Publ. Math. Debrecen
74/3-4 (2009), 369–382

P-Berwald manifolds

By SÁNDOR BÁCSÓ (Debrecen) and ZOLTÁN SZILASI (Debrecen)

Dedicated to Professor Lajos Tamássy on the occasion of his 85th birthday

Abstract. We introduce a new class of special Finsler manifolds, the class of p-Berwald manifolds. P-Berwald manifolds are defined as Finsler manifolds for which the projected Berwald curvature vanishes. We show that an at least 3-dimensional Finsler manifold is a p-Berwald manifold if and only if it is a weakly Berwald Douglas manifold. 2-dimensional p-Berwald manifolds are characterized by means of a differential equation concerning the main scalar. We prove that a p-Berwald manifold is R-quadratic if and only if its stretch tensor vanishes.

1. Introduction

By a p-Berwald manifold we mean a Finsler manifold whose projected Berwald curvature vanishes. The concept of a "projected Finsler tensor" was first systematically investigated by M. Matsumoto under the quite strange term "indicatorizaion", using the arsenal of classical tensor calculus [8]. An index-free description of Matsumoto's indicatorization was presented by Sz. Vattamány [18], working on TTM and using the Frölicher–Nijenhuis calculus of vector-valued forms. It seems to us that the pull-back bundle $\mathring{\tau}^*TM$ is a more economical framework for these constructions, and the Berwald derivative arising naturally from a Finsler structure is an adequate tool for calculations in this setting. For the

Mathematics Subject Classification: 53B40, 53C60.
Key words and phrases: p-Berwald manifolds, Douglas manifolds, weakly Berwald manifolds, stretch tensor.
The first author was supported by National Science Research Foundation OTKA No. T48878.

readers' convenience, we briefly summarize these basic technicalities in Section 2, and, partly, in Section 3. We follow the notation and conventions of reference [17] and, with some modifications, [5]. These papers also give some links to the classical approach. In Section 4 we discuss basic curvature relations in a Finsler manifold. The most interesting is formulated in Proposition 4.2; it has a "converse" (see (25)) in p-Berwald manifolds. In Section 5 it turns out that in $n > 2$ dimensions p-Berwald manifolds form the intersection of the class of Douglas manifolds and the class of weakly Berwald manifolds – of two classes of special Finsler manifolds which have been investigated extensively [1]–[4], [6].

2. Preliminaries

Throughout the paper M will be an n-dimensional ($n \geq 1$), second countable, Hausdorff, smooth manifold. $C^\infty(M)$ is the ring of real-valued smooth functions on M; the $C^\infty(M)$-module of smooth vector fields on M is denoted by $\mathfrak{X}(M)$. d is the operator of exterior derivative, i_X is the substitution operator induced by $X \in \mathfrak{X}(M)$.

If TM is the $2n$-dimensional manifold of all tangent vectors to M, and $\tau : TM \to M$ is the natural projection, the "foot map", then τ is said to be the tangent bundle of M, TM is the total space of the tangent bundle. The complete lift of a function $f \in C^\infty(M)$ is

$$f^{\mathsf{c}} : v \in TM \longmapsto f^{\mathsf{c}}(v) := v(f).$$

The complete lift of a vector field $X \in \mathfrak{X}(M)$ is the unique vector field $X^{\mathsf{c}} \in \mathfrak{X}(TM)$ such that

$$X^{\mathsf{c}} f^{\mathsf{c}} = (Xf)^{\mathsf{c}}, \quad f \in C^\infty(M).$$

Let $\widetilde{TM} \subset TM$ be an open subset satisfying $\tau(\widetilde{TM}) = M$, and let $\widetilde{\tau} := \tau \restriction \widetilde{TM}$. If

$$\widetilde{\tau}^* TM =: \widetilde{TM} \times_M TM := \left\{ (u, v) \in \widetilde{TM} \times TM \mid \widetilde{\tau}(u) = \tau(v) \right\}$$

and $\widetilde{\pi}(u, v) := u$ for $(u, v) \in \widetilde{\tau}^* TM$, then $\widetilde{\pi}$ is a vector bundle of rank n, the *pull-back of* τ *over* $\widetilde{\tau}$. The most important special cases arise when $\widetilde{TM} := TM$, $\widetilde{\tau} := \tau$ and $\widetilde{TM} := \mathring{T}M := TM \backslash o(M)$ ($o \in \mathfrak{X}(M)$ is the zero vector field), $\widetilde{\tau} := \mathring{\tau} := \tau \restriction \mathring{T}M$. Then we get the pull-back bundles $\pi : TM \times_M TM \to TM$ and $\mathring{\pi} : \mathring{T}M \times_M TM \to \mathring{T}M$.

