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Dependence of the Gauss–Codazzi equations and the Ricci equation of Lorentz surfaces

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Abstract. The fundamental equations of Gauss, Codazzi and Ricci provide the conditions for local isometric embeddability. In general, the three fundamental equations are independent for surfaces in Riemannian 4-manifolds. In contrast, we prove in this article that for arbitrary Lorentz surfaces in Lorentzian Kaehler surfaces the equation of Ricci is a consequence of the equations of Gauss and Codazzi.

1. Introduction

Let $\tilde{M}^n$ be a complex n-dimensional indefinite Kaehler manifold, that means $\tilde{M}^n$ is endowed with an almost complex structure J and with an indefinite Riemannian metric $\tilde{g}$, which is J-Hermitian, i.e., for all $p \in \tilde{M}^n$, we have

$$\tilde{g}(JX, JY) = \tilde{g}(X, Y), \quad \forall X, Y \in T_p M^n, \tag{1.1}$$

$$\tilde{\nabla} J = 0, \tag{1.2}$$

where $\tilde{\nabla}$ is the Levi–Civita connection of $\tilde{g}$. It follows that J is integrable.

The complex index of $\tilde{M}^n$ is defined as the complex dimension of the largest complex negative definite subspace of the tangent space. When the complex index is one, we denote the indefinite Kaehler manifold by $\tilde{M}^n_1$, which is called a *Lorentzian Kaehler manifold* (cf. [1]).

The curvature tensor $\tilde{R}$ of an indefinite Kaehler manifold $\tilde{M}^n$ satisfies

$$\tilde{R}(X, Y; Z, W) = -\tilde{R}(Y, X; Z, W), \tag{1.3}$$

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$$\tilde{R}(X,Y;Z,W) = \tilde{R}(Z,W;X,Y), \tag{1.4}$$

$$\tilde{R}(X,Y;JZ,W) = -\tilde{R}(X,Y;Z,JW), \tag{1.5}$$

where $\tilde{R}(X,Y;Z,W) = \tilde{g}(\tilde{R}(X,Y)Z,W)$.

It is well-known that the three fundamental equations of Gauss, Codazzi and Ricci play fundamental roles in the theory of submanifolds. For surfaces in Riemannian 4-manifolds, the three equations of Gauss, Codazzi and Ricci are independent in general.

On the other hand, we prove in this article a fundamental result for Lorentz surfaces; namely, *for any Lorentz surface in any Lorentzian Kaehler surface the equation of Ricci is a consequence of the equations of Gauss and Codazzi.*

2. Basic formulas and fundamental equations

Let M_1^2 be a Lorentz surface in a Lorentzian Kaehler surface $\tilde{M}_1^2$ with an almost complex structure J and Lorentzian Kaehler metric $\tilde{g}$. Let g denote the induced metric on M_1^2. Denote by ∇ and $\tilde{\nabla}$ the Levi–Civita connection on g and $\tilde{g}$, respectively; and by R the curvature tensor of M.

The formulas of Gauss and Weingarten are given respectively by (cf. [2], [9])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for vector fields X, Y tangent to M_1^2 and ξ normal to M, where h, A and D are the second fundamental form, the shape operator and the normal connection.

For a normal vector ξ of M_1^2 at $x \in M_1^2$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_x M_1^2$. The shape operator and the second fundamental form are related by

$$\tilde{g}(h(X,Y),\xi) = g(A_\xi X, Y) \tag{2.3}$$

for X, Y tangent to M_1^2.

The three fundamental equations of Gauss, Codazzi and Ricci are given by

$$R(X,Y;Z,W) = \tilde{R}(X,Y;Z,W) + \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle, \tag{2.4}$$

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z), \tag{2.5}$$

$$\tilde{g}(R^D(X,Y)\xi,\eta) = \tilde{R}(X,Y;\xi,\eta) + g([A_\xi, A_\eta]X,Y), \tag{2.6}$$

where X, Y, Z, W are vector tangent to M_1^2, and $\bar{\nabla}h$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.7}$$

The following lemma is an easy consequence of a result of [7].

Lemma 2.1. *Locally there exists a coordinate system $\{x,y\}$ on a Lorenz surface M_1^2 such that the metric tensor is given by*

$$g = -m^2(x,y)^2(dx \otimes dy + dy \otimes dx) \tag{2.8}$$

for some positive function $m(x,y)$.

