Publ. Math. Debrecen
74/3-4 (2009), 307–320

Spectral subspaces on hypergroup algebras

By ALI REZA MEDGHALCHI (Tehran) and S. M. TABATABAIE (Tehran)

Abstract. In this paper we develop the concepts of Arveson spectrum and spectral subspaces on hypergroups and extend some of their basic properties to commutative hypergroups.

1. Introduction and notation

Let K be a commutative locally compact hypergroup. We denote by $M(K)$ the space of all bounded regular complex Borel measures on K, by $M^+(K)$ the subset of positive measures in $M(K)$, by E° the interior of $E \subseteq K$, and by δ_x the Dirac measure at the point x. Hypergroups were introduced in a series of papers by R. I. Jewett [9], C. F. Dunkl [6], and R. Spector [14] in 70's. They are in fact extensions of topological groups. Roughly speaking, a hypergroup is a locally compact space which has enough structure so that a convolution on the space of finite regular Borel measures can be defined. Therefore, the extension of Fourier analysis on hypergroups is made with more difficulties and sometimes with different proofs to that of groups. Examples include locally compact groups, double-coset hypergroups, G_H hypergroups, polynomial hypergroups, etc. We refer to [2] for more examples. In this paper, we develop the notion of Arveson spectrum and extend spectral subspaces to hypergroups. In [12] we have used this notion to give an extension of the spectral mapping theorem.

The conjugate space of a Banach space Y is denoted by Y^*. If a Banach space X is the conjugate space of a Banach space Y, we shall say that Y is the *predual* of X, and write $Y = X_*$.

Mathematics Subject Classification: 43A62, 46L99.
Key words and phrases: hypergroups, Arveson spectrum, spectral subspaces, W^*-algebras.

A C^*-algebra M is called W^*-algebra if for a Banach algebra M_*, $(M_*)^* = M$. Any W^*-algebra is unitary (with unit 1_M). The famous examples of W^*-algebras are von Neumann algebras. We can consider $\sigma(M, M_*)$ topology on M [11]. In this paper M is always a W^*-algebra. We denote by $B_\sigma(M)$ the set of all $(\sigma(M, M_*), \sigma(M, M_*))$- continuous operators on M.

Let $\sigma : M(K) \to B_\sigma(M)$ be a norm-decreasing algebra-homomorphism. For any $t \in K$ we denote $\sigma_t = \sigma(\delta_t)$. Suppose that σ has the following properties,

(1) For any $t \in K$, $\sigma_t : M \to M$ is an $*$-automorphism.

(2) For every $x \in M$ and $\rho \in M_*$, the function $t \mapsto \langle \sigma_t(x), \rho \rangle$ is continuous.

(3) $\sigma_e = I_M$, where e is the identity of K and I_M is the identity mapping on M.

For any $\mu \in M(K)$ we have

$$\langle \sigma(\mu)(x), \rho \rangle = \int_K \langle \sigma_t(x), \rho \rangle \, d\mu(t),$$

where $x \in M$ and $\rho \in M_*$. A proof of this formula is given in Section 2.

Spectral subspaces were introduced by R. Godement [7], which may be viewed as an attempt to extend the Stone theorem. A systematic study of spectral subspaces and their applications to dynamic systems was presented by W. Arveson [1]. In this paper, we define the Arveson spectrum and spectral subspaces for hypergroups and study their properties. The hypergroups we study are commutative, strong, and we also assume that $X_b(K) = \hat{K}$. We give some examples of this type in Section 2. The proofs of some results are completely different from the group case. For instance, the proof of Lemma 3.4 in group case is based on $(fg)_x = f_x g_x$, while this is not true in general for hypergroups. The Lemma 3.4 helps us to extend some significant results of Arveson spectrum, from A. Connes [4], to hypergroups. In Section 2, we will first give definition and some basic properties of commutative locally compact hypergroups.

2. Some basic properties of hypergroups

First we recall the definition and basic properties of a hypergroup. The main references are [2] and [9].

Definition 2.1. Let K be a locally compact Hausdorff space. The space K is a hypergroup if there exists a binary mapping $(x, y) \mapsto \delta_x * \delta_y$ from $K \times K$ into $M^+(K)$ satisfying the following conditions,

(1) The mapping $(\delta_x, \delta_y) \mapsto \delta_x * \delta_y$ extends to a bilinear associative operator $*$ from $M(K) \times M(K)$ into $M(K)$ such that

$$\int_K f d(\mu * \nu) = \int_K \int_K \int_K f d(\delta_x * \delta_y) d\mu(x) d\nu(y)$$

for all continuous functions f on K vanishing at infinity.

(2) For each $x, y \in K$ the measure $\delta_x * \delta_y$ is a probability measure with compact support.

(3) The mapping $(\mu, \nu) \mapsto \mu * \nu$ is continuous from $M^+(K) \times M^+(K)$ into $M^+(K)$; the topology on $M^+(K)$ being the cone topology.

(4) There exists an $e \in K$ such that $\delta_e * \delta_x = \delta_x = \delta_x * \delta_e$ for all $x \in K$.

(5) There exists a homeomorphism involution $x \mapsto x^-$ from K onto K such that, for all $x, y \in K$, we have $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$ where for $\mu \in M(K)$, μ^- is defined by $\int_K f(t) d\mu^-(t) = \int_K f(t^-) d\mu(t)$, and also, $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $y = x^-$, where $\operatorname{supp}(\delta_x * \delta_y)$ is the support of the measure $\delta_x * \delta_y$.

(6) The mapping $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $K \times K$ into the space $\mathbf{C}(K)$ of compact subsets of K is continuous, where $\mathbf{C}(K)$ is given the topology whose subbasis is given by all $\mathbf{C}_{U,V} = \{A \in \mathbf{C}(K) : A \cap U \neq \emptyset \text{ and } A \subseteq V\}$, where U, V are open subsets of K.