We denote by $\Gamma(\widetilde{\pi})$ the $C^\infty(\widetilde{TM})$-module of smooth sections of $\widetilde{\pi}$. A typical element of $\Gamma(\widetilde{\pi})$ is of the form

$$\widetilde{X} : v \in \widetilde{TM} \longmapsto \widetilde{X}(v) = (v, \underline{X}(v)) \in \widetilde{TM} \times_M TM,$$

where $\underline{X} : \widetilde{TM} \to TM$ is a smooth map such that $\tau \circ \underline{X} = \widetilde{\tau}$. Any vector field X on M yields a section

$$\widehat{X} : v \in \widetilde{TM} \longmapsto \widehat{X}(v) = (v, X \circ \widetilde{\tau}(v)) \in \widetilde{TM} \times_M TM,$$

of $\widetilde{\pi}$, called a *basic vector field*. Basic vector fields generate the $C^\infty(\widetilde{TM})$-module $\Gamma(\widetilde{\pi})$. The *canonical section* δ of $\widetilde{\pi}$ sends $v \in \widetilde{TM}$ to $(v, v) \in \widetilde{\tau}^*TM$.

We denote by $\mathcal{T}^k_l(\widetilde{\pi})$ the $C^\infty(\widetilde{TM})$-module of all tensors of type (k, l) over $\Gamma(\widetilde{\pi})$ ($(k, l) \in \mathbb{N} \times \mathbb{N}$; $\mathcal{T}^0_0(\widetilde{\pi}) := C^\infty(\widetilde{TM})$). Elements of $\mathcal{T}^1_l(\widetilde{\pi})$ may naturally be interpreted as $\Gamma(\widetilde{\pi})$-valued $C^\infty(\widetilde{TM})$-multilinear maps. The unit tensor in $\mathcal{T}^1_1(\widetilde{\pi})$ will simply be denoted by $\mathbf{1}$. We note that $\mathcal{T}^k_l(\pi)$ may (and will) be considered as a submodule of $\mathcal{T}^k_l(\mathring{\pi})$.

$\mathbf{i}$ denotes the canonical bundle injection $\widetilde{TM} \times_M TM \to T\widetilde{TM}$, $\mathbf{j}$ is the canonical bundle surjection of $T\widetilde{TM}$ onto $\widetilde{TM} \times_M TM$. Then $\mathbf{j} \circ \mathbf{i} = 0$, while $\mathbf{J} := \mathbf{i} \circ \mathbf{j}$ is another canonical bundle map, the *vertical endomorphism* of $T\widetilde{TM}$. $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{J}$ induce the $C^\infty(\widetilde{TM})$-homomorhpisms

$$\begin{aligned} \Gamma(\widetilde{\pi}) &\longrightarrow \mathfrak{X}(TM), & \widetilde{X} &\longmapsto \mathbf{i}\widetilde{X} := \mathbf{i} \circ \widetilde{X}, \\ \mathfrak{X}(\widetilde{TM}) &\longrightarrow \Gamma(\pi), & \xi &\longmapsto \mathbf{j}\xi := \mathbf{j} \circ \xi, \\ \mathfrak{X}(\widetilde{TM}) &\longrightarrow \mathfrak{X}(TM), & \xi &\longmapsto \mathbf{J}\xi := \mathbf{J} \circ \xi. \end{aligned}$$

Then

$$\mathfrak{X}^{\mathsf{v}}(\widetilde{TM}) := \mathbf{i}(\Gamma(\widetilde{\pi})) = \operatorname{Im}(\mathbf{J}) = Ker(\mathbf{J})$$

is the $C^\infty(\widetilde{TM})$-module of *vertical vector fields* on $\widetilde{TM}$, $X^{\mathsf{v}} := \mathbf{i}\widehat{X}$ is the *vertical lift* of $X \in \mathfrak{X}(M)$. $C := \mathbf{i}\delta$ is a canonical vertical vector field on $\widetilde{TM}$, the *Liouville vector field*. For any vector field X on M we have

$$[C, X^{\mathsf{v}}] = -X^{\mathsf{v}}, \quad [C, X^{\mathsf{c}}] = 0. \tag{1}$$

We define the *vertical differential* $\nabla^{\mathsf{v}} F \in \mathcal{T}^0_1(\widetilde{\pi})$ of a function $F \in C^\infty(\widetilde{TM})$ by

$$\nabla^{\mathsf{v}} F(\widetilde{X}) := (\mathbf{i}\widetilde{X})F, \quad \widetilde{X} \in \Gamma(\widetilde{\pi}). \tag{2}$$

