(f), \quad f \in \mathcal{F}. \tag{9}$$

Let $f \in \mathcal{F}$ be fixed; we consider the mapping:

$$S \ni y \mapsto L({}_yf) \in Y. \tag{10}$$

From condition (9) and (8) we obtain immediately that

$$L({}_yf) \in F({}_yf) \subseteq F(f), \quad y \in S, \tag{11}$$

which ensures that function (10) is bounded, i.e. belongs to the space $B(S,Y)$ of all bounded functions transforming S into Y.

Let $\tilde{\mathcal{F}}$ denote the space of all functions $\tilde{f} : S \to Y$ of the form:

$$\tilde{f}(y) = L({}_yf), \quad y \in S \tag{12}$$

for some function $f \in \mathcal{F}$. Then the space $\tilde{\mathcal{F}}$ is a linear subspace of the space $B(S,Y)$ and left invariant. To proved that it is left invariant we observe first that:

$$_y({}_zf) = {}_{zy}f \tag{13}$$

for all $f \in \mathcal{F}$ and $y, z \in S$. Indeed, for every $x \in S$ we get:

$$_y({}_zf)(x) ={}_z f(yx) = f(zyx) ={}_{zy} f(x),$$

which means that (13) holds.

Now, if $\tilde{f} \in \tilde{\mathcal{F}}$ is of the form

$$\tilde{f}(y) = L({}_yf), \quad y \in S,$$

for some $f \in \mathcal{F}$, then for any $z \in S$ we have:

$${}_z\tilde{f}(y) = \tilde{f}(zy) = L({}_{zy}f) = L({}_y({}_zf)), \quad y \in S.$$

Consequently, the function ${}_z\tilde{f}$ is of the form (12) (with the function ${}_zf$ as f).

Let $\tilde{f} \in \tilde{\mathcal{F}}$ be fixed; from condition (11) we obtain:

$$\tilde{f}(S) \subseteq F(f) \tag{14}$$

for each $f \in \mathcal{F}$ such that

$$\tilde{f}(y) = L({}_y f), \quad y \in S.$$

Therefore, the family

$$\{F(f) : f \in \mathcal{F} \text{ and } \tilde{f}(y) = L({}_y f),\ y \in S\}$$

has non-empty intersection $\tilde{F}(\tilde{f})$. Thereby, we have defined a map $\tilde{F}$ from $\tilde{\mathcal{F}}$ into the family $\tilde{\mathcal{C}}$ of all non-empty intersections of some subfamilies of $\mathcal{C}$. By Lemma 1 the family $\tilde{C}$ has also the binary intersection property. By Lemma 2 and the assumed properties of F we obtain that the map $\tilde{F}$ satisfies condition (5) and (6). Moreover, condition (14) implies that the map $\tilde{F}$ satisfies condition (7).

Hence, we can apply Theorem 1 to the space $\tilde{\mathcal{F}}$, the family $\tilde{\mathcal{C}}$ and the map $\tilde{F}$ defined above. Now let $\tilde{M}$ denote a left invariant $\tilde{F}$-mean on $\tilde{\mathcal{F}}$.

We define an operator $M : \mathcal{F} \to Y$ by the formula:

$$M(f) = \tilde{M}_y(L({}_y f)), \quad f \in \mathcal{F},$$

where the subscript y next to $\tilde{M}$ indicates that the mean $\tilde{M}$ is applied to a function of the variable y.

It is clear that the operator M is linear, whereas from (11) we infer that

$$M(f) \in F(f), \quad f \in \mathcal{F}.$$

The identity (13) combined with the left invariance of $\tilde{M}$ yields:

$$M({}_z f) = \tilde{M}_y(L({}_y({}_z f))) = \tilde{M}_y(L({}_{zy} f)) = \tilde{M}_y(L({}_y f)) = M(f)$$

for all $f \in \mathcal{F}$ and $z \in S$. Thus, the operator M has all the desired properties for a left invariant F-mean and the proof is finished.

Now, we will give an example of a situation in which all the assumptions of Theorem 3 are satisfied.