Note that $\delta_x * \delta_y$ is not necessarily a Dirac measure. The set $Z(K) := \{x \in K : \text{for all } y \in K,\ \operatorname{supp}(\delta_x * \delta_y) \text{ is a singleton}\}$ is called the center of K. A hypergroup K is commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all x, y in K. Each commutative hypergroup K carries a Haar measure m such that $\delta_x * m = m$ for all $x \in K$, as shown by Spector [15]. Let f, g be Borel functions on K and $\mu \in M(K)$. For any $x, y \in K$ we denote $f_x(y) = f(x * y) := \int_K f d(\delta_x * \delta_y)$. Also we define

$$(\mu * f)(x) := \int_K f(y^- * x)\, d\mu(y) \quad \text{and} \quad (f * g)(x) := \int_K f(x * y) g(y^-)\, dm(y),$$

where $x \in K$. If $x, y \in K$ and $A, B \subseteq K$ we denote $A^- = \{x^- : x \in A\}$, $\{x\} * \{y\} = \operatorname{supp}(\delta_x * \delta_y)$, and $A * B = \bigcup_{x \in A, y \in B} \{x\} * \{y\}$.

A complex continuous function ξ on K is said to be multiplicative if $\xi(x*y) = \xi(x)\xi(y)$ holds for all $x, y \in K$. The space of all multiplicative functions on K is denoted by $X_b(K)$. A nonzero multiplicative function ξ on K is called a character if $\xi(x^-) = \overline{\xi(x)}$ for all x in K. The dual $\hat{K}$ of K is the locally compact Hausdorff space of all characters with the topology of uniform convergence on compacta. In general $\hat{K}$ is not necessarily a hypergroup. A hypergroup K is called strong if its dual $\hat{K}$ is also a hypergroup with complex conjugation as involution, pointwise product as convolution, that is

$$\eta(x)\chi(x) = \int_{\hat{K}} \xi(x)\, d\delta_\eta * \delta_\chi(\xi)$$

for all $\eta, \chi \in \hat{K}$ and $x \in K$, and have the constant function 1 as the identity element. By [13], $\xi \in Z(\hat{K})$ if and only if $|\xi| = 1$.

We denote

$$L^p(K) = L^p(K, m) \quad \text{and} \quad L^p(\hat{K}) = L^p(\hat{K}, \pi),$$

where π is the Plancherel measure on $\hat{K}$ associated with m. In despite of the group case, the structure space $\Delta(L^1(K))$ of Banach algebra $L^1(K)$ does not necessarily equal to $\hat{K}$ and we only have $\Delta(L^1(K)) = X_b(K)$, while $\hat{K} \subseteq X_b(K)$. Throughout this paper, we assume that K is a commutative strong hypergroup (i.e. $\hat{K}$ is also a hypergroup) and $\hat{K} = X_b(K)$ (we refer to these conditions by notation $(\wp)$). Observe that any locally compact abelian group has these properties. Also, if G is a locally compact abelian group and H is a compact subgroup of $\mathrm{Aut}(G)$, then the space G_H containing all H-orbits is a commutative hypergroup satisfying $(\wp)$. In fact, $(G_H)\hat{} \cong (\hat{G})_H$. We refer to [13] for more details. As another example, let G be a group such that G/Z is compact, where $Z = \{x \in G : \text{for any } y \in G,\ xy = yx\}$. If K is the hypergroup containing all conjugacy classes of G, then K and its dual $\hat{K}$ satisfy $(\wp)$ [2]. On the contrary, an interesting example of Naimark given in [9] does not satisfy conditions $(\wp)$.

For any $f \in L^1(K)$ and $\mu \in M(K)$, the Fourier–Stieltjes transform $\hat{\mu}$ of μ and the Fourier transform $\hat{f}$ of f are defined by

$$\hat{\mu}(\xi) = \int_K \overline{\xi(t)} d\mu(t) \quad \text{and} \quad \hat{f}(\xi) = \int_K \overline{\xi(t)} f(t) dm(t),$$

where $\xi \in \hat{K}$. For any $f, g \in L^2(K)$, we have $\hat{f}, \hat{g} \in L^2(\hat{K})$ and $\hat{f} * \hat{g} = (fg)\hat{}$ [2].

Let M be a W^*-algebra with predual M_*. Let $x \in M$ and $\rho \in M_*$. Obviously for any measure $\upsilon \in M(K)$ with finite support, we have $\langle \sigma(\upsilon)(x), \rho \rangle = \int_K \langle \sigma_t(x), \rho \rangle\, d\upsilon(t)$. Let $\mu \in M(K)$. Since the set E containing all measures in $M(K)$ that have finite support is dense in $M(K)$, there exists a net $(\upsilon_\beta) \subseteq E$ such that $\upsilon_\beta \longrightarrow \mu$ in $M(K)$. Then by ([9], 2.2C) we have $\int_K f\, d\upsilon_\beta \longrightarrow \int_K f\, d\mu$, where $f(t) = \langle \sigma_t(x), \rho \rangle$ $(t \in K)$. On the other hand by continuity of σ, $\int_K f\, d\upsilon_\beta = \langle \sigma(\upsilon_\beta)(x), \rho \rangle \longrightarrow \langle \sigma(\mu)(x), \rho \rangle$. Then for any $\mu \in M(K)$ we have

$$\langle \sigma(\mu)(x), \rho \rangle = \int_K \langle \sigma_t(x), \rho \rangle\, d\mu(t),$$

where $x \in M$ and $\rho \in M_*$.

3. Arveson spectrum on hypergroups

If A is a commutative Banach algebra and $E \subseteq A$, the hull of E is defined by $\operatorname{hull}(E) := \{\varphi \in \Delta(A) : \text{for any } a \in E,\ \hat{a}(\varphi) = 0\}$, where $\Delta(A)$ is the structure space of A.

Definition 3.1. Let $\sigma : M(K) \to B_\sigma(M)$ be the function introduced in Section 1.

(i) The Arveson spectrum of σ is defined by $\operatorname{sp}\sigma := \operatorname{hull}(\{f \in L^1(K) : \sigma(f) = 0\})$. Trivially $\operatorname{sp}\sigma = \bigcap\{\hat{f}^{-1}(0) : f \in L^1(K) \text{ and } \sigma(f) = 0\}$.
Since for any $f \in L^1(K)$, $\hat{f} \in C_0(\hat{K})$, so $\operatorname{sp}\sigma$ is a closed subset of $\hat{K}$. Also for any $\rho \in M_*$ and $f \in L^1(K)$,

$$\langle \sigma(f)(1_M), \rho\rangle = \int_K \langle \sigma_t(1_M), \rho\rangle\, f(t)\, dm(t) = \langle \hat{f}(1) 1_M, \rho\rangle.$$

Therefore for each $f \in L^1(K)$, $\sigma(f)(1_M) = \hat{f}(1)1_M$, and so if $\sigma(f) = 0$ then $\hat{f}(1) = 0$. In other words $1 \in \operatorname{sp}\sigma$.

