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The stability of a general quadratic functional equation in distributions

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Abstract. Making use of the fundamental solution of the heat equation we reformulate and prove the stability of a general quadratic functional equation with n-independent variables in the space of tempered distributions. Moreover, using the Dirac sequence of regularizing functions we extend this result to the space of distribution.

1. Introduction

Functional equations can be solved by reducing them to differential equations. This is one of the easiest methods for solving functional equations. However, we need to assume differentiability up to a certain order of the unknown functions, which is not required in the use of direct methods. This leads to investigations on the regularity properties of functional equations. Járai [18] showed that, for certain general functional equations, measurability implies continuity and continuity implies differentiability. From this point of view, there have been several works dealing with functional equations based on the theory of distributions. Actually, using distributional operators, it was shown that some functional equations in distributions reduce to the classical ones when the solutions are locally integrable functions (see [11], [13], [20], [23]). Another approach to distributional analogue

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for functional equations is via use of the regularizing functions [6], [9]. In fact, this method gives essentially the same formulation as in [11], [13], [20], [23], but it can be applied to stability problems of the functional equations in the space of distributions (see [5], [7], [8]).

One of the interesting questions concerning the stability problems of functional equations is as follows:

> When is it true that a mapping satisfying a functional equation approximately must be close to the solution of the given functional equation?

Such an idea was suggested in 1940 by Ulam [28]. The case of approximately additive mappings was solved by Hyers [16]. In 1978, Rassias [24] generalized Hyers' result to the unbounded Cauchy difference. During the last decades stability problems of various functional equations have been extensively studied and generalized by a number of authors (see [12], [14], [17], [27]). For instance, Bae and Jun [2] investigated stability properties of the following functional equation

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{1\le i<j\le n} f(x_i - x_j) = n\sum_{i=1}^{n} f(x_i). \tag{1.1}$$

We notice that (1.1) is a generalization of the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \tag{1.2}$$

It is well-known [21] that if $f : \mathbb{R} \to \mathbb{C}$ satisfies (1.2) for all $x, y \in \mathbb{R}$ and f is bounded on some subset of $\mathbb{R}$ having positive inner Lebesgue measure, then there exists $c \in \mathbb{C}$ such that $f(x) = cx^2$ for all $x \in \mathbb{R}$. Moreover, a function f between real vector spaces satisfies (1.2) if and only if there exists a unique symmetric biadditive function B such that $f(x) = B(x, x)$ [1].

In this paper, using the notions as in [5], [7], [8] we reformulate and prove the stability of (1.1) in the space of generalized functions such as tempered distributions by virtue of the heat kernel. Also making use of the regularizing functions we extend this result to the space of distributions. Recently, using the theory of distributions Chung [5], [8] proved the stability of (1.2) in the space of distributions. As a matter of fact our approaches are based on the methods as in [5], [8]. We reformulate the stability problem of (1.1) in the space of distributions as follows:

$$\left\| u\circ A + \sum_{1\le i<j\le n} u\circ B_{ij} - n\sum_{i=1}^{n} u\circ P_i \right\| \le \epsilon, \tag{1.3}$$

where A, B_{ij} and P_i are the functions defined by

$$A(x_1, \dots, x_n) = x_1 + \cdots + x_n,$$

$$B_{ij}(x_1, \dots, x_n) = x_i - x_j, \qquad 1 \le i < j \le n,$$

$$P_i(x_1, \dots, x_n) = x_i, \qquad 1 \le i \le n.$$

Here $\circ$ denotes the pullback of generalized functions and the inequality $\|v\| \le \epsilon$ in (1.3) means that $|\langle v, \varphi\rangle| \le \epsilon\|\varphi\|_{L^1}$ for all test functions φ. We refer to [15] for pullbacks and to [3], [7], [8], [9] for more details of the space of distributions and some related stability results.

We prove as results that every solution u in distributions of the inequality (1.3) can be written uniquely in the form

$$u = q(x) + \mu(x),$$

where $q(x)$ is a quadratic function satisfying (1.1) and μ is a bounded measurable function such that $\|\mu\|_{L^\infty} \le \frac{n^2+n-4}{n^2+n-2}\epsilon$.