The vertical differential of a section $\widetilde{Y} \in \Gamma(\widetilde{\pi})$ is the $(1,1)$ tensor $\nabla^{\mathsf{v}}\widetilde{Y} \in \mathcal{T}^1_1(\widetilde{\pi})$ given by

$$\nabla^{\mathsf{v}}\widetilde{Y}(\widetilde{X}) =: \nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} := \mathbf{j}\big[\mathbf{i}\widetilde{X}, \eta\big], \quad \widetilde{X} \in \Gamma(\widetilde{\pi}), \tag{3}$$

where $\eta \in \mathfrak{X}(\widetilde{TM})$ is such that $\mathbf{j}\eta = \widetilde{Y}$. (It is easy to check that the result does not depend on the choice of η.) Using the Leibnizian product rule as a guiding principle, the operators $\nabla^{\mathsf{v}}_{\widetilde{X}}$ may uniquely be extended to a tensor derivation of the tensor algebra of $\Gamma(\widetilde{\pi})$. Forming the vertical differential of a tensor over $\Gamma(\widetilde{\pi})$, we use the following convention: if, e.g., $\mathbf{A} \in \mathcal{T}^1_2(\widetilde{\pi})$, then $\nabla^{\mathsf{v}}(\mathbf{A}) \in \mathcal{T}^1_3(\widetilde{\pi})$, given by

$$\nabla^{\mathsf{v}}\mathbf{A}(\widetilde{X},\widetilde{Y},\widetilde{Z}) := (\nabla^{\mathsf{v}}_{\widetilde{X}}\mathbf{A})(\widetilde{Y},\widetilde{Z}) = \nabla^{\mathsf{v}}_{\widetilde{X}}\mathbf{A}(\widetilde{Y},\widetilde{Z}) - \mathbf{A}(\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) - \mathbf{A}(\widetilde{Y},\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Z}).$$

3. Finsler functions and associated objects

Let m_λ, where λ is a real number, denote the map $v \in TM \mapsto \lambda v \in TM$. By a *Finsler function* we mean a function $F : TM \to \mathbb{R}$ satisfying:

(F1) F is smooth on $\overset{\circ}{T}M$.

(F2) $F \circ m_\lambda = \lambda F$ for all real numbers $\lambda \geq 0$.

(F3) $F \geq 0$ and equals 0 only on $o(M)$.

(F4) The $(0,2)$ tensor $g := \frac{1}{2}\nabla^{\mathsf{v}}\nabla^{\mathsf{v}}F^2 \in \mathcal{T}^0_2(\overset{\circ}{\pi})$ is (fibrewise) positive definite.

A *Finsler manifold* is a pair (M,F) consisting of a manifold M and a Finsler function on TM. By Euler's theorem on homogeneous functions, condition (F2) may equivalently be written in the form $CF = F$. $E := \frac{1}{2}F^2$ is the *energy function* of the Finsler manifold. It is positive-homogeneous of degree 2, i.e., $CE = 2E$, smooth on $\overset{\circ}{T}M$ and identically zero on $o(M)$. It may be shown (see e.g. [19]) that, actually, E is C^1 on TM and is C^2, if and only if, E is the norm associated with a Riemannian structure on M in which case E is smooth on TM. $g = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}}E$ is said to be the *metric tensor* of (M,F). For any vector fields X, Y on M we have

$$g(\widehat{X},\widehat{Y}) = X^{\mathsf{v}}(Y^{\mathsf{v}}E). \tag{4}$$

Since $[X^{\mathsf{v}},Y^{\mathsf{v}}] = 0$, this implies that g is symmetric. It would have been sufficient to assume only the (fibrewise) non-singularity of this tensor for positive definiteness is then a consequence of the other conditions on F.

Now we list some basic data arising immediately from a Finsler function.

(i) $\delta_\flat : \widetilde{X} \in \Gamma(\overset{\circ}{\pi}) \longmapsto \delta_\flat(\widetilde{X}) := g(\widetilde{X},\delta)$ - the *canonical 1-form* of (M,F),

(ii) $\ell := \frac{1}{F}\delta \in \Gamma(\overset{\circ}{\pi})$ - the *normalized support element field*,

(iii) $\ell_\flat := \frac{1}{F}\delta_\flat \in \mathcal{T}^0_1(\overset{\circ}{\pi})$ - the dual form of ℓ,

(iv) $\eta := g - \ell_\flat \otimes \ell_\flat$ - the *angular metric tensor*.