(ii) Let $x \in M$. We define $\operatorname{sp}_\sigma(x) := \operatorname{hull}(\{f \in L^1(K) : \sigma(f)(x) = 0\})$. Then $\operatorname{sp}_\sigma(x)$ is a closed subset of $\hat{K}$.

(iii) Let E be a closed subset of $\hat{K}$. We define the associated spectral subspace by $M(\sigma, E) := \{x \in M : \operatorname{sp}_\sigma(x) \subseteq E\}$.

The following lemma is very useful in the sequel.

Lemma 3.2. *Let $\xi \in \hat{K}$ and U be a closed neighborhood of ξ. Then there exists a function $k \in L^1(K)$ such that $0 \leq \hat{k} \leq 1$, $\hat{k}(\xi) = 1$ and $\operatorname{supp}\hat{k} \subseteq U$.*

Proof. We consider a symmetric neighborhood V of e such that $\{\xi\} * V * V \subseteq U$ and $0 < \pi(V) < \infty$ (this neighborhood exists by Definition 2.1(6)). Since $\hat{}\, : L^2(K) \to L^2(\hat{K})$ is surjective [2] and $\chi_V, \chi_{\{\xi\}*V} \in L^2(\hat{K})$, then there are $g, h \in L^2(K)$ such that $\hat{g} = \chi_{\{\xi\}*V}$ and $\hat{h} = \chi_V$. Put $k := \frac{gh}{\pi(V)}$. Then $k \in L^1(K)$ and $\hat{k} = \frac{1}{\pi(V)}(gh)\hat{} = \frac{1}{\pi(V)}\hat{g} * \hat{h}$ ([2], 2.2.23). So

$$\begin{aligned} \hat{k}(\eta) &= \frac{1}{\pi(V)}\hat{g} * \hat{h}(\eta) = \frac{1}{\pi(V)} \int_K \chi_V(\gamma)\chi_{\{\xi\}*V}(\eta * \gamma^-)\, d\pi(\gamma) \\ &= \frac{1}{\pi(V)} \int_V \int_{\{\xi\}*V} d\delta_\eta * \delta_{\gamma^-}\, d\pi(\gamma). \end{aligned}$$

If $\hat{k}(\eta) \neq 0$, then there are characters γ, λ such that $\gamma \in V$ and $\lambda \in \{\xi\}*V \cap \{\eta\} * \{\gamma^-\}$. Therefore by ([9], 4.1B), $\eta^- \in \{\lambda^-\}*\{\gamma^-\} \subseteq \{\xi^-\}*V*\{\gamma^-\} \subseteq \{\xi^-\}*V*V$, and so $\eta \in \{\xi\} * V * V \subseteq U$. This implies that $\operatorname{supp}(\hat{k}) \subseteq U$. Clearly we have $\hat{k}(\xi) = 1$. $\square$

Theorem 3.3. (i) $\mathrm{sp}_\sigma(x) = \emptyset$ *if and only if* $x = 0$.

(ii) *For any closed subset* $E \subseteq \hat{K}$, $0 \in M(\sigma, E)$.

(iii) $\mathrm{sp}_\sigma(x^*) = \mathrm{sp}_\sigma(x)^- \ (= \{\bar{\xi} : \xi \in \mathrm{sp}_\sigma(x)\})$.

(iv) *If* $t \in K$ *is such that for any* $\xi \in \hat{K}$, $\xi(t) \neq 0$, *then for any* $x \in M$, $\mathrm{sp}_\sigma(x) \subseteq \mathrm{sp}_\sigma(\sigma_t(x))$.

(v) *For any* $f \in L^1(K)$ *and* $x \in M$, $\mathrm{sp}_\sigma(\sigma(f)(x)) \subseteq \mathrm{sp}_\sigma(x) \bigcap \mathrm{supp}(\hat{f})$.

(vi) *Let* E *be a closed subset of* $\hat{K}$. *Then* $x \in M(\sigma, E)$ *if and only if for any* $f \in L^1(K)$ *with* $\hat{f} \equiv 0$ *on a neighborhood of* E, $\sigma(f)(x) = 0$.

(vii) *If* $t \in K$ *is such that for each* $\xi \in \hat{K}$, $\xi(t) \neq 0$, *then for any closed subset* $E \subseteq \hat{K}$, $M(\sigma, E) \subseteq \sigma_t(M(\sigma, E))$.

(viii) *If* $x \in M$ *and* $\mu \in M(K)$ *and* $\hat{\mu} \equiv 0$ *on a neighborhood of* $\mathrm{sp}_\sigma(x)$ *then* $\sigma(\mu)(x) = 0$. *Also if* $f \in L^1(K)$ *and* $\hat{f} \equiv 1$ *on a neighborhood of* $\mathrm{sp}_\sigma(x)$ *then* $\sigma(f)(x) = x$.

(ix) $\mathrm{sp}\,\sigma = \overline{\cup_{x \in M} \mathrm{sp}_\sigma(x)}$.