2. Stability in $\mathcal{S}'(\mathbb{R}^m)$

In this section we consider the stability problem of (1.1) in the space of tempered distributions. We first introduce the space of tempered distributions. Here we use the multi-index notations, $|\alpha| = \alpha_1 + \dots + \alpha_m$, $\alpha! = \alpha_1! \dots \alpha_m!$, $\zeta^\alpha = \zeta_1^{\alpha_1} \dots \zeta_m^{\alpha_m}$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_m^{\alpha_m}$, for $\zeta = (\zeta_1, \dots, \zeta_m) \in \mathbb{R}^m$, $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}_0^m$, where $\mathbb{N}_0$ is the set of non-negative integers and $\partial_j = \frac{\partial}{\partial \zeta_j}$.

Definition 2.1 ([15], [25]). We denote by $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of all infinitely differentiable functions φ in $\mathbb{R}^m$ satisfying

$$\|\varphi\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^m} |x^\alpha \partial^\beta \varphi(x)| < \infty$$

for all α, $\beta \in \mathbb{N}_0^m$, equipped with the topology defined by the seminorms $\|\cdot\|_{\alpha,\beta}$. A linear functional u on $\mathcal{S}(\mathbb{R}^m)$ is said to be tempered distribution if there exists constant $C \ge 0$ and nonnegative integer N such that

$$|\langle u, \varphi\rangle| \le C \sum_{|\alpha|, |\beta| \le N} \sup_{x \in \mathbb{R}^m} |x^\alpha \partial^\beta \varphi|$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^m)$. The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R}^m)$.

In order to prove the stability problem in the space $\mathcal{S}'(\mathbb{R}^m)$ we employ the m-dimensional heat kernel

$$E_t(x) = E(x,t) = \begin{cases} (4\pi t)^{-m/2}\exp(-|x|^2/4t), & x \in \mathbb{R}^m,\ t > 0, \\ 0, & x \in \mathbb{R}^m,\ t \le 0. \end{cases}$$

Since for each $t > 0$, $E(\cdot, t)$ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^m)$, the convolution

$$\tilde{u}(x,t) = (u * E)(x,t) = \langle u_y, E_t(x-y)\rangle, \quad x \in \mathbb{R}^m,\ t > 0$$

is well defined for all $u \in \mathcal{S}'(\mathbb{R}^m)$, which is called the Gauss transform of u. It is well-known that semigroup property of the heat kernel

$$(E_t * E_s)(x) = E_{t+s}(x)$$

holds for convolution. Semigroup property will be useful to convert inequality (1.3) into the classical functional inequality defined on upper-half plane. We also use the following famous result, so called heat kernel method [22], which states as follows:

Let $u \in \mathcal{S}'(\mathbb{R}^m)$. Then its Gauss transform $\tilde{u}$ is a C^∞-solution of the heat equation

$$(\partial/\partial t - \Delta)\tilde{u}(x,t) = 0$$

satisfying

(i) There exist positive constants C, M and N such that

$$|\tilde{u}(x,t)| \le Ct^{-M}(1+|x|)^N \text{ in } \mathbb{R}^m \times (0,\delta). \tag{2.1}$$

(ii) $\tilde{u}(x,t) \to u$ as $t \to 0^+$ in the sense that for every $\varphi \in \mathcal{S}(\mathbb{R}^m)$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+}\int \tilde{u}(x,t)\varphi(x)dx.$$

Conversely, every C^∞-solution $U(x,t)$ of the heat equation satisfying the growth condition (2.1) can be uniquely expressed as $U(x,t) = \tilde{u}(x,t)$ for some $u \in \mathcal{S}'(\mathbb{R}^m)$.

We are now going to consider the stability of (1.1) in the space of tempered distributions. Since (1.1) is a generalization of (1.2), we shall first see the stability results of (1.2). The classical stability of (1.2) was proved by Skof [26] and generalized in [4], [10], [19] as follows:

Theorem 2.2. *Let $f : G \to E$ be a mapping from a group G to a Banach space E satisfying*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \epsilon$$

for all $x, y \in G$. Then there exists a unique function $q : G \to E$ satisfying

$$q(x+y) + q(x-y) = 2q(x) + 2q(y)$$

such that

$$\|f(x) - q(x)\| \le \frac{\epsilon}{2}$$

for all $x \in G$.