Let $(S, \cdot)$ be a semigroup. A non-empty family $\Im$ of subsets of S will be called a *proper set ideal* if:

$$S \notin \Im; \tag{15}$$

$$A, B \in \Im \text{ implies } A \cup B \in \Im; \tag{16}$$

$$A \in \Im \text{ and } B \subseteq A \text{ imply } B \in \Im. \tag{17}$$

Moreover, if the set ${}_y A = \{x \in S : yx \in A\}$ belongs to the family $\Im$ whenever $A \in \Im$ and $y \in S$, then the set ideal $\Im$ is said to be *proper* *and left quasi-invariant* (in short: *p.l.q.i.*). Analogously, the set ideal $\Im$ is said to be *proper and right quasi-invariant* (in short: *p.r.q.i.*) if the set $A_y = \{x \in S : xy \in A\}$ belongs to the family $\Im$ whenever $A \in \Im$ and $y \in S$. In the case where the ideal $\Im$ satisfies both these conditions we shall call it *proper and quasi-invariant* (*p.q.i.*).

The sets belonging to the ideal are intuitively regarded as small sets. For example, if S is a second category subsemigroup of a topological group G then the family of all first category subsets of S is a p.q.i. ideal. Also, if G is a locally compact topological group equipped with the left or right Haar measure μ and if S is a semigroup of G with positive measure μ then the family of all subsets of S which have zero measure μ is a p.q.i. ideal (see also M. Kuczma [21]).

Let $(Y, \|\cdot\|)$ be a normed space having the binary intercestion property and let $\mathcal{C}_b$ denote the family of all non-empty intersections of closed balls of Y. This family has also the binary intersection property (see Lemma 1.) Let $\Im$ be a set ideal subsets of a semigroup S. For a function f from S into Y we define $\Im_f$ to be the family of all sets $A \in \Im$ such that f is bounded on the complement of A.

A function f from S into Y is called $\Im$-*essentially bounded* if and only if the family $\Im_f$ is non-empty. The space of all $\Im$-essentially bounded functions from S into Y will be denoted by $B^{\Im}(S, Y)$.

It is obvious that, in general, the space $B^{\Im}(S, Y)$ is essentially larger than the space $B(S, Y)$ of all bounded functions from S into Y.

Let $f \in B^{\Im}(S, Y)$ be fixed. For $A \in \Im_f$ we denote by $B_A(f)$ the intersection of all closed balls B of Y such that $f(S \setminus A) \subseteq B$. The family $\mathcal{C}_f = \{B_A(f) : A \in \Im_f\}$ is a subfamily of $\mathcal{C}_b$ and for any $A, \tilde{A} \in \Im_f$ we have:

$$\emptyset \neq f(S \setminus (A \cup \tilde{A})) \subseteq B_A(f) \cap B_{\tilde{A}}(f).$$

Hence, the intersection of $\mathcal{C}_f$ is non-empty and belongs to $\mathcal{C}_b$.

Now, we define a map $F_b^{\Im} : B^{\Im}(S, Y) \to \mathcal{C}_b$ by the following formula:

$$F_b^{\Im}(f) = \bigcap \mathcal{C}_f, \quad f \in B^{\Im}(S, Y). \tag{18}$$

Lemma 3. *If $\Im$ is a p.l.q.i. (p.r.q.i) ideal of subsets of S then*

(i) *the space $B^{\Im}(S, Y)$ is a linear left (right) invariant space;*
(ii) *the map $F_b^{\Im}$ defined by* (18) *satisfies conditions* (5), (6) *and* (8).

Proof. (i) The fact that $B^{\Im}(S, Y)$ is closed under the pointwise addition and scalar multiplication is a direct consequence of the observation that: for every $f, g \in B^{\Im}(S, Y)$

$$A \in \Im_f,\ B \in \Im_g \text{ imply that } A \cup B \in \Im_{f+g} \tag{19}$$

and

$$A \in \Im_{tf} \text{ for every real } t \text{ and } A \in \Im_f. \tag{20}$$

For the proof of the left invariance of $B^{\Im}(S,Y)$ we fix $f \in B^{\Im}(S,Y)$ and $y \in S$. Choosing arbitriarily an $A \in \Im_f$ we notice that ${}_yA \in \Im$. Moreover, if $x \in S \setminus_y A$ then $yx \in S \setminus A$. Consequently,

$${}_yf(x) = f(yx) \in f(S \setminus A),\ x \in S \setminus_y A. \tag{21}$$

Since f is bounded on $S \setminus A$, we infer that ${}_yf$ is bounded on $S \setminus_y A$. Thus ${}_yf \in B^{\Im}(S,Y)$.