Proof. (i) Let $\xi \in \hat{K}$. There exists an $h \in C_c(\hat{K})$ such that $\xi \in \mathrm{supp}(h)$. Since the set $\{\hat{f} : f \in C_c(K)\}$ is a dense self-adjoint subalgebra of $C_0(\hat{K})$, there is a function $f \in C_c(K)$ such that $|\hat{f}| > 0$ on $\mathrm{supp}(h)$. Then $\hat{f}(\xi) \neq 0$. Obviously $\sigma(f)(0) = 0$. This implies that $\xi \notin \mathrm{sp}_\sigma(0)$, i.e. $\mathrm{sp}_\sigma(0) = \emptyset$. Conversely let $\mathrm{sp}_\sigma(x) = \emptyset$. Then $\mathrm{hull}(\{f \in L^1(K) : \sigma(f)(x) = 0\}) = \emptyset$. By abstract Tauberian Theorem [8], $\emptyset$ is a spectral synthesis subset of $\hat{K}$. Then $L^1(K) = \{f \in L^1(K) : \sigma(f)(x) = 0\}$. So for any $f \in L^1(K)$ and $\rho \in M_*$, $0 = \langle \sigma(f)(x), \rho \rangle = \int_K \langle \sigma_t(x), \rho \rangle f(t)\, dm(t)$. But the mapping $t \mapsto \langle \sigma_t(x), \rho \rangle$ is bounded and continuous ($|\langle \sigma_t(x), \rho \rangle| \leq \|\rho\| \, \|\sigma_t(x)\| = \|\rho\| \, \|x\|$). So for any $t \in K$ and $\rho \in M_*$, $\langle \sigma_t(x), \rho \rangle = 0$, i.e. $x = 0$.

(ii) is obvious by (i).

(iii) First we note that $\sigma(f)(x)^* = \sigma(\bar{f})(x^*)$, because by ([5], Proposition 1.12 page 240) for any $\rho \in M_*$,

$$\begin{aligned}\langle \sigma(\bar{f})(x^*), \rho \rangle &= \int_K \langle \sigma_t(x^*), \rho \rangle \, \bar{f}(t)\, dm(t) = \int_K \langle \sigma_t(x)^*, \rho \rangle \, \bar{f}(t)\, dm(t) \\ &= \int_K \overline{\langle \sigma_t(x), \rho \rangle} \, \bar{f}(t)\, dm(t) = \overline{\langle \sigma(f)(x), \rho \rangle} = \langle \sigma(f)(x)^*, \rho \rangle.\end{aligned}$$

Now by some calculations one can conclude that

$$\begin{aligned}\mathrm{sp}_\sigma(x^*) &= \{\xi \in \hat{K} : \text{for any } f \in L^1(K), \sigma(\bar{f})(x)^* = 0 \text{ implies } \hat{f}(\xi) = 0\} \\ &= \{\xi \in \hat{K} : \text{for any } f \in L^1(K), \sigma(f)(x) = 0 \text{ implies } \overline{\hat{f}(\bar{\xi})} = 0\} = \mathrm{sp}_\sigma(x)^-.\end{aligned}$$

(iv) For any $t \in K$ and $\xi \in \hat{K}$ we have $\hat{f}_t(\xi) = \hat{f}(\xi)\xi(t)$ because

$$\hat{f}_t(\xi) = \int_K \overline{\xi(s)} f_t(s)\, dm(s) = \int_K \overline{\xi(s)} f(t * s)\, dm(s) = \int_K \overline{\xi(t^- * s)} f(s)\, dm(s)$$
$$= \int_K \xi(t)\overline{\xi(s)} f(s)\, dm(s) = \hat{f}(\xi)\xi(t).$$

Since $\xi \in \hat{K}$ implies $\xi(t) \neq 0$, we have $\hat{f}(\xi) = 0$ if and only if $\hat{f}_t(\xi) = 0$. Now we show that $\sigma(f)(\sigma_t(x)) = \sigma(\delta_t * f)(x)$. For $x \in K$ and $\rho \in M_*$ put $h := \langle \sigma(.)(x), \rho\rangle$. Then

$$\langle \sigma(f)(\sigma_t(x)), \rho\rangle = \int_K \langle \sigma_s(\sigma_t(x)), \rho\rangle f(s)\, dm(s) = \int_K \langle \sigma(s * t)(x), \rho\rangle f(s)\, dm(s)$$
$$= \int_K h(s * t) f(s)\, dm(s) = \int_K h(s) f(s * t^-)\, dm(s)$$
$$= \int_K \langle \sigma(s)(x), \rho\rangle \delta_t * f(s)\, dm(s) = \langle \sigma(\delta_t * f)(x), \rho\rangle.$$

Therefore,

$$\mathrm{sp}_\sigma(\sigma_t(x)) = \{\xi \in \hat{K} : \text{for any } f \in L^1(K),\ \sigma(\delta_t * f)(x) = 0 \text{ implies } \hat{f}(\xi)\xi(t^-) = 0\}$$

and then it clearly shows that $\mathrm{sp}_\sigma(x) \subseteq \mathrm{sp}_\sigma(\sigma_t(x))$.

(v) Let $\xi \in \hat{K} \setminus \mathrm{supp}(\hat{f})$. Since $\mathrm{supp}(\hat{f})$ is closed then by the regularity of $L^1(K)$ [3], there exists a function $g \in L^1(K)$ such that $\hat{g}(\xi) = 1$ and $\hat{g} \equiv 0$ on $\mathrm{supp}(\hat{f})$. Then we have $(g * f)\hat{} = \hat{g}\hat{f} = 0$ and so that by injectivity of the Fourier transform, $g * f = 0$. So for any $\rho \in M_*$,

$$\langle \sigma(g)(\sigma(f)(x)), \rho\rangle = \langle \sigma(g * f)(x), \rho\rangle = \int_K \langle \sigma_t(x), \rho\rangle\, g * f(t)\, dm(t) = 0.$$

Then $\sigma(g)(\sigma(f)(x)) = 0$, while $\hat{g}(\xi) \neq 0$. So $\xi \notin \mathrm{sp}_\sigma(\sigma(f)(x))$. Therefore $\mathrm{sp}_\sigma(\sigma(f)(x)) \subseteq \mathrm{supp}(\hat{f})$. On the other hand if $\sigma(g)(x) = 0$ then $\sigma(g)(\sigma(f)(x)) = \sigma(g * f)(x) = \sigma(f * g)(x) = \sigma(f)(\sigma(g)(x)) = 0$ and so $\mathrm{sp}_\sigma(\sigma(f)(x)) \subseteq \mathrm{sp}_\sigma(x)$.