We have the following relation:

$$\delta_\flat = F\nabla^{\mathsf{v}} F = \nabla^{\mathsf{v}} E. \tag{5}$$

Indeed for any vector field X on M, $\delta_\flat(\widehat{X}) := g(\widehat{X},\delta) = g(\delta,\widehat{X}) = \nabla^{\mathsf{v}}\nabla^{\mathsf{v}} E(\delta,\widehat{X}) = \nabla^{\mathsf{v}}_\delta\nabla^{\mathsf{v}} E(\widehat{X}) = C(X^{\mathsf{v}}E) - \nabla^{\mathsf{v}} E(\nabla^{\mathsf{v}}_\delta \widehat{X}) \overset{(3)}{=} [C,X^{\mathsf{v}}]E + X^{\mathsf{v}}(CE) - \nabla^{\mathsf{v}} E(\mathbf{j}[C,X^{\mathsf{c}}]) \overset{(1)}{=} -X^{\mathsf{v}}E + 2X^{\mathsf{v}}E = \frac{1}{2}X^{\mathsf{v}}F^2 = F(X^{\mathsf{v}}F) \overset{(3)}{=} F\nabla^{\mathsf{v}} F(\widehat{X})$, which proves the formula.

From this observation relations

$$g(\delta,\delta) = \delta_\flat(\delta) = F^2, \quad \ell_\flat(\ell) = g(\ell,\ell) = 1, \tag{6}$$

$$\eta = g - \nabla^{\mathsf{v}} F \otimes \nabla^{\mathsf{v}} F \tag{7}$$

are immediately deduced.

If (M,F) is a Finsler manifold, then there is a unique vector field S on TM defined to be zero on $o(M)$, and defined on $\overset{\circ}{T}M$ to be the unique vector field such that

$$i_S d(\nabla^{\mathsf{v}} F^2 \circ \mathbf{j}) = -dF^2.$$

Then S is C^1 on TM, smooth on $\overset{\circ}{T}M$ and has the properties

$$JS = C, \quad [C,S] = S, \tag{8}$$

therefore S is a spray, called the *canonical spray* of the Finsler manifold. It is less known, but a proof of this really fundamental fact may also be found in Warner's above cited paper [19]. The canonical spray induces an Ehresmann connection $\mathcal{H} : \overset{\circ}{T}M \times_M TM \longrightarrow T\overset{\circ}{T}M$ such that for any vector field X on M,

$$X^{\mathsf{h}} := \mathcal{H}\widehat{X} := \mathcal{H}\circ\widehat{X} := \frac{1}{2}(X^{\mathsf{c}} + [X^{\mathsf{v}},S]). \tag{9}$$

$\mathcal{H}$ is said to be the *Barthel connection* of (M,F), X^{h} is the *horizontal lift* of X. $\mathcal{H}$ is *homogeneous* in the sense that

$$\left[C,X^{\mathsf{h}}\right] = 0, \quad X \in \mathfrak{X}(M). \tag{10}$$

Indeed, $2[C,X^{\mathsf{h}}] = [C,X^{\mathsf{c}}] + [C,[X^{\mathsf{v}},S]] \overset{(1)}{=} [C,[X^{\mathsf{v}},S]] = -[X^{\mathsf{v}},[S,C]] - [S,[C,X^{\mathsf{v}}]] \overset{(1)),((8)}{=} [X^{\mathsf{v}},S] + [S,X^{\mathsf{v}}] = 0$.

An important property of the Barthel connection is that the Finsler function is a first integral for the horizontal lifts, i.e.,

$$X^{\mathsf{h}}F = 0, \quad X \in \mathfrak{X}(M). \tag{11}$$

Equivalently, $dF\circ\mathcal{H} = 0$. For a recent, simple proof of this fact we refer to [16].

To the Barthel connection (as to any Ehresmann connection) we associate

(i) the *horizontal projector* $\mathbf{h} := \mathcal{H} \circ \mathbf{j}$,

(ii) the *vertical projector* $\mathbf{v} := 1_{T\overset{\circ}{T}M} - \mathbf{h}$,

(iii) the *vertical map* $\mathcal{V} : T\overset{\circ}{T}M \to \overset{\circ}{T}M \times_M TM$ such that $\mathbf{i} \circ \mathcal{V} = \mathbf{v}$.

We define the *h-Berwald differentials* $\nabla^{\mathsf{h}} F \in \mathcal{T}^0_1(\overset{\circ}{\pi})$ $(F \in C^\infty(\overset{\circ}{T}M))$ and $\nabla^{\mathsf{h}} \widetilde{Y} \in \mathcal{T}^1_1(\overset{\circ}{\pi})$ $(\widetilde{Y} \in \Gamma(\overset{\circ}{\pi}))$ by the following rules:

$$\nabla^{\mathsf{h}} F(\widetilde{X}) := (\mathcal{H}\widetilde{X})F, \quad \widetilde{X} \in \Gamma(\overset{\circ}{\pi}); \tag{12}$$

$$\nabla^{\mathsf{h}} \widetilde{Y}(\widetilde{X}) := \nabla^{\mathsf{h}}_{\widetilde{X}} \widetilde{Y} := \mathcal{V}\big[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}\big], \quad \widetilde{X} \in \Gamma(\overset{\circ}{\pi}). \tag{13}$$