(ii) Let $f, g \in B^{\Im}(S,Y)$ be fixed. For any $A \in \Im_f$, $\tilde{A} \in \Im_g$ and for any two closed balls B and $\tilde{B}$ of Y such that

$$f(S \setminus A) \subseteq B \quad \text{and} \quad g(S \setminus \tilde{A}) \subseteq \tilde{B}$$

we get

$$(f+g)(S \setminus (A \cup \tilde{A})) \subseteq B + \tilde{B}.$$

Because $B + \tilde{B}$ is a closed ball of Y (see [2], Remark 1) we conclude that

$$F_b^{\Im}(f+g) \subseteq B + \tilde{B}.$$

Hence, by Lemma 2, we get

$$F_b^{\Im}(f+g) \subseteq F_b^{\Im}(f) + F_b^{\Im}(g),$$

which proves (5).

Condition (6) is a result of the fact that

$$\Im_{tf} = \Im_f$$

for every $f \in B^{\Im}(S,Y)$ and $t \in \mathbb{R} \setminus \{0\}$. Then

$$tB_A(f) = B_A(tf)$$

and

$$\begin{aligned} tF_b^{\Im}(f) &= t\bigcap \mathcal{C}_f = t\bigcap\{B_A(f) : A \in \Im_f\} \\ &= \bigcap\{tB_A(f) : A \in \Im_f\} = \bigcap\{B_A(tf) : A \in \Im_f\} \\ &= \bigcap\{B_A(tf) : A \in \Im_{tf}\} = \bigcap \mathcal{C}_{tf} = F_b^{\Im}(tf) \end{aligned}$$

for all $f \in B^{\Im}(S,Y)$ and $t \in \mathbb{R} \setminus \{0\}$.

It remains to prove condition (8). Let $f \in B^{\Im}(S,Y)$ and $y \in S$ be fixed. From the proof of (i) we obtain

$$A \in \Im_f \text{ implies } {}_yA \in \Im_{{}_yf}$$

and, by condition (21),

$$ {}_yF(S \setminus_y A) \subseteq f(S \setminus A). $$

Therefore,

$$ B_{{}_yA}({}_yf) \subseteq B_A(f) $$

and

$$ \begin{aligned} F_b^{\Im}({}_yf) &= \bigcap\{B_{\tilde{A}}({}_yf) : \tilde{A} \in \Im_{{}_yf}\} \subseteq \bigcap\{B_{{}_yA}({}_yf) : A \in \Im_f\} \\ &\subseteq \bigcap\{B_A(f) : A \in \Im_f\} = F_b^{\Im}(f). \end{aligned} $$

The proof of the "right-hand side version" of this lemma is analogous to that presented above; thus the proof of our lemma is complete.

Consequently, the space $B^{\Im}(S, Y)$ and the map $F_b^{\Im}$ fulfil all the assumptions of Theorem 3. We will sum up these observations in the form of a theorem which will be applied in the sequel.

For $f \in B^{\Im}(S, Y)$ we will denote by $b^{\Im}(f)$ the value of $F_b^{\Im}$ at f.

Theorem 4. *Let $(S, \cdot)$ be a left (right) amenable semigroup and let $(Y, \|\cdot\|)$ be a real normed space which has the binary intersection property. Then there exists a linear operator $M^{\Im} : B^{\Im}(S, Y) \to Y$ such that:*

(22) $$ M^{\Im}(f) \in b^{\Im}(f), \quad f \in B^{\Im}(S, Y) $$

and

(23) $$ M^{\Im}({}_yf) = M^{\Im}(f) \quad (M^{\Im}(f_y) = M^{\Im}(f)), \quad f \in B^{\Im}(S, Y),\ y \in S. $$

This theorem is an abstract version of Theorem 1 from [1].

3. An application

Let $(S, \cdot)$ be a semigroup and let $(Y, \|\cdot\|)$ be a normed space. A function $f : S \to Y$ is called *additive* if and only if it satisfies Cauchy's functional equation:

(24) $$ f(xy) = f(x) + f(y) $$

for all $x, y \in S$.

The problem of stability of equation (24) (for mappings from a Banach space into another Banach space) was first considered by D. H. Hyers [19] (see also J. Rätz [25] and Z. Moszner [22]).

In 1960 P. Erdös [10] raised the following problem:

Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ satisfies relation (24) for almost all $(x, y) \in \mathbb{R}^2$ (in the sence of the planar Lebesgue measure). Does there exist an additive function $a : \mathbb{R} \to \mathbb{R}$ such that $f(x) = a(x)$ almost everywhere in $\mathbb{R}$ (in the sense of the linear Lebesgue measure)? A positive answer to this question was given, in a more general setting, by N. G. Bruijn [3] (see also W. B. Jurkat [20], J. L. Denny [7], [8], R. Ger [13], [14]).