(vi) Suppose that $\hat{f} \equiv 0$ on a neighborhood of E and $x \in M(\sigma, E)$ i.e. $\mathrm{sp}_\sigma(x) \subseteq E$. Then $\hat{f}$ vanishes on a neighborhood of $\mathrm{hull}(\{g \in L^1(K) : \sigma(g)(x) = 0\})$. Now a well-known result of Shilov states that an ideal I in a regular Banach algebra A contains all elements a in A that $\hat{a}$ vanishes on a neighborhood of $\mathrm{hull}(I)$ in $\Delta(A)$ [10]. Then $f \in \{g \in L^1(K) : \sigma(g)(x) = 0\}$, that is $\sigma(f)(x) = 0$. Conversely suppose that for any $f \in L^1(K)$ with $f \equiv 0$ on a neighborhood of E we have $\sigma(f)(x) = 0$. If $x \notin M(\sigma, E)$ then we can consider an element $\xi \in \mathrm{sp}_\sigma(x) \setminus E$ and a closed neighborhood U of ξ such that $U \bigcap E = \emptyset$. By Lemma 3.2 there exists a function $k \in L^1(K)$ such that $\hat{k}(\xi) = 1$ and $\mathrm{supp}(\hat{k}) \subseteq U$. Then $\hat{k} \equiv 0$ on the neighborhood $\hat{K}\backslash U$ of E, and so that $\sigma(k)(x) = 0$. Since $\xi \in \mathrm{sp}_\sigma(x)$, $\hat{k}(\xi) = 0$, a contradiction. So $x \in M(\sigma, E)$.

(vii) Let $x \in \sigma_t^{-1}(M(\sigma, E))$. Then we have $\mathrm{sp}_\sigma(\sigma_t(x)) \subseteq E$. By (iv), $\mathrm{sp}_\sigma(x) \subseteq E$ and so $x \in M(\sigma, E)$. Since σ_t is bijective, (vii) holds.

(viii) Suppose that $\hat{\mu} \equiv 0$ on a neighborhood of $\mathrm{sp}_\sigma(x)$. For any $f \in L^1(K)$ we show that $\sigma(f)(\sigma(\mu)(x)) = 0$ and this proves the first claim in (viii). For any $f \in L^1(K)$, $f * \mu \in L^1(K)$ and $(f*\mu)\hat{} = \hat{f}\hat{\mu} = 0$ on a neighborhood of $\mathrm{sp}_\sigma(x)$. So by the regularity of $L^1(K)$ (as in the proof of (vi)), $\sigma(f)(\sigma(\mu)(x)) = \sigma(f * \mu)(x) = 0$. Now let $f \in L^1(K)$ and $\hat{f} \equiv 1$ on a neighborhood of $\mathrm{sp}_\sigma(x)$. Since $\hat{\delta_e}(\xi) = \xi(e) = 1$ $(\xi \in \hat{K})$, then $(f - \delta_e)\hat{} \equiv 0$ on a neighborhood of $\mathrm{sp}_\sigma(x)$. So $\sigma(f - \delta_e)(x) = 0$ and therefore $\sigma(f)(x) = \sigma(\delta_e)(x) = I_M(x) = x$.

(ix) It is clear that $\overline{\cup_{x\in M} \mathrm{sp}_\sigma(x)} \subseteq \mathrm{sp}\,\sigma$. Conversely if $\xi \notin \overline{\cup_{x\in M} \mathrm{sp}_\sigma(x)}$ then we choose a closed neighborhood U of ξ such that $U \cap (\overline{\cup_{x\in M} \mathrm{sp}_\sigma(x)}) = \emptyset$. By Lemma 3.2 there exists a function $k \in L^1(K)$ such that $\hat{k}(\xi) = 1$ and $\mathrm{supp}(\hat{k}) \subseteq U$. Then $\hat{k} \equiv 0$ on a neighborhood of $\overline{\cup_{x\in M} \mathrm{sp}_\sigma(x)}$. So for any $x \in M$, $\hat{k} \equiv 0$ on a neighborhood of $\mathrm{sp}_\sigma(x)$. On the other hand since always we have $x \in M(\sigma, \mathrm{sp}_\sigma(x))$, by (vi) for any $x \in M$ we have $\sigma(k)(x) = 0$, i.e. $\sigma(k) = 0$. Now because of $\hat{k}(\xi) = 1$, $\xi \notin \mathrm{sp}\,\sigma$. □

Lemma 3.4. *Let $\epsilon > 0$, $\xi \in Z(\hat{K})$, $f \in L^1(K)$ and $\hat{f}(\xi) = 0$. Then there exists a function $k \in L^1(K)$ such that $\|f * k\|_1 < \epsilon$ and $\hat{k} \equiv 1$ on a neighborhood of ξ.*

Proof. We consider positive arbitrary numbers ϵ_1, ϵ_2, ϵ_3. Since $f \in L^1(K)$, there exists a compact set $E \subseteq K$ such that $\int_{K\backslash E} |f(s)|\, dm(s) < \epsilon_1$. Then the set $W = \{\eta \in \hat{K} : |\eta(x) - \xi(x)| < \epsilon_2,$ for any $x \in E^-\}$ is an open neighborhood of ξ in $\hat{K}$. Since $\hat{K}$ is Hausdorff and locally compact, there are compact neighborhoods U_1 and V of ξ such that $V \subseteq U_1^\circ \subseteq U_1 \subseteq W$. We consider a symmetric open neighborhood U of 1 in $\hat{K}$ such that $V * U * U \subseteq U_1^\circ$, $0 < \pi(U) < \infty$, $0 < \pi(V * U) < \infty$, and $U \subseteq \{\eta \in \hat{K} : |\eta(x) - 1| < \epsilon_3,$ for any $x \in E^-\}$. Since $\hat{} : L^2(K) \to L^2(\hat{K})$ is bijective [2], there are $g, h \in L^2(K)$ such that $\hat{g} = \chi_{V*U}$, $\hat{h} = \chi_U$. If $k := \frac{gh}{\pi(U)}$, similar to the proof of Lemma 3.2, $\hat{k} \equiv 1$ on V, and also $\|k\|_1 \leq \|g\|_2\, \|h\|_2\, \pi(U)^{-1} = \pi(V * U)^{\frac{1}{2}} \pi(U)^{\frac{-1}{2}}$. For any $x \in K$ we have

$$(k * f)(x) = \int_K f(y)k(x * y^-)\, dm(y) = \int_K f(y)k(x * y^-)\, dm(y) - k(x)\hat{f}(\xi)$$
$$= \int_K f(y)[k(x * y^-) - \overline{\xi(y)}k(x)]\, dm(y).$$