Proof. It is known that locally there exist isothermal coordinates (u,v) on a Lorentz surface M_1^2 such that the metric tensor takes the form:

$$g = E(u,v)^2(-du \otimes du + dv \otimes dv) \tag{2.9}$$

for some positive function E (see [7] (see, also [5]). Thus, after putting

$$x = u + v, \quad y = u - v,$$

we obtain (2.8) from (2.9) with $m(x,y) = E(x,y)/\sqrt{2}$. □

3. Main theorem

The main purpose of this article is to prove the following fundamental result for Lorentz surfaces.

Theorem 3.1. *The equation of Ricci is a consequence of the equations of Gauss and Codazzi for any Lorentz surface in any Lorentzian Kaehler surface.*

Proof. Assume that $\phi : M_1^2 \to \tilde{M}_1^2$ is an isometric immersion of a Lorentz surface M_1^2 into a Lorentzian Kaehler surface $\tilde{M}_1^2$. According to Lemma 2.1, we may assume that locally M_1^2 is equipped with the following Lorentzian metric:

$$g = -m^2(x,y)(dx \otimes dy + dy \otimes dx) \tag{3.1}$$

for some positive function m. The Levi–Civita connection of g satisfies

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \frac{2m_x}{m}\frac{\partial}{\partial x}, \quad \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0, \quad \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{2m_y}{m}\frac{\partial}{\partial y} \tag{3.2}$$

and the Gaussian curvature K is given by

$$K = \frac{2mm_{xy} - 2m_x m_y}{m^4}. \tag{3.3}$$

If we put

$$e_1 = \frac{1}{m}\frac{\partial}{\partial x}, \quad e_2 = \frac{1}{m}\frac{\partial}{\partial y}, \tag{3.4}$$

then $\{e_1, e_2\}$ is a pseudo-orthonormal frame satisfying

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0, \quad \langle e_1, e_2\rangle = -1. \tag{3.5}$$

From (3.2) and (3.4) we find

$$\begin{aligned} \nabla_{e_1} e_1 &= \frac{m_x}{m^2} e_1, & \nabla_{e_2} e_1 &= -\frac{m_y}{m^2} e_1, \\ \nabla_{e_1} e_2 &= -\frac{m_x}{m^2} e_2, & \nabla_{e_2} e_2 &= \frac{m_y}{m^2} e_2. \end{aligned} \tag{3.6}$$

For each tangent vector X of M_1^2, we put

$$JX = PX + FX, \tag{3.7}$$

where PX and FX are the tangential and the normal components of JX. For the pseudo-orthonormal frame $\{e_1, e_2\}$ defined by (3.4), it follows from (1.1), (3.5), and (3.7) that

$$Pe_1 = (\sinh\alpha)e_1, \quad Pe_2 = -(\sinh\alpha)e_2 \tag{3.8}$$

for some function α. We call this function α the *Wirtinger angle*.

If we put

$$e_3 = (\operatorname{sech}\alpha)Fe_1, \quad e_4 = (\operatorname{sech}\alpha)Fe_2, \tag{3.9}$$

then we may derive from (3.7)-(3.9) that

$$Je_1 = \sinh\alpha e_1 + \cosh\alpha e_3, \qquad Je_2 = -\sinh\alpha e_2 + \cosh\alpha e_4, \tag{3.10}$$

$$Je_3 = -\cosh\alpha e_1 - \sinh\alpha e_3, \quad Je_4 = -\cosh\alpha e_2 + \sinh\alpha e_4, \tag{3.11}$$

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 0, \quad \langle e_3, e_4\rangle = -1. \tag{3.12}$$

We call such a frame $\{e_1, e_2, e_3, e_4\}$ an *adapted pseudo-orthonormal frame* for M_1^2.