An interesting combination of these two problems was considered for the first time by R. Ger [15] (see also J. Tabor [28]). The main result of [15] states that any function f defined on an Abelian group S with values in some linear space Y and fulfilling condition

$$f(xy) - f(x) - f(y) \in V \tag{25}$$

for almost all pairs $(x, y) \in S^2$, where the set $V \subseteq Y$ is, in a sense, small (such a function f is called *almost approximately additive*) is almost everywhere uniformly close to an additive mappings.

Now, we are going to present an application of the theory developed in the preceding chapter to a further study of almost approximately additive functions.

Our approach is motivated by the result of Z. Gajda [12] who pointed out that Ger's theorem holds true for functions transforming an amenable semigroup into a boundedly complete linear lattice or into a semi-reflexive locally convex linear topological space (see also L. Székelyhidi [26], [27]). We will show that the result of Z. Gajda can be carried over to almost approximately additive functions with values in a normed space having the binary intersection property. Moreover, we reduce slightly the assumption imposed on the set V.

Now, we assume that S is a subsemigroup of a group $(G, \cdot)$ such that:

$$G = S \cdot S^{-1}, \tag{26}$$

where $S \cdot S^{-1} = \{x \cdot y^{-1} : x, y \in S\}$.

Further assume that we are given a proper ideal $\Im$ of subsets of G with the following property:

$$y \in G \text{ and } A \in \Im \text{ imply } y \cdot A,\ A \cdot y,\ A^{-1} \in \Im, \tag{27}$$

where $y \cdot A = \{yx : x \in A\}$, $A \cdot y = \{xy : x \in A\}$ and $A^{-1} = \{x^{-1} : x \in A\}$. This condition, taking into account that the ideal $\Im$ is in a group, implies that the ideal $\Im$ is p.l.q.i. and p.r.q.i.. Proper set ideals in G satisfying (27) are known in the literature under the name "*proper linearly invariant* (in short: *p.l.i.*) *ideals*" (see R. Ger [15] and M. Kuczma [21]).

The relation between the subsemigroup S of G and the p.l.i. ideal $\Im$ is experssed in the following supplementary assumption:

$$S \notin \Im. \tag{28}$$

Then the family
$$\Im(S) = \{A \cap S : A \in \Im\}$$
is a proper ideal of subsets of S which is both left and right quasi-invariant.

Now, we are going to itroduce the notion "*almost everywhere*". Given a proper ideal $\Im$ of subsets of some non-empty set X, we say that a given condition is satisfied $\Im$-*almost everywhere in* X (written $\Im$-*a.e.* on X) if and only if there exists a set $A \in \Im$ such that the condition in question is satisfied for every $x \in X \setminus A$.

Next, given a subset N of $G \times G$ and an element $x \in G$, we put
$$N_{[x]} = \{y \in G : \ (x,y) \in N\}.$$

In accordance with the notation applied in R. Ger [15] and M. Kuczma [21], for a p.l.i. ideal $\Im$ of subsets of G the symbol $\Omega(\Im)$ will stand for the family of all sets $N \subseteq G \times G$ with the property that
$$N_{[x]} \in \Im \quad \Im - a.e. \text{ on } G. \tag{29}$$

It is not difficult to check that $\Omega(\Im)$ is a p.l.i. ideal in the product group $G \times G$. Any p.l.i. ideal in $G \times G$ consisting of sets N satisfying (29) is said to be *conjugate to* $\Im$. In this sence $\Omega(\Im)$ is the maximal p.l.i. ideal conjugate to $\Im$.

We are now in a position formulate and to prove the main result of this section. We shall formulate this result in the case corresponding to left invariant mean only. It will be quite obvious now to rephrase the result so as to obtain its right-handed version. The proof of this alternative theorem require only minor changes and, therefore, will be omitted.