Then the operators $\nabla^{\mathsf{h}}_{\widetilde{X}}$ $(\widetilde{X} \in \Gamma(\overset{\circ}{\pi}))$ may uniquely be extended to the whole tensor algebra of $\Gamma(\overset{\circ}{\pi})$ as tensor derivations. Forming the h-Berwald differential of an arbitrary tensor, we adopt the same convention as in the vertical case. We note that the homogeneity of the Barthel connection implies

$$\nabla^{\mathsf{h}} \delta = 0. \tag{14}$$

From the operators ∇^{v} and ∇^{h} we build the *Berwald derivative*

$$\nabla : (\xi, \widetilde{Y}) \in \mathfrak{X}(\overset{\circ}{T}M) \times \Gamma(\overset{\circ}{\pi}) \longmapsto \nabla_\xi \widetilde{Y} := \nabla^{\mathsf{v}}_{\mathcal{V}\xi} \widetilde{Y} + \nabla^{\mathsf{h}}_{\mathbf{j}\xi} \widetilde{Y} \in \Gamma(\overset{\circ}{\pi}).$$

Then, by (3) and (13),

$$\nabla_\xi \widetilde{Y} = \mathbf{j}\big[\mathbf{v}\xi, \mathcal{H}\widetilde{Y}\big] + \mathcal{V}\big[\mathbf{h}\xi, \mathbf{i}\widetilde{Y}\big].$$

In particular,

$$\begin{aligned} &\nabla_{\mathbf{i}\widetilde{X}} \widetilde{Y} = \nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y}, \quad \nabla_{\mathcal{H}\widetilde{X}} \widetilde{Y} = \nabla^{\mathcal{H}}_{\widetilde{X}} \widetilde{Y}; \qquad && \widetilde{X}, \widetilde{Y} \in \Gamma(\overset{\circ}{\pi}); \\ &\nabla_{X^{\mathsf{v}}} \widehat{Y} = 0, \qquad \nabla_{X^{\mathsf{h}}} \widehat{Y} = \mathcal{V}\left[X^{\mathsf{h}}, Y^{\mathsf{v}}\right]; && X, Y \in \mathfrak{X}(M). \end{aligned} \tag{15}$$

4. Curvature properties

We assume for the remainder of the paper that (M, F) is a fixed n-dimensional Finsler manifold. To introduce some curvature data in (M, F), we start from the classical curvature tensor R^∇ of the Berwald derivative on M given by

$$R^\nabla(\xi, \eta)\widetilde{Z} := \nabla_\xi \nabla_\eta \widetilde{Z} - \nabla_\eta \nabla_\xi \widetilde{Z} - \nabla_{[\xi,\eta]} \widetilde{Z}, \quad (\xi, \eta \in \mathfrak{X}(\overset{\circ}{T}M),\ \widetilde{Z} \in \Gamma(\overset{\circ}{\pi})).$$

By the *affine curvature tensor* of (M,F) we mean the tensor $\mathbf{H} \in \mathcal{T}^1_3(\overset{\circ}{\pi})$ given by

$$\mathbf{H}(\widetilde{X},\widetilde{Y})\widetilde{Z} := R^{\nabla}(\mathcal{H}\widetilde{X},\mathcal{H}\widetilde{Y})\widetilde{Z};\quad \widetilde{X},\widetilde{Y},\widetilde{Z} \in \Gamma(\overset{\circ}{\pi}).$$

Here we followed L. Berwald's terminology. According to Z. Shen's usage, we say that (M,F) is *R-quadratic* if $\nabla^{\mathsf{v}}\mathbf{H} = 0$, i.e., the affine curvature "depends only on the position".

The type $(1,3)$ tensor $\mathbf{B}$ given by

$$\mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z} := R^{\nabla}(\mathbf{i}\widetilde{X},\mathcal{H}\widetilde{Y})\widetilde{Z};\quad \widetilde{X},\widetilde{Y},\widetilde{Z} \in \Gamma(\overset{\circ}{\pi})$$

is said to be the *Berwald curvature* of (M,F). Evaluating on basic vector fields, we find that

$$\mathbf{B}(\widehat{X},\widehat{Y})\widehat{Z} = \mathcal{V}\left[X^{\mathsf{v}},\left[Y^{\mathsf{h}},Z^{\mathsf{v}}\right]\right] \quad \text{or} \quad \mathbf{iB}(\widehat{X},\widehat{Y})\widehat{Z} = \left[X^{\mathsf{v}},\left[Y^{\mathsf{h}},Z^{\mathsf{v}}\right]\right].$$

It is then a straightforward matter to check that $\mathbf{B}$ is totally symmetric. We also have:

$$\delta \in \left\{\widetilde{X},\widetilde{Y},\widetilde{Z}\right\} \Rightarrow \mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z} = 0. \tag{16}$$

A Finsler manifold is said to be a *Berwald manifold* if its Berwald curvature vanishes. (M,F) is a *weakly Berwald manifold* provided $\operatorname{tr}\mathbf{B} = 0$, where tr denotes the trace of the $C^\infty(\overset{\circ}{T}M)$-linear map $\widetilde{X} \mapsto \mathbf{B}(\widetilde{X},\widetilde{Y})\widetilde{Z}$.