Let us put $\nabla_X e_j = \sum_{k=1}^2 \omega_j^k(X)e_k; j, k = 1, 2$. Then we deduce from (3.5) that

$$\nabla_X e_1 = \omega(X)e_1, \quad \nabla_X e_2 = -\omega(X)e_2, \ \ \omega = \omega_1^1. \tag{3.13}$$

Similarly, if we put $D_X e_r = \omega_r^s(X)e_s; r, s = 3, 4$, then (3.12) yields

$$D_X e_3 = \Phi(X)e_3, \quad D_X e_4 = -\Phi(X)e_4, \quad \Phi = \omega_3^3. \tag{3.14}$$

For the second fundamental form h, we put $h(e_i, e_j) = h^3_{ij} e_3 + h^4_{ij} e_4$. Then, by applying $\tilde{\nabla}_X(JY) = J\tilde{\nabla}_X Y$, (3.10)-(3.14), we may obtain the following:

$$A_{e_3} e_j = h^4_{j2} e_1 + h^4_{1j} e_2, \quad A_{e_4} e_j = h^3_{j2} e_1 + h^3_{1j} e_2, \tag{3.15}$$

$$e_j \alpha = (\omega_j - \Phi_j) \coth \alpha - 2h^3_{1j}, \tag{3.16}$$

$$e_1 \alpha = h^4_{12} - h^3_{11}, \ \ e_2 \alpha = h^4_{22} - h^3_{12}, \tag{3.17}$$

$$\omega_j - \Phi_j = (h^3_{1j} + h^4_{j2}) \tanh \alpha, \tag{3.18}$$

where $\omega_j = \omega(e_j)$ and $\Phi_j = \Phi(e_j)$ for $j = 1, 2$.

For simplicity, let us put

$$h(e_1, e_1) = \beta e_3 + \gamma e_4, \quad h(e_1, e_2) = \delta e_3 + \varphi e_4, \quad h(e_2, e_2) = \lambda e_3 + \mu e_4. \tag{3.19}$$

In view of (3.12), and (3.19), equation (2.4) of Gauss can be expressed as

$$\gamma\lambda + \beta\mu - 2\delta\varphi = \frac{2(m m_{xy} - m_x m_y)}{m^4} - \tilde{K}, \tag{3.20}$$

where $\tilde{K} = -\tilde{R}(e_1, e_2; e_2, e_1)$ is the sectional curvature of the ambient space $\tilde{M}^2_1$ with respect to the 2-plane spanned by e_1, e_2.

By using (3.6), (3.14), and (3.18) we find

$$\begin{aligned}
D_{e_1} e_3 &= \Big(\frac{m_x}{m^2} - (\beta + \varphi) \tanh \alpha\Big) e_3, \\
D_{e_2} e_3 &= -\Big(\frac{m_y}{m^2} + (\delta + \mu) \tanh \alpha\Big) e_3, \\
D_{e_1} e_4 &= \Big((\beta + \varphi) \tanh \alpha - \frac{m_x}{m^2}\Big) e_4, \\
D_{e_2} e_4 &= \Big(\frac{m_y}{m^2} + (\delta + \mu) \tanh \alpha\Big) e_4.
\end{aligned} \tag{3.21}$$

So, it follows from (3.6), (3.19) and (3.21) that

$$\begin{aligned}
(\bar{\nabla}_{e_1} h)(e_1, e_1) &= \bigg(\frac{\beta_x}{m} - \frac{\beta m_x}{m^2} - \beta(\beta + \varphi) \tanh \alpha\bigg) e_3 \\
&\quad + \bigg(\frac{\gamma_x}{m} - \frac{3\gamma m_x}{m^2} + \gamma(\beta + \varphi) \tanh \alpha\bigg) e_4, \\
(\bar{\nabla}_{e_1} h)(e_1, e_2) &= \bigg(\frac{\delta_x}{m} + \frac{\delta m_x}{m^2} - \delta(\beta + \varphi) \tanh \alpha\bigg) e_3 \\
&\quad + \Big(\frac{\varphi_x}{m} - \frac{\varphi m_x}{m^2} + \varphi(\beta + \varphi) \tanh \alpha\Big) e_4,
\end{aligned}$$