Theorem 5. *Let S be a left amenable subsemigroup of a group $(G,\cdot)$ subject to condition (26) and assume that $\Im$ is a p.l.i. ideal of subsets of G fulfilling (28). Let $(Y,\|\cdot\|)$ be a real normed space having the binary intersection property. Moreover, let $f : S \to Y$ be a function such that for a certain set $N \in \Omega(\Im)$ relation:*
$$f(xy) - f(x) - f(y) = B(x,y) \tag{30}$$
holds whenever $(x,y) \in S \times S \setminus N$, where $B : S \times S \to Y$ is a map such that the functions:
$$B(x,\cdot) : S \to Y$$
belong to the space $B^{\Im(S)}(S,Y)$ for $\Im(S)$-almost all x from S. Then there exists an additive mapping $a : G \to Y$ such that
$$a(x) - f(x) \in b^{\Im(S)}(B(x,\cdot)) \quad \Im(S) - a.e. \ \text{ on } \ S. \tag{31}$$

Proof. Since $N \in \Omega(\Im)$, one can find a set $U' \in \Im$ such that
$$N_{[x]} \in \Im, \quad x \in G \setminus U'.$$

Moreover, from our assumption on the map B, it follows that there exists a set U'' in $\Im(S)$ such that the functions $B(x,\cdot) : S \to Y$ belong to $B^{\Im(S)}(S,Y)$ for each $x \in S \setminus U''$. Let $U = U' \cup U''$.

First we observe that for any fixed $x \in S \setminus U$ the function

$$S \ni y \mapsto f(xy) - f(y) \in Y \tag{32}$$

is $\Im(S)$-essentially bounded. Inded, given an element $x \in S \setminus U$ we have:

$$f(xy) - f(x) = f(x) + B(x,y), \quad y \in S \setminus (N_{[x]} \cap S). \tag{33}$$

Moreover, the set $N_{[x]} \cap S$ is evidently an element of the ideal $\Im(S)$ and map

$$S \ni y \mapsto f(x) + B(x,y) \in Y \tag{34}$$

belongs to $B^{\Im(S)}(S,Y)$ (because $x \in S \setminus U \subseteq S \setminus U''$). Therefore, function (32) is in the space $B^{\Im(S)}(S,Y)$.

Let $M^{\Im(S)}$ stand for an operator on $B^{\Im(S)}(S,Y)$ into Y which fulfils conditions (22) and (23), and whose existence results from Theorem 4. We define a function $g : S \to Y$ by the formula:

$$g(x) = \begin{cases} M_y^{\Im(S)}(f(xy) - f(y)) & \text{for all } x \in S \setminus U \\ 0 & \text{for } x \in U, \end{cases}$$

where the subscript y next to $M^{\Im(S)}$ indicates the fact that $M^{\Im(S)}$ is applied to a function of the variable y.

Now choose $u, v \in S \setminus U$ in such a manner that $uv \in S \setminus U$, too. Then, by the definition of g and the left invariance of $M^{\Im(S)}$, we get:

$$\begin{aligned} g(u) + g(v) &= M_y^{\Im(S)}(f(uy) - f(y)) + M_y^{\Im(S)}(f(vy) - f(y)) \\ &= M_y^{\Im(S)}(f(uvy) - f(vy)) + M_y^{\Im(S)}(f(vy) - f(y)) \\ &= M_y^{\Im(S)}(f(uvy) - f(y)) = g(uv). \end{aligned}$$

This means that

$$g(uv) = g(u) + g(v)$$

for all $(u,v) \in S^2 \setminus N'$, where

$$N' = (U \times G) \cup (G \times U) \cup \{(u,v) \in G \times G : uv \in U\}.$$

It is clear that $N' \in \Omega(\Im)$ and, consequently, the function g is almost additive with respect to the ideal $\Omega(\Im) \cap S \times S$. By a theorem of R. Ger (see [13], Theorem 1; see also M. Kuczma [21], Chapter XVIII, §7) there exists an additive mapping $a : G \to Y$ and a set $V \in \Im(S)$ such that

$$a(x) = g(x) \tag{35}$$

for all $x \in S \setminus V$.

Choosing again an $x \in S \setminus U$ we infer that $N_{[x]} \in \Im$ and from condition (34) we conclude that function (32) and (34) are $\Im(S)$-a.e. equal on S. Therefore, the map

$$S \ni y \mapsto f(xy) - f(y) - f(x) - B(x, y) \in Y$$

is equal to zero $\Im(S)$-a.e. on S and

$$\begin{aligned} 0 &= M_y^{\Im(S)}(f(xy) - f(y) - f(x) - B(x, y)) \\ &= M_y^{\Im(S)}(f(xy) - f(y)) - M_y^{\Im(S)}(f(x) + B)x, y)) \\ &= M_y^{\Im(S)}(f(xy) - f(y)) - f(x) - M_y^{\Im(S)}(B(x, y)). \end{aligned}$$

Consequently,

$$g(x) - f(x) = M_y^{\Im(S)}(B(x, y)) \in b^{\Im(S)}(B(x, \cdot))$$

for every $x \in S \setminus U$, which combined with (35) yields (31) and completes the proof.