So,

$$\begin{aligned}\|k*f\|_1 &\le \int_K\int_K |f(y)|\,|k(x*y^-)-\overline{\xi(y)}k(x)|\,dm(y)\,dm(x)\\ &= \int_K |f(y)|\int_K |k(x*y^-)-\overline{\xi(y)}k(x)|\,dm(x)\,dm(y)\\ &= \int_E |f(y)|\,\|k_{y^-}-\overline{\xi(y)}k\|_1\,dm(y)+\int_{K\setminus E}|f(y)|\,\|k_{y^-}-\overline{\xi(y)}k\|_1\,dm(y).\end{aligned}$$

In the sequel we compute two latter integrals. Since

$$\|k_{y^-}-\overline{\xi(y)}k\|_1 \le \|k_{y^-}\|_1+\|\overline{\xi(y)}k\|_1 \le 2\|k\|_1,$$

then

$$\int_{K\setminus E}|f(y)|\,\|k_{y^-}-\overline{\xi(y)}k\|_1\,dm(y)\le 2\|k\|_1\,\epsilon_1 \le 2\pi(V*U)^{\frac{1}{2}}\pi(U)^{\frac{-1}{2}}\,\epsilon_1.$$

For the other integral,

$$\begin{aligned}\int_E |f(y)|\,\|k_z-\overline{\xi(y)}k\|_1\,dm(y) &\le \|f\|_1\sup_{y\in E}\|k_z-\overline{\xi(y)}k\|_1\\ &= \|f\|_1\sup_{y\in E}\int_K |k(x*z)-\overline{\xi(y)}k(x)|\,dm(x),\end{aligned}$$

where $z=y^-$. But

$$\begin{aligned}\pi(U)\,[k(x*z)-\xi(z)k(x)] &= \int_K g(t)h(t)\,d\delta_x*\delta_z(t)-\xi(z)g(x)h(x)\\ &= \int_K ((h(t)-h(x))(g(t)-\xi(z)g(x))d\delta_x*\delta_z(t)\\ &\quad+\xi(z)g(x)[h_z(x)-h(x)]+h(x)[g_z(x)-\xi(z)g(x)].\end{aligned}$$

Then

$$\pi(U)\,\|k_z-\xi(z)k\|_1 \le \int_K |H_1(x)|\,dm(x)+\int_K |H_2(x)|\,dm(x)+\int_K |H_3(x)|\,dm(x),$$

where

$$H_1(x)=\int_K ((h(t)-h(x))(g(t)-\xi(z)g(x))d\delta_x*\delta_z(t),$$

$$H_2(x)=\xi(z)g(x)[h_z(x)-h(x)] \quad \text{and} \quad H_3(x)=h(x)[g_z(x)-\xi(z)g(x)].$$

Now we compute these three integrals. Since $|\xi| = 1$,

$$\begin{aligned}\int_K |H_2(x)|\,dm(x) &= \int_K |g(x)|\,|h_z(x) - h(x)|\,dm(x)\\ &= \|g(h_z - h)\|_1 \le \|g\|_2\,\|h_z - h\|_2 \le \pi(V * U)^{\frac{1}{2}}\,\pi(U)^{\frac{1}{2}}\,\epsilon_3,\end{aligned}$$

because $\|g\|_2^2 = \|\hat{g}\|_2^2 = \pi(V * U)$, and for each $z \in E^-$

$$\begin{aligned}\|h_z - h\|_2^2 = \|\hat{h_z} - \hat{h}\|_2^2 &= \int_K |\hat{h_z}(\gamma) - \hat{h}(\gamma)|^2\,d\pi(\gamma) = \int_K |\hat{h}(\gamma)|^2\,|\gamma(z) - 1|^2\,d\pi(\gamma)\\ &= \int_U |\gamma(z) - 1|^2\,d\pi(\gamma) \le \pi(U)\,\epsilon_3^2.\end{aligned}$$

Also we have

$$\begin{aligned}\int_K |H_3(x)|\,dm(x) &= \int_K |h(x)|\,|g_z(x) - \xi(z)g(x)|\,dm(x) = \|h(g_z - \xi(z)g)\|_1\\ &\le \|h\|_2\,\|g_z - \xi(z)g\|_2 \le \pi(U)^{\frac{1}{2}}\,\pi(V * U)^{\frac{1}{2}}\,\epsilon_2,\end{aligned}$$

because $\|h\|_2^2 = \|\hat{h}\|_2^2 = \pi(U)$ and for each $z \in E^-$

$$\begin{aligned}\|g_z - \xi(z)g\|_2^2 = \|\hat{g_z} - \xi(z)\hat{g}\|_2^2 &= \int_K |\hat{g_z}(\gamma) - \xi(z)\hat{g}(\gamma)|^2\,d\pi(\gamma)\\ &= \int_K |\hat{g}(\gamma)|^2\,|\gamma(z) - \xi(z)|^2\,d\pi(\gamma)\\ &= \int_{V*U} |\gamma(z) - \xi(z)|^2\,d\pi(\gamma) \le \pi(V * U)\,\epsilon_2^2.\end{aligned}$$

The computing of $\int_K |H_1(x)|\,dm(x)$ needs more intricacies, because for any functions f, g on a hypergroup we do not have in general $(fg)_x = f_x g_x$. First note that by Hölder inequality we have

$$|H_1(x)|^2 = |\int_K (h(t) - h(x))(g(t) - \xi(z)g(x))\,d\delta_x * \delta_z(t)|^2 \le B_1(x)B_2(x),$$

where

$$B_1(x) = \int_K |h(t) - h(x)|^2\,d\delta_x * \delta_z(t) \ \text{ and } \ B_2(x) = \int_K |g(t) - \xi(z)g(x)|^2\,d\delta_x * \delta_z(t).$$

So

$$\begin{aligned}\int_K |H_1(x)|\,dm(x) &\le \int_K B_1(x)^{\frac{1}{2}} B_2(x)^{\frac{1}{2}}\,dm(x)\\ &\le \Big(\int_K B_1(x)\,dm(x)\Big)^{\frac{1}{2}} \Big(\int_K B_2(x)\,dm(x)\Big)^{\frac{1}{2}}.\end{aligned}$$