$$(\bar{\nabla}_{e_2}h)(e_1,e_1)=\left(\frac{\beta_y}{m}+\frac{\beta m_y}{m^2}-\beta(\delta+\mu)\tanh\alpha\right)e_3$$
$$+\left(\frac{\gamma_y}{m}+\frac{3\gamma m_y}{m^2}+\gamma(\delta+\mu)\tanh\alpha\right)e_4,$$
$$(\bar{\nabla}_{e_1}h)(e_2,e_2)=\left(\frac{\lambda_x}{m}+\frac{3\lambda m_x}{m^2}-\lambda(\beta+\varphi)\tanh\alpha\right)e_3$$
$$+\left(\frac{\mu_x}{m}+\frac{\mu m_x}{m^2}+\mu(\beta+\varphi)\tanh\alpha\right)e_4,$$
$$(\bar{\nabla}_{e_2}h)(e_1,e_2)=\left(\frac{\delta_y}{m}-\frac{\delta m_y}{m^2}-\delta(\delta+\mu)\tanh\alpha\right)e_3$$
$$+\left(\frac{\varphi_y}{m}+\frac{\varphi m_y}{m^2}+\varphi(\delta+\mu)\tanh\alpha\right)e_4,$$
$$(\bar{\nabla}_{e_2}h)(e_2,e_2)=\left(\frac{\lambda_y}{m}-\frac{3\lambda m_y}{m^2}-\lambda(\delta+\mu)\tanh\alpha\right)e_3$$
$$+\left(\frac{\mu_y}{m}-\frac{\mu m_y}{m^2}+\mu(\delta+\mu)\tanh\alpha\right)e_4. \tag{3.22}$$

On the other hand, from (3.10) we also find

$$(\tilde{R}(e_1,e_2)e_2)^{\perp}=-\operatorname{sech}\alpha\tilde{R}(e_1,e_2;e_2,Je_2)e_3$$
$$-\{\tanh\alpha\tilde{K}+\operatorname{sech}\alpha\tilde{R}(e_1,e_2;e_2,Je_1)\}e_4,$$
$$(\tilde{R}(e_2,e_1)e_1)^{\perp}=\{\tanh\alpha\tilde{K}-\operatorname{sech}\alpha\tilde{R}(e_2,e_1;e_1,Je_2)\}e_3$$
$$-\operatorname{sech}\alpha\tilde{R}(e_2,e_1;e_1,Je_1)e_4. \tag{3.23}$$

By applying (3.4), (3.12), (3.22), (3.23), and the equation of Codazzi we get

$$\lambda_x-\delta_y=(\lambda\beta+\lambda\varphi-\delta^2-\delta\mu)m\tanh\alpha-\frac{\delta m_y+3\lambda m_x}{m}$$
$$-m\operatorname{sech}\alpha\tilde{R}(e_1,e_2;e_2,Je_2),$$
$$\mu_x-\varphi_y=(\delta\varphi-\beta\mu)m\tanh\alpha+\frac{\varphi m_y-\mu m_x}{m}$$
$$-m\operatorname{sech}\alpha\tilde{R}(e_1,e_2;e_2,Je_1)-m(\tanh\alpha)\tilde{K},$$
$$\beta_y-\delta_x=(\beta\mu-\delta\varphi)m\tanh\alpha+\frac{\delta m_x-\beta m_y}{m}$$
$$-m\operatorname{sech}\alpha\tilde{R}(e_2,e_1;e_1,Je_2)+m(\tanh\alpha)\tilde{K},$$
$$\gamma_y-\varphi_x=(\beta\varphi+\varphi^2-\delta\gamma-\gamma\mu)m\tanh\alpha-\frac{\varphi m_x+3\gamma m_y}{m}$$
$$-m\operatorname{sech}\alpha\tilde{R}(e_2,e_1;e_1,Je_1). \tag{3.24}$$

Also, from (3.4), (3.5), (3.15), (3.17) and (3.19) we have

$$A_{e_3} = \begin{pmatrix} \varphi & \mu \\ \gamma & \varphi \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} \delta & \lambda \\ \beta & \delta \end{pmatrix}, \tag{3.25}$$

$$\alpha_x = m(\varphi - \beta), \qquad \alpha_y = m(\mu - \delta). \tag{3.26}$$

By applying (3.10), (3.11) and (3.25) we derive that

$$\begin{aligned}\tilde{R}(e_1, e_2; e_3, e_4) = {} & (\operatorname{sech}^2 \alpha - \tanh^2 \alpha)\tilde{K} \\ & - 2 \operatorname{sech} \alpha \tanh \alpha \tilde{R}(e_1, e_2; e_2, Je_1),\end{aligned} \tag{3.27}$$

$$\langle [A_{e_3}, A_{e_4}] e_1, e_2 \rangle = \gamma\lambda - \beta\mu. \tag{3.28}$$