We shall need the following Bianchi identity:

$$\nabla^{\mathsf{v}}\mathbf{H}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) + \nabla^{\mathsf{h}}\mathbf{B}(\widetilde{Y},\widetilde{Z},\widetilde{X},\widetilde{U}) - \nabla^{\mathsf{h}}\mathbf{B}(\widetilde{Z},\widetilde{Y},\widetilde{X},\widetilde{U}) = 0 \tag{17}$$

$(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U} \in \Gamma(\overset{\circ}{\pi}))$; see [14], p. 1331.

The *Landsberg tensor* of (M,F) is

$$\mathbf{P} := -\frac{1}{2}\nabla^{\mathsf{h}} g. \tag{18}$$

As a special case of 2.50, Lemma 5 in [14], we obtain

Lemma 4.1. *The Berwald curvature and the Landsberg tensor of a Finsler manifold are related by*

$$\nabla^{\mathsf{v}}E \circ \mathbf{B} = -2\mathbf{P}, \tag{19}$$

where E is the energy function.

Notice that relation (19) implies immediately that Berwald manifolds have vanishing Landsberg tensor.

By the *stretch tensor* of (M,F) we mean the tensor $\mathbf{\Sigma} \in \mathcal{T}^0_4(\mathring{\pi})$ given by

$$\frac{1}{2}\mathbf{\Sigma}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) := \nabla^{\mathsf{h}}\mathbf{P}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) - \nabla^{\mathsf{h}}\mathbf{P}(\widetilde{Y},\widetilde{X},\widetilde{Z},\widetilde{U}). \tag{20}$$

The next important observation gives an index-free reformulation of relation (3.3.2.5) in [10]. For completeness we present an immediate (and also index-free) proof, which differs essentially from Matsumoto's argument based on classical tensor calculus.

Proposition 4.2. *For any sections $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$, $\widetilde{U}$ in $\Gamma(\mathring{\pi})$,*

$$\nabla^{\mathsf{v}}E \circ \nabla^{\mathsf{v}}\mathbf{H}(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) = \mathbf{\Sigma}(\widetilde{Y},\widetilde{X},\widetilde{Z},\widetilde{U}). \tag{21}$$

Proof. It is enough to check the relation for basic vector fields $\widehat{X}$, $\widehat{Y}$, $\widehat{Z}$, $\widehat{U}$.

$$\begin{aligned}\nabla^{\mathsf{v}}E(\nabla^{\mathsf{v}}\mathbf{H}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U})) &\overset{(2)}{=} (\mathbf{i}\nabla^{\mathsf{v}}\mathbf{H}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U}))E \\ &\overset{(17)}{=} \mathbf{i}(-\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U}) + \nabla^{\mathsf{h}}\mathbf{B}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U}))E.\end{aligned}$$

Here

$$\begin{aligned}\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U}) &= (\nabla_{Y^{\mathsf{h}}}\mathbf{B})(\widehat{Z},\widehat{X},\widehat{U}) = \nabla_{Y^{\mathsf{h}}}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U} \\ &\quad - \mathbf{B}(\nabla_{Y^{\mathsf{h}}}\widehat{Z},\widehat{X})\widehat{U} - \mathbf{B}(\widehat{Z},\nabla_{Y^{\mathsf{h}}}\widehat{X})\widehat{U} - \mathbf{B}(\widehat{Z},\widehat{X})\nabla_{Y^{\mathsf{h}}}\widehat{U},\end{aligned}$$

and by (15)

$$\nabla_{Y^{\mathsf{h}}}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U} = \mathcal{V}\big[Y^{\mathsf{h}}, \mathbf{i}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U}\big].$$

Therefore, applying (19) we get

$$\begin{aligned}\mathbf{i}\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U})E &= \big[Y^{\mathsf{h}}, \mathbf{i}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U}\big]E + 2\mathbf{P}(\nabla_{Y^{\mathsf{h}}}\widehat{Z},\widehat{X},\widehat{U}) \\ &\quad + 2\mathbf{P}(\widehat{Z},\nabla_{Y^{\mathsf{h}}}\widehat{X},\widehat{U}) + 2\mathbf{P}(\widehat{Z},\widehat{X},\nabla_{Y^{\mathsf{h}}}\widehat{U}).\end{aligned}$$