Generalizing the above stability theorem to the space of tempered distributions Chung [5], [8] proved the following:

Theorem 2.3. *Let $u \in \mathcal{S}'(\mathbb{R}^m)$ satisfy the inequality*

$$\|u \circ A_1 + u \circ A_2 - 2u \circ P_1 - 2u \circ P_2\| \le \epsilon,$$

where $A_1(x, y) = x + y$, $A_2(x, y) = x - y$, $P_1(x, y) = x$ and $P_2(x, y) = y$. Then there exists a unique quadratic form

$$q(x) = \sum_{1 \le i \le j \le m} a_{ij} p_i p_j, \quad x = (p_1, \dots, p_m)$$

such that

$$\|u - q(x)\| \le \frac{\epsilon}{2}.$$

Following the notions as in [5], [6], [7], [8], [9] we convert inequality (1.3) into the classical functional inequality. Convolving the tensor product $E_{t_1}(x_1) \dots E_{t_n}(x_n)$ of the heat kernels in both sides of (1.3) and using the semigroup property of the heat kernel we have

$$\left[(u \circ A) * \left(E_{t_1}(x_1) \dots E_{t_n}(x_n)\right)\right](\xi_1, \dots, \xi_n) = \tilde{u}(\xi_1 + \dots + \xi_n, t_1 + \dots + t_n),$$

where $\tilde{u}$ is the Gauss transform of u. Similarly we get

$$\left[(u \circ B_{ij}) * \left(E_{t_1}(x_1) \dots E_{t_n}(x_n)\right)\right](\xi_1, \dots, \xi_n) = \tilde{u}(\xi_i - \xi_j, t_i + t_j), \quad 1 \le i < j \le n,$$

$$\left[(u \circ P_i) * \left(E_{t_1}(x_1) \dots E_{t_n}(x_n)\right)\right](\xi_1, \dots, \xi_n) = \tilde{u}(\xi_i, t_i), \qquad 1 \le i \le n.$$

Thus, inequality (1.3) is converted into the classical functional inequality

$$\left|\tilde{u}\left(\sum_{i=1}^n x_i, \sum_{i=1}^n t_i\right) + \sum_{1 \le i < j \le n} \tilde{u}(x_i - x_j, t_i + t_j) - n \sum_{i=1}^n \tilde{u}(x_i, t_i)\right| \le \epsilon$$

for all $x_1, \dots, x_n \in \mathbb{R}^m, t_1, \dots, t_n > 0$.

Lemma 2.4. *Suppose that $f : \mathbb{R}^m \times (0,\infty) \to \mathbb{C}$ is a continuous function satisfying*

$$f\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) + \sum_{1\le i<j\le n} f(x_i - x_j, t_i + t_j) = n\sum_{i=1}^{n} f(x_i, t_i) \quad (n \ge 2) \tag{2.2}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m$, $t_1, \ldots, t_n > 0$. Then the solution f is of the form

$$f(x,t) = \sum_{1\le i\le j\le m} a_{ij} p_i p_j + bt, \quad x = (p_1, \ldots, p_m)$$

for some $a_{ij}, b \in \mathbb{C}$.

Proof. Define $F(x,t) := f(x,t) - f(0,t)$ for all $x \in \mathbb{R}^m, t > 0$. Then F satisfies $F(0,t) = 0$ for all $t > 0$ and

$$F\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) + \sum_{1\le i<j\le n} F(x_i - x_j, t_i + t_j) = n\sum_{i=1}^{n} F(x_i, t_i) \tag{2.3}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m, t_1, \ldots, t_n > 0$. Putting $x_1 = x$ and $x_2 = \cdots = x_n = 0$ in (2.3) we have

$$F(x, t_1 + \cdots + t_n) + \sum_{i=2}^{n} F(x, t_1 + t_i) = nF(x, t_1). \tag{2.4}$$

Letting $t_3 = \cdots = t_n \to 0$ in (2.4) we get

$$F(x, t_1 + t_2) = F(x, t_1)$$

for all $x \in \mathbb{R}^m, t_1, t_2 > 0$. This shows that $F(x,t)$ is independent of $t > 0$ and we verify that $q(x) := F(x,1) = F(x,t)$ satisfies (1.1). Putting $x_3 = \ldots, x_n = 0$ in (1.1) we see that $q(x)$ satisfies (1.2). Since q is a continuous function in $\mathbb{R}^m$, we may write

$$q(x) = \sum_{1\le i\le j\le m} a_{ij} p_i p_j, \quad x = (p_1, \ldots, p_m)$$

for some $aij \in \mathbb{C}$.