From (3.6), (3.21), and (3.28), we find

$$\begin{aligned}\tilde{g}(R^D(e_1, e_2)e_3, e_4) = {} & \frac{2mm_{xy} - 2m_x m_y}{m^4} + \{(\delta + \mu)\alpha_x - (\beta + \varphi)\alpha_y\} \frac{\operatorname{sech}^2 \alpha}{m} \\ & + \{(\delta + \mu)m_x - (\beta + \varphi)m_y + m(\delta_x + \mu_x - \beta_y - \varphi_y)\} \frac{\tanh \alpha}{m^2}.\end{aligned} \tag{3.29}$$

Therefore, the equation of Ricci is given by

$$\begin{aligned} & \frac{2mm_{xy} - 2m_x m_y}{m^4} + \{(\delta + \mu)\alpha_x - (\beta + \varphi)\alpha_y\} \frac{\operatorname{sech}^2 \alpha}{m} \\ & \quad + \{(\delta + \mu)m_x - (\beta + \varphi)m_y + m(\delta_x + \mu_x - \beta_y - \varphi_y)\} \frac{\tanh \alpha}{m^2} \\ & = \gamma\lambda - \beta\mu + (\operatorname{sech}^2 \alpha - \tanh^2 \alpha)\tilde{K} - 2 \operatorname{sech} \alpha \tanh \alpha \tilde{R}(e_1, e_2; e_2, Je_1).\end{aligned} \tag{3.30}$$

On the other hand, using (3.4) and (3.17) we find

$$(\delta + \mu)\alpha_x - (\beta + \varphi)\alpha_y = 2m(\delta\varphi - \beta\mu). \tag{3.31}$$

Also, by applying (3.24), we get

$$\begin{aligned} & (\delta + \mu)m_x - (\beta + \varphi)m_y + m(\delta_x + \mu_x - \beta_y - \varphi_y) \\ & \quad = 2(\delta\varphi - \beta\mu)m^2 \tanh \alpha - 2m^2 \tanh \alpha \tilde{K} \\ & \qquad + m^2 \operatorname{sech} \alpha \big\{ R(e_2, e_1; e_1, Je_2) - \tilde{R}(e_1, e_2; e_2, Je_1) \big\}.\end{aligned} \tag{3.32}$$

Substituting (3.31) and (3.32) into equation (3.30) gives

$$\begin{aligned}\gamma\lambda + \beta\mu - 2\delta\varphi = {} & \frac{2mm_{xy} - 2m_x m_y}{m^4} - \tilde{K} \\ & - \tanh \alpha \operatorname{sech} \alpha \big\{ \tilde{R}(e_2, e_1; e_1, Je_2) + \tilde{R}(e_1, e_2; e_2, Je_1) \big\}.\end{aligned} \tag{3.33}$$

On the other hand, by applying the curvature identities (1.3) and (1.5), we find

$$\tilde{R}(e_2, e_1; e_1, Je_2) = -\tilde{R}(e_1, e_2; e_2, Je_1).$$

Combining this with (3.33) shows that equation (3.33) becomes equation (3.20) of Gauss. Consequently, the equation of Ricci is a consequence of Gauss and Codazzi for arbitrary Lorentz surfaces in any Lorentzian Kaehler surface. □

From the proof of Theorem 1 we also have the following.

Theorem 3.2. *The equation of Gauss is a consequence of the equations of Codazzi and Ricci for Lorentz surfaces in Lorentzian Kaehler surfaces.*

Remark 1. Some special cases of Theorem 1 are obtained in [3], [4].

Remark 2. Theorem 1 is false in general if the Lorentz surface in a Lorentzian Kaehler surface were replaced by a spatial surface in a Lorentzian Kaehler surface.

Remark 3. Since the three fundamental equations of Gauss, Codazzi and Ricci provide the conditions for local isometric embeddability, these equations also play some important role in physics; in particular in the Kaluza–Klein theory (cf. [6], [8], [10]).

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