References

[1] R. Badora, On an invariant mean for $\Im$-essentially bounded functions, *Ser. Mathematics and Informatics* **6** (1991), 95–106, *Facta Universitatis.*

[2] R. Badora, On some generalized invariant means and their application to the stability of Hyres–Ulam type, *Ann. Polon. Math.* **58.2** (1993), 147–159.

[3] N. G. de Brujin, On almost additive functions, *Colloq. Math.* **15** (1966), 59–63.

[4] M. M. Day, Amenable semigroups, *Illinois J. Math.* **1** (1975), 509–544.

[5] M. M. Day, Fixed point theorems for compact convex sets, *Illinois J. Math.* **5** (1961), 585–590.

[6] M. M. Day, Normed linear spaces, *Springer Verlag*, *Berline – Heidelberg – New York*, 1973.

[7] J. L. Denny, Sufficient conditions for a family of probabilities to be exponential, *Proc. Nat. Acad. Sci. U.S.A.* **57** (1967), 1184–1187.

[8] J. L. Denny, Cauchy's equation and sufficient statistics on arcwise connected spaces, *Ann. Math. Statist.* **41** (1970), 401–411.

[9] J. Dixmier, Les moyennes invariantes dans les semigroups et leur applications, *Acta Math. Sci. Szeged* **12** (1950), 213–227.

[10] P. Erdös, P 310, *Collo. Math.* **7** (1960), 311.

[11] Z. Gajda, Invariant means and representations of semigroups in the theory of functional equations, *Prace Naukowe Uniwersytetu Śląskiego 1273, Silesian University Press*, *Katowice*, 1992.

[12] Z. Gajda, Generalized invariant means and their applications to the stability of homomorphisms, (*submitted*).

[13] R. Ger, Note on almost additive functions, *Aequationes Math.* **17** (1978), 73–76.

[14] R. Ger, Almost additive functions on semigroups and a functional equation, *Publ. Math. Debrecen* **26** (1979), 219–228.

[15] R. Ger, Almost approximately additive mappings, General Inequalities 3, ISNM 64, *Birkhäuser Verlag*, *Basel – Boston –Stuttgart*, 1983, pp. 263–276.

[16] F. P. Greenleaf, Invariant means on topological groups and their applications, Van Nostrand Mathematical Studies 16, *New York – Toronto – London – Melbourne*, 1969.

[17] S. Hartman, A remark about Cauchy's equation, *Collo. Math.* **18** (1961), 77–79.

[18] E. Hewitt and K. A. Ross, Abstract harmonic analysis, vol. I, *Springer Verlag, Berlin – Göttingen – Heidelberg*, 1963.

[18] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U.S.A.* **27** (1941), 222–224.

[20] W. B. Jurkat, On Cauchy's functional equation, *Proc. Amer. Math. Soc.* **16** (1965), 683–686.

[21] M. Kuczma, An introduction to the theory of functional equations and inequalities, *Polish Scientific Publishers (PWN) and Silesian University Press, Warszawa – Kraków – Katowice*, 1985.

[22] Z. Moszner, Sur la stabilité de l'équation d'homomorphisme, *Aequationes Math.* **29** (1985), 290–306.

[23] L. Nachbin, A theorem of the Hahn – Banach type for linear transformations, *Trans. Amer. Math. Soc.* **68** (1950), 28–46.

[24] J. von Neuman, Zur allgemeinen Theorie der Masses, *Fund. Math.* **13** (1929), 73–116.

[25] J. Rätz, On approximately additive mappings, General Inequalities 2, ISNM 47, *Birkhäuser Verlag, Basel – Boston – Stuttgart*, 1980, pp. 233–251.

[26] L. Székelyhidi, Remark 17, Report of Meeting, *Aequations Math.* **29** (1985), 95–96.

[27] L. Székelyhidi, Note on Hyers' theorem, *C. R. Math. Rep. Sci. Canada* **8** (1986), 127–129.

[28] J. Tabor, Ideal stability of the Cauchy equation, Proceedings of the 23-rd International Symposium on Functional Equations, Gargnano, Italy, June 2–11. 1985, *Published by the Centre for Information Theory, University of Waterloo, Waterloo, Ontario, Canada.*

ROMAN BADORA
INSTITUTE OF MATHEMATICS
SILESIAN UNIVERSITY
BANKOWA 14
PL 40–007 KATOWICE
POLAND

(Received June 8, 1992)