On the other hand, setting $x_1 = \cdots = x_n = 0$ in (2.2) we get

$$f\left(0, \sum_{i=1}^{n} t_i\right) + \sum_{1\le i<j\le n} f(0, t_i + t_j) = n\sum_{i=1}^{n} f(0, t_i) \tag{2.5}$$

for all $t_1, \ldots, t_n > 0$. By virtue of (2.5) we see that $c := \lim_{t\to 0^+} f(0,t)$ exists. Letting $t_1 = \cdots = t_n \to 0$ in (2.5) we get $c = 0$. Taking $t_3 = \cdots = t_n \to 0$ in (2.5) we obtain

$$f(0, t_1 + t_2) = f(0, t_1) + f(0, t_2)$$

for all $t_1, t_2 > 0$. Given the continuity, we must have $f(0,t) = bt$ for some $b \in \mathbb{C}$. Therefore the solution of (2.2) is of the form

$$f(x,t) = F(x,t) + f(0,t) = q(x) + bt.$$

This completes the proof. □

As a consequence of the above lemma we state and prove the stability theorem of (1.1) in the space of tempered distributions. This is a generalization of Theorem 2.3.

Theorem 2.5. *Suppose that $u \in \mathcal{S}'(\mathbb{R}^m)$ satisfies the inequality*

$$\|u \circ A + \sum_{1\le i<j\le n} u \circ B_{ij} - n\sum_{i=1}^{n} u \circ P_i\| \le \epsilon \quad (n \ge 2). \tag{2.6}$$

Then there exists a unique quadratic form

$$q(x) = \sum_{1\le i\le j\le m} a_{ij} p_i p_j, \quad x = (p_1, \ldots, p_m)$$

such that

$$\|u - q(x)\| \le \frac{n^2+n-4}{n^2+n-2}\epsilon.$$

Proof. Convoluting with the tensor product $E_{t_1}(x_1)\ldots E_{t_n}(x_n)$ of the heat kernels in both sides of (2.6) we have the classical functional inequality

$$\left|\tilde{u}\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) + \sum_{1\le i<j\le n} \tilde{u}(x_i - x_j, t_i + t_j) - n\sum_{i=1}^{n} \tilde{u}(x_i, t_i)\right| \le \epsilon \tag{2.7}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^m, t_1, \ldots, t_n > 0$. Putting $x_1 = \cdots = x_n = 0$ in (2.7) yields

$$\left|\tilde{u}\left(0, \sum_{i=1}^{n} t_i\right) + \sum_{1\le i<j\le n} \tilde{u}(0, t_i + t_j) - n\sum_{i=1}^{n} \tilde{u}(0, t_i)\right| \le \epsilon \tag{2.8}$$

for all $t_1, \ldots, t_n > 0$. In view of (2.8) it is easy to see that

$$c := \limsup_{s\to 0^+} \tilde{u}(0,s)$$

exists. Letting $t_1 = \cdots = t_n \to 0^+$ in (2.8) we have $|c| \leq \frac{2\epsilon}{n^2+n-2}$. Setting $t_1 = t_2 = t$, $t_3 = \cdots = t_n \to 0^+$ in (2.8), and then dividing the result by 4 we obtain

$$\left|2^{-1}\tilde{u}(0,2t) - \tilde{u}(0,t)\right| \leq \frac{n^2+n-4}{2(n^2+n-2)}\epsilon.$$