But

$$B_1(x) = \int_K |h(t)|^2 + |h(x)|^2 - 2\Re(h(t)\overline{h(x)})\, d\delta_x * \delta_z(t)$$
$$= (|h^2|)_z(x) + |h(x)|^2 - 2\Re(\overline{h(x)}h_z(x))$$
$$= (|h^2|)_z(x) + |h_z(x) - h(x)|^2 - |h_z(x)|^2.$$

By ([9], 3.3B), $\|h_z\|_2 \le \|h\|_2$, so

$$\int_K [(|h^2|)_z(x) - |h_z(x)|^2]\, dm(x) \le \|h\|_2^2 - \|h_z\|_2^2 = (\|h\|_2 + \|h_z\|_2)(\|h\|_2 - \|h_z\|_2)$$
$$\le 2\|h\|_2\, \|h_z - h\|_2 \le 2\pi(U)\, \epsilon_3.$$

Then $\int_K B_1(x)\, dm(x) \le \pi(U)\, (\epsilon_3^2 + 2\, \epsilon_3)$. Similarly

$$B_2(x) = \int_K |g(t)|^2 + |\xi(z)g(x)|^2 - 2\Re(g(t)\overline{\xi(z)g(x)})\, d\delta_x * \delta_z(t)$$
$$= (|g^2|)_z(x) + |\xi(z)g(x)|^2 - 2\Re(\overline{\xi(z)g(x)}g_z(x))$$
$$= (|\xi(z)g|^2)_z(x) + |g_z(x) - \xi(z)g(x)|^2 - |g_z(x)|^2.$$

Also for each $z \in E^-$ we have

$$\int_K |\xi(z)g(x) - g_z(x)|^2\, dm(x) = \|\xi(z)g - g_z\|_2^2 = \|\xi(z)\hat{g} - \hat{g_z}\|_2^2$$
$$= \int_K |\hat{g}(\gamma)|^2\, |\xi(z) - \gamma(z)|^2\, d\pi(\gamma) = \int_{V*U} |\xi(z) - \gamma(z)|^2\, d\pi(\gamma) \le \pi(V * U)\, \epsilon_2^2,$$

and

$$\int_K (|\xi(z)g|^2)_z(x) - |g_z(x)|^2\, dm(x) \le \|\xi(z)g\|_2^2 - \|g_z\|_2^2$$
$$\le (\|\xi(z)g\|_2 + \|g_z\|_2)(\|\xi(z)g - g_z\|_2) \le 2\|g\|_2\, \|\xi(z)g - g_z\|_2 \le 2\pi(V * U)\, \epsilon_2.$$

Then $\int_K B_2(x)\, dm(x) \le \pi(V * U)\, \epsilon_2^2 + 2\pi(V * U)\, \epsilon_2$. This completes the proof. □

Lemma 3.5. *Let $\xi \in Z(\hat{K})$ and $\epsilon > 0$. For every compact set $E \subseteq K$ there exists a compact neighborhood V of ξ in $\hat{K}$ such that for any $s \in E$ and $x \in M(\sigma, V)$,*

$$\|\sigma_s(x) - \overline{\xi(s)}x\| < \epsilon\, \|x\|.$$

Proof. Since $\hat{K}$ is locally compact, we can consider compact neighborhoods W_0 and W_1 of ξ such that $W_1 \subseteq W_0^\circ \subseteq W_0$. By (a similar argument to) Lemma 3.2

there exists a function $f \in L^1(K)$ such that $\hat{f} \equiv 1$ on W_1 and $\operatorname{supp} \hat{f} \subseteq W_0^\circ$. For any $s \in K$, we put $F_s := \delta_s * f - \overline{\xi(s)} f$. It is clear that $F_s \in L^1(K)$, and

$$\hat{F_s}(\xi) = (\delta_s * f)\hat{}(\xi) - \overline{\xi(s)}\, \hat{f}(\xi) = \hat{f}(\xi)\, \xi(s^-) - \xi(s^-)\, \hat{f}(\xi) = 0.$$

Then by Lemma 3.4 for any $\delta > 0$ and any $s \in K$ there exist a function $h_s \in L^1(K)$ and a compact neighborhood W_s of ξ such that $\|h_s * F_s\|_1 < \delta$ and $\hat{h}_s \equiv 1$ on W_s. The mapping $s \mapsto F_s$ from K into $L^1(K)$ is continuous. Then for any $s \in K$ and $\alpha > 0$ there exists a neighborhood U_s of s in K such that for any $t \in U_s$, $\|F_s - F_t\|_1 < \alpha$. By compactness of E, there are $s_1, \ldots, s_n \in E$ such that $E \subseteq U_{s_1} \cup \cdots \cup U_{s_n}$. We put $W_2 = W_{s_1} \cap \cdots \cap W_{s_n}$. Clearly W_2 is a neighborhood of ξ and for any $s \in E$ there exists a $j \in \{1, 2, \ldots, n\}$ such that $\|F_s - F_{s_j}\| < \alpha$. Also since the set $\{\|h_s\|_1 : s \in K\}$ is bounded (say by a bound M),

$$\|h_{s_j} * F_s\|_1 \leq \|h_{s_j} * F_{s_j}\|_1 + \|h_{s_j} * F_s - h_{s_j} * F_{s_j}\|_1 < \delta + M\alpha.$$

It is clear that $\hat{h}_{s_j} \equiv 1$ on W_2. We can consider δ, α so small that $\delta + M\alpha < \epsilon$. Thus for any $s \in E$ there exists a function $k \in L^1(K)$ such that $\|F_s * k\|_1 < \epsilon$ and $\hat{k} \equiv 1$ on W_2. Now we put $W = W_1 \cap W_2$ and consider a compact neighborhood V of ξ such that $V \subseteq W^\circ \subseteq W$. Let $s \in E$ and $x \in M(\sigma, V)$. So $\operatorname{sp}_\sigma(x) \subseteq W^\circ$. Since $\hat{k}, \hat{f} \equiv 1$ on W°, we have $(f * k)\hat{} \equiv 1$ on a neighborhood of $\operatorname{sp}_\sigma(x)$. Then by Theorem 3.3(viii) $\sigma(f * k)(x) = x$. So

$$\begin{aligned}
\|\sigma_s(x) - \overline{\xi(s)} x\| &= \|\sigma_s(\sigma(f * k)(x)) - \overline{\xi(s)} \sigma(f * k)(x)\| \\
&= \|\sigma(\delta_s * f * k)(x) - \sigma(\xi(s^-) f * k)(x)\| \\
&= \|\sigma([\delta_s * f - \overline{\xi(s)} f] * k)(x)\| \\
&= \|\sigma(F_s * k)(x)\| \leq \|F_s * k\|_1\, \|x\| < \epsilon\, \|x\|.
\end{aligned}$$

□

Summing up, we have the following characterization of $\operatorname{sp} \sigma$.