Since $Y^{\mathsf{h}}E = 0$ by (11), at the right-hand side the first term is

$$Y^{\mathsf{h}}((\mathbf{i}\mathbf{B}(\widehat{Z},\widehat{X})\widehat{U})E) \overset{(19)}{=} -2Y^{\mathsf{h}}\mathbf{P}(\widehat{Z},\widehat{X},\widehat{U}),$$

therefore the right-hand side is just $-2\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U})$. In the same way we find that

$$\mathbf{i}\nabla^{\mathsf{h}}\mathbf{B}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U})E = -2\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U}).$$

Hence

$$\begin{aligned}\nabla^{\mathsf{v}}E\big(\nabla^{\mathsf{v}}\mathbf{H}(\widehat{X},\widehat{Y},\widehat{Z},\widehat{U})\big) &= 2\big(\nabla^{\mathsf{h}}\mathbf{P}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U}) - \nabla^{\mathsf{h}}\mathbf{P}(\widehat{Z},\widehat{Y},\widehat{X},\widehat{U})\big) \\ &\overset{(20)}{=} \mathbf{\Sigma}(\widehat{Y},\widehat{Z},\widehat{X},\widehat{U}),\end{aligned}$$

as was to be proved. $\square$

Corollary 4.3. *R-quadratic Finsler manifolds have vanishing stretch tensor.*

5. P-Berwald manifolds

Lemma 5.1. *If*
$$\mathbf{p} := \mathbf{1} - \frac{1}{2E}\nabla^{\mathsf{v}} E \otimes \delta, \tag{22}$$
then $\mathbf{p}(\delta) = 0$*, and* $\mathbf{p}$ *is a projection operator on* $\Gamma(\overset{\circ}{\pi})$*, i.e.,* $\mathbf{p}^2 = \mathbf{p}$.

Proof. Since the energy function is positive-homogeneous of degree 2,
$$\mathbf{p}(\delta) := \delta - \frac{1}{2E}\nabla^{\mathsf{v}} E(\delta)\delta = \delta - \frac{1}{2E}(CE)\delta = \delta - \delta = 0.$$
Using this observation, for any section $\widetilde{X}$ in $\Gamma(\overset{\circ}{\pi})$,
$$\mathbf{p}^2(\widetilde{X}) = \mathbf{p}(\widetilde{X} - \frac{1}{2E}(\mathbf{i}\widetilde{X})E\delta) = \mathbf{p}(\widetilde{X}),$$
thus proving the claim. □

By the *projected tensor* of a tensor $\mathbf{K} \in \mathcal{T}^0_k(\overset{\circ}{\pi})$ or $\mathbf{L} \in \mathcal{T}^1_k(\overset{\circ}{\pi})$ we mean the tensors $\mathbf{pK}$ and $\mathbf{pL}$ given by
$$\mathbf{pK}(\widetilde{X}_1, \dots, \widetilde{X}_k) := \mathbf{K}(\mathbf{p}\widetilde{X}_1, \dots, \mathbf{p}\widetilde{X}_k)$$
and
$$\mathbf{pL}(\widetilde{X}_1, \dots, \widetilde{X}_k) := \mathbf{p}(\mathbf{L}(\mathbf{p}\widetilde{X}_1, \dots, \mathbf{p}\widetilde{X}_k)).$$

Corollary 5.2. *Let* $\mathbf{K} \in \mathcal{T}^0_k(\overset{\circ}{\pi})$, $\mathbf{L} \in \mathcal{T}^1_k(\overset{\circ}{\pi})$ *. If*
$$\delta \in \left\{\widetilde{X}_1, \dots, \widetilde{X}_k\right\} \Rightarrow \mathbf{K}(\widetilde{X}_1, \dots, \widetilde{X}_k) = 0, \quad \mathbf{L}(\widetilde{X}_1, \dots, \widetilde{X}_k) = 0,$$
then $\mathbf{pK} = \mathbf{K}$*,* $\mathbf{pL} = \mathbf{p} \circ \mathbf{L}$*.*