Using the iterative methods gives

$$\left|2^{-k}\tilde{u}(0,2^k t) - \tilde{u}(0,t)\right| \leq \frac{n^2+n-4}{n^2+n-2}\epsilon$$

for all $k \in \mathbb{N}, t > 0$. By virtue of this inequality $h(t) := \lim_{k\to\infty} 2^{-k}\tilde{u}(0,2^k t)$ converges uniformly and is the unique function satisfying

$$h(t+s) = h(t) + h(s), \tag{2.9}$$

$$|\tilde{u}(0,t) - h(t)| \leq \frac{n^2+n-4}{n^2+n-2}\epsilon \tag{2.10}$$

for all t, $s > 0$. It follows from (2.9) and (2.10) that

$$\left|\sum_{j=1}^{k} 4^{-j}\tilde{u}(0,2^j t) - (1-2^{-k})h(t)\right| \leq \frac{n^2+n-4}{3(n^2+n-2)}\epsilon. \tag{2.11}$$

On the other hand, putting $x_1 = x_2 = x$, $x_3 = \cdots = x_n = 0$ and letting $t_1 = t_2 = t$, $t_3 = \ldots t_n \to 0^+$ in (2.8) we have

$$|\tilde{u}(2x,2t) + \tilde{u}(0,2t) - 4\tilde{u}(x,t)| \leq \frac{2(n^2+n-4)}{n^2+n-2}\epsilon.$$

Using the induction arguments yields

$$\left|\tilde{u}(x,t) - 4^{-k}\tilde{u}(2^k x, 2^k t) - \sum_{j=1}^{k} 4^{-j}\tilde{u}(0,2^j t)\right| \leq \frac{2(n^2+n-4)}{3(n^2+n-2)}\epsilon. \tag{2.12}$$

Plugging (2.11) into (2.12) and letting $F(x,t) := \tilde{u}(x,t) - h(t)$ we obtain

$$|F(x,t) - 4^{-k}F(2^k x, 2^k t)| \leq \frac{n^2+n-4}{n^2+n-2}\epsilon. \tag{2.13}$$

Now we verify that

$$g(x,t) := \lim_{k\to\infty} 4^{-k}F(2^k x, 2^k t)$$

is the unique function satisfying

$$g\left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} t_i\right) + \sum_{1\le i<j\le n} g(x_i - x_j, t_i + t_j) = n\sum_{i=1}^{n} g(x_i, t_i) \tag{2.14}$$

such that

$$|F(x,t) - g(x,t)| \le \frac{n^2+n-4}{n^2+n-2}\epsilon. \tag{2.15}$$

Let us define the function $G(x,t) := g(x,t) + h(t)$. Then, G is a continuous function satisfying (2.14). By Lemma 2.4 the function G has the form

$$G(x,t) = \sum_{1\le i\le j\le m} a_{ij}p_ip_j + bt, \quad x = (p_1,\ldots,p_n) \in \mathbb{R}^m$$

for some $a_{ij}, b \in \mathbb{C}$. Thus, it follows from (2.15) that

$$|\tilde{u}(x,t) - q(x) - bt| \le \frac{n^2+n-4}{n^2+n-2}\epsilon. \tag{2.16}$$

Letting $t \to 0^+$ in (2.16), finally we have

$$\|u - q(x)\| \le \frac{n^2+n-4}{n^2+n-2}\epsilon.$$

This completes the proof. □

Remark 2.6. The above norm inequality $\|u - q(x)\| \le \frac{n^2+n-4}{n^2+n-2}\epsilon$ implies that $u - q(x)$ belongs to $(L^1)' = L^\infty$. Thus all the solutions u of the inequality (2.6) in $\mathcal{S}'(\mathbb{R}^m)$ can be rewritten uniquely in the form

$$u = q(x) + \mu(x),$$

where μ is a bounded measurable function such that $\|\mu\|_{L^\infty} \le \frac{n^2+n-4}{n^2+n-2}\epsilon$.