Theorem 3.6. *Let ξ be in the center of $\hat{K}$. The followings are equivalent.*

(i) $\xi \in \operatorname{sp} \sigma$.

(ii) *For any closed neighborhood V of ξ in $\hat{K}$, $M(\sigma, V) \neq \{0\}$.*

(iii) *There is a net (x_ι) in M such that for any ι, $\|x_\iota\| = 1$ and $\lim_\iota \|\sigma_s(x_\iota) - \xi(s) x_\iota\| = 0$, uniformly on compacta.*

(iv) *For any $f \in L^1(K)$, $|\hat{f}(\xi)| \leq \|\sigma(f)\|$.*

Proof. (i)⇒(ii) Let $\xi \in \operatorname{sp}\sigma$. If for a closed neighborhood V of ξ we have $M(\sigma, V) = \{0\}$ then by Lemma 3.2 there exists a function $f \in L^1(K)$ such that $\operatorname{supp}\hat{f} \subseteq V$ and $\hat{f}(\xi) = 1$. Since $\operatorname{sp}_\sigma(\sigma(f)(x)) \subseteq \operatorname{supp}\hat{f} \subseteq V$, $\sigma(f)(x) \in M(\sigma, V)$. Then $\sigma(f) = 0$ and $\hat{f}(\xi) \neq 0$. This implies that $\xi \notin \operatorname{sp}\sigma$, a contradiction.

(ii)⇒(iii) For any closed neighborhood V of ξ we consider a non-zero element $t_\iota \in M(\sigma, V)$ and put $x_\iota = \frac{t_\iota}{\|t_\iota\|}$. By Lemma 3.5 the net (x_ι) satisfies (iii).

(iii)⇒(iv) Let (x_ι) be the net in (iii). Then for any $f \in C_c(K)$ and x_ι,

$$\begin{aligned}\|\sigma(f)\| \geq \|\sigma(f)(x_\iota)\| &= \|\int_K \sigma_s(x_\iota) f(s)\, dm(s)\| \\ &\geq |\int_K \xi(s) f(s)\, dm(s)|\, \|x_\iota\| - \int_K \|\sigma_s(x_\iota) - \xi(s)x_\iota\|\, |f(s)|\, dm(s).\end{aligned}$$

Then by limiting and using the compactness of support of f we have $\|\sigma(f)\| \geq |\hat{f}(\xi)|$. Now (iv) holds by density of $C_c(K)$ in $L^1(K)$.

(iv)⇒(i) is obvious by definition of Arveson spectrum. □

References

[1] W. B. Arveson, On groups of automorphisms of operator algebras, *J. Funct. Anal.* **15** (1974), 217–243.

[2] W. R. Bloom and H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, *De Gruyter*, *Berlin*, 1995.

[3] A. K. Chilana and K. A. Ross, Spectral synthesis in hypergroups, *Pac. J. Math.* **76** (1978), 313–328.

[4] A. Connes, Une classification des facteurs de type (III), *Ann. Sci. Ecole Norm.* Sup. (4) **6** (1973), 133–252.

[5] J. B. Conway, A Course in Functional Analysis, *Springer-Verlag*, *New York*, 1985.

[6] C. F. Dunkl, The measure algebra of a locally compact hypergroup, *Trans. Amer. Math. Soc.* **179** (1973), 331–348.

[7] R. Godement, Theoremes tauberiens et theorie spectrale, *Ann. Sci. Ecole Norm.* Sup. (3) **63** (1947), 119–138.

[8] E. Hewitt and K. A. Ross, Abstract harmonic analysis I, II, *Springer-Verlag*, *New York, Heidelberg, Berlin*, 1963, 1970.

[9] R. I. Jewett, Spaces with an abstract convolution of measures, *Advan. in Math.* **18** (1975), 1–101.

[10] Y. Katznelson, An Introduction to Harmonic Analysis, *Dover Publ.*, *New York*, 1976.

[11] B. R. Li, Introduction to Operator Algebras, *World Scientific*, *Hong Kong, London, New Jersey, Singapore*, 1992.

[12] A. R. Medghalchi and S. M. Tabatabaie, An extension of the spectral mapping theorem, *IJMMS* (*in print*).

[13] K. A. Ross, Centers of hypergroups, *Trans. Amer. Math. Soc.* **243** (1978), 251–269.

[14] R. Spector, Apercu de la theorie des hypergroups, In: Analyse Harmonique sur les Groupes de Lie, Vol. 497, Lect. Notes in Math., *Springer*, 1975, 643–673.

[15] R. Spector, Measures invariantes sur les hypergroups, *Trans. Amer. Math. Soc.* **239** (1978), 147–165.

ALI REZA MEDGHALCHI
FACULTY OF MATHEMATICS AND COMPUTER SCIENCES
TARBIAT MOALLEM UNIVERSITY
TEHRAN, P.O.B. 15618
IRAN

E-mail: a_medghalchi@saba.tmu.ac.ir

S. M. TABATABAIE
FACULTY OF MATHEMATICS AND COMPUTER SCIENCES
TARBIAT MOALLEM UNIVERSITY
TEHRAN, P.O.B. 15618
IRAN
AND
DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF QOM
QOM, P.O.B.3716146611
IRAN

E-mail: sm-tabatabaei@qom.ac.ir

(Received July 4, 2008, revised December 7, 2008)