3. Stability in $\mathcal{D}'(\mathbb{R}^m)$

In this section we shall extend the previous result to the space of distributions. Recall that a distribution u is a linear functional on $C_c^\infty(\mathbb{R}^m)$ of infinitely differentiable functions on $\mathbb{R}^m$ with compact supports such that for every compact set $K \subset \mathbb{R}^m$ there exist constants $C > 0$ and $N \in \mathbb{N}_0$ satisfying

$$|\langle u, \varphi\rangle| \le C \sum_{|\alpha|\le N} \sup|\partial^\alpha \varphi|$$

for all $\varphi \in C_c^\infty(\mathbb{R}^m)$ with supports contained in K. The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R}^m)$. It is well-known that the following topological inclusions hold:

$$C_c^\infty(\mathbb{R}^m) \hookrightarrow \mathcal{S}(\mathbb{R}^m), \quad \mathcal{S}'(\mathbb{R}^m) \hookrightarrow \mathcal{D}'(\mathbb{R}^m).$$

As we see in [5], [7], by virtue of the semigroup property of the heat kernel, the inequality (1.3) can be controlled easily in the space $\mathcal{S}'(\mathbb{R}^m)$. But we can not employ the heat kernel in the space $\mathcal{D}'(\mathbb{R}^m)$. Instead of the heat kernel, we use the function $\psi_t(x) := t^{-m}\psi(\frac{x}{t}),\ x \in \mathbb{R}^m,\ t > 0$, where $\psi(x) \in C_c^\infty(\mathbb{R}^m)$ such that

$$\psi(x) \geq 0, \quad \operatorname{supp}\psi(x) \subset \{x \in \mathbb{R}^m : |x| \leq 1\}, \quad \int \psi(x)dx = 1.$$

For example, let

$$\psi(x) = \begin{cases} A\exp(-(1-|x|^2)^{-1}), & |x| < 1 \\ 0, & |x| \geq 1, \end{cases}$$

where

$$A = \left(\int_{|x|<1} \exp(-(1-|x|^2)^{-1})dx\right)^{-1},$$

then it is easy to see $\psi(x)$ is an infinitely differentiable function with support $\{x : |x| \leq 1\}$. Now we employ the function $\psi_t(x) := t^{-m}\psi(x/t), t > 0$. If $u \in \mathcal{D}'(\mathbb{R}^m)$, then for each $t > 0$, $(u * \psi_t)(x) = \langle u_y, \psi_t(x-y)\rangle$ is a smooth function in $\mathbb{R}^m$ and $(u * \psi_t)(x) \to u$ as $t \to 0^+$ in the sense of distributions, that is, for every $\varphi \in C_c^\infty(\mathbb{R}^m)$

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int (u * \psi_t)(x)\varphi(x)dx.$$

Making use of the regularizing functions Chung [8] extended Theorem 2.3 to the space $\mathcal{D}'(\mathbb{R}^m)$. Similarly, we generalize Theorem 2.5 to the space of distributions. This is a main result of this paper.

Theorem 3.1. *Let $u \in \mathcal{D}'(\mathbb{R}^m)$ satisfy the inequality*

$$\|u \circ A + \sum_{1\leq i<j\leq n} u \circ B_{ij} - n\sum_{i=1}^n u \circ P_i\| \leq \epsilon \quad (n \geq 2). \tag{3.1}$$

Then there exists a unique quadratic form

$$q(x) = \sum_{1\leq i\leq j\leq m} a_{ij}p_ip_j, \quad x = (p_1, \ldots, p_m)$$

such that

$$\|u - q(x)\| \le \frac{n^2+n-4}{n^2+n-2}\epsilon.$$

Proof. In view of Theorem 2.5, it suffices to show that every distribution satisfying (3.1) belongs to the space $\mathcal{S}'(\mathbb{R}^m)$. Convoluting with $\psi_{t_1}(x_1)\dots\psi_{t_n}(x_n)$ in both sides of (3.1) we have

$$\Big|(u*\psi_{t_1}*\cdots*\psi_{t_n})(x_1+\cdots+x_n) + \sum_{1\le i<j\le n}(u*\psi_{t_i}*\psi_{t_j})(x_i-x_j) - n\sum_{i=1}^{n}(u*\psi_{t_i})(x_i)\Big| \le \epsilon \quad (3.2)$$

for all $x_1,\dots,x_n \in \mathbb{R}^m, t_1,\dots,t_n > 0$. By virtue of (3.2) it is easy to see that for each fixed x,

$$f(x) := \limsup_{s\to 0^+}(u*\psi_s)(x)$$

exists. Letting $x_1 = \cdots = x_n = 0$ and $t_1 = \cdots = t_n = s \to 0^+$ so that $(u*\psi_s)(0) \to f(0)$ in (3.2) we get

$$|f(0)| \le \frac{2\epsilon}{n^2+n-2}.$$

Setting $x_1 = x, x_2 = y, x_3 = \cdots = x_n = 0$ and $t_3 = \cdots = t_n = s \to 0^+$ so that $(u*\psi_s)(0) \to f(0)$ in (3.2) we obtain

$$|(u*\psi_{t_1}*\psi_{t_2})(x+y) + (u*\psi_{t_1}*\psi_{t_2})(x-y) - 2(u*\psi_{t_1})(x) - 2(u*\psi_{t_2})(y) - \frac{n^2+n-6}{2}f(0)| \le \epsilon.$$

Since $|f(0)| \le \frac{2\epsilon}{n^2+n-2}$, the above inequality can be rewritten as

$$|(u*\psi_{t_1}*\psi_{t_2})(x+y) + (u*\psi_{t_1}*\psi_{t_2})(x-y) - 2(u*\psi_{t_1})(x) - 2(u*\psi_{t_2})(y)| \le \frac{2(n^2+n-4)}{n^2+n-2}\epsilon. \quad (3.3)$$

Putting $y = 0$ in (3.3) and dividing the result by 2 we have

$$|(u*\psi_{t_1}*\psi_{t_2})(x) - (u*\psi_{t_1})(x) - (u*\psi_{t_2})(0)| \le \frac{n^2+n-4}{n^2+n-2}\epsilon. \quad (3.4)$$

Letting $t_1 \to 0^+$ so that $(u * \psi_{t_1})(x) \to f(x)$ in (3.4) we get

$$|(u * \psi_{t_2})(x) - f(x) - (u * \psi_{t_2})(0)| \leq \frac{n^2 + n - 4}{n^2 + n - 2}\epsilon. \tag{3.5}$$

From the inequality (3.3), (3.4) and (3.5) and the triangle inequality we obtain

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y)| \leq \frac{10(n^2 + n - 4)}{n^2 + n - 2}\epsilon$$

for all $x, y \in \mathbb{R}^m$. According to the result as in [26] there exists a unique quadratic function $q : \mathbb{R}^m \to \mathbb{C}$ satisfying

$$q(x+y) + q(x-y) = 2q(x) + 2q(y)$$

such that

$$|f(x) - q(x)| \leq \frac{5(n^2 + n - 4)}{n^2 + n - 2}\epsilon \tag{3.6}$$

for all $x \in \mathbb{R}^m$. It follows from (3.5) and (3.6) that

$$|(u * \psi_{t_2})(x) - q(x) - (u * \psi_{t_2})(0)| \leq \frac{6(n^2 + n - 4)}{n^2 + n - 2}\epsilon. \tag{3.7}$$

Letting $t_2 \to 0^+$ so that $(u * \psi_{t_2})(0) \to f(0)$ in (3.7) we have

$$||u - q(x)|| \leq \frac{2(3n^2 + 3n - 11)}{n^2 + n - 2}\epsilon. \tag{3.8}$$

By virtue of the inequality (3.8) $h(x) := u - q(x)$ belongs to $(L^1)' = L^\infty$. Thus we conclude that $u = q(x) + h(x) \in \mathcal{S}'(\mathbb{R}^m)$. This completes the proof. □

As an immediate consequence of Theorem 3.1, we have the following corollary [8].

Corollary 3.2. *Let $u \in \mathcal{D}'(\mathbb{R}^m)$ satisfy the inequality*

$$\|u \circ A_1 + u \circ A_2 - 2u \circ P_1 - 2u \circ P_2\| \leq \epsilon,$$

where $A_1(x,y) = x + y$, $A_2(x,y) = x - y$, $P_1(x,y) = x$ and $P_2(x,y) = y$. Then there exists a unique quadratic form

$$q(x) = \sum_{1 \leq i \leq j \leq m} a_{ij} p_i p_j, \quad x = (p_1, \ldots, p_m)$$

such that

$$\|u - q(x)\| \leq \frac{\epsilon}{2}.$$

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