Example. The projected tensor of the metric tensor g *is the angular metric tensor* η. Indeed, for any vector fields X, Y on M,
$$\begin{aligned}\mathbf{p}g(\widehat{X}, \widehat{Y}) &:= g\big(\mathbf{p}(\widehat{X}), \mathbf{p}(\widehat{Y})\big) = g\left(\widehat{X} - \frac{1}{2E}(X^{\mathsf{v}}E)\delta, \widehat{Y} - \frac{1}{2E}(Y^{\mathsf{v}}E)\delta\right)\\
&= g(\widehat{X}, \widehat{Y}) - \frac{1}{2E}(X^{\mathsf{v}}E)g(\delta, \widehat{Y}) - \frac{1}{2E}(Y^{\mathsf{v}}E)g(\widehat{X}, \delta)\\
&\quad + \frac{1}{4E^2}(X^{\mathsf{v}}E)(Y^{\mathsf{v}}E)g(\delta, \delta) \overset{(5),(6)}{=} g(\widehat{X}, \widehat{Y}) - \frac{1}{F^2}(X^{\mathsf{v}}E)\nabla^{\mathsf{v}}E(\widehat{Y})\\
&\quad - \frac{1}{F^2}(Y^{\mathsf{v}}E)\nabla^{\mathsf{v}}E(\widehat{X}) + \frac{1}{F^2}(X^{\mathsf{v}}E)(Y^{\mathsf{v}}E)\\
&= \left(g - \frac{1}{F^2}\nabla^{\mathsf{v}}E \otimes \nabla^{\mathsf{v}}E\right)(\widehat{X}, \widehat{Y}) = (g - \nabla^{\mathsf{v}}F \otimes \nabla^{\mathsf{v}}F)(\widehat{X}, \widehat{Y}) = \eta(\widehat{X}, \widehat{Y}).\end{aligned}$$

Lemma 5.3. *The projected tensor of the Berwald curvature of a Finsler manifold is*

$$\mathbf{pB} = \mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta. \tag{23}$$

Proof. By (16) and Corollary 5.2, $\mathbf{pB} = \mathbf{p} \circ \mathbf{B}$. Now, for any vector fields X, Y, Z on M,

$$(\mathbf{pB})(\widehat{X}, \widehat{Y}, \widehat{Z}) = \mathbf{p}(\mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z}) \overset{(22)}{=} \mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} - \frac{1}{2E}(\mathbf{iB}(\widehat{X}, \widehat{Y})\widehat{Z})E\delta$$

$$\overset{(19)}{=} \mathbf{B}(\widehat{X}, \widehat{Y})\widehat{Z} + \frac{1}{E}\mathbf{P}(\widehat{X}, \widehat{Y}, \widehat{Z})\delta = \left(\mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta\right)(\widehat{X}, \widehat{Y}, \widehat{Z}),$$

hence our statement. □

Definition. By a *p-Berwald manifold* we mean a Finsler manifold in which the projected Berwald curvature vanishes, i.e., which has the property

$$\mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta = 0. \tag{24}$$

Proposition 5.4. *Any p-Berwald manifold is a weakly Berwald manifold.*

Proof. We have to show that if (M, F) is a p-Berwald manifold, then $\operatorname{tr}\mathbf{B} = 0$. By (24) and Lemma 1 of [15], $\operatorname{tr}\mathbf{B} = -\frac{1}{E}\operatorname{tr}(\mathbf{P} \otimes \delta) = -\frac{1}{E}i_\delta\mathbf{P}$. Here $i_\delta\mathbf{P} = -\frac{1}{2}i_\delta\nabla^{\mathsf{h}}g = 0$; for an index-free proof of this well-known fact we refer to [14], 3.11 (p. 1381). □

Theorem 5.5. *A p-Berwald manifold is R-quadratic, if and only if, its stretch tensor vanishes.*

Proof. The necessity of the condition is a consequence of Corollary 4.3. To prove the sufficiency, we show that in a p-Berwald manifold we have

$$\nabla^{\mathsf{v}}\mathbf{H}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) = \frac{1}{F^2}\mathbf{\Sigma}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U}) \otimes \delta; \quad \widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U} \in \Gamma(\mathring{\pi}). \tag{25}$$

Observe first that

$$\nabla^{\mathsf{h}}\mathbf{B} \overset{(24)}{=} -\nabla^{\mathsf{h}}\left(\frac{1}{E}\mathbf{P} \otimes \delta\right) \overset{(11),(14)}{=} -\frac{1}{E}\nabla^{\mathsf{h}}\mathbf{P} \otimes \delta.$$

Now, applying Bianchi identity (17), we get

$$\nabla^{\mathsf{v}}\mathbf{H}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) = \nabla^{\mathsf{h}}\mathbf{B}(\widetilde{Z}, \widetilde{Y}, \widetilde{X}, \widetilde{U}) - \nabla^{\mathsf{h}}\mathbf{B}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U}) = -\frac{1}{E}(\nabla^{\mathsf{h}}\mathbf{P}(\widetilde{Z}, \widetilde{Y}, \widetilde{X}, \widetilde{U})$$

$$- \nabla^{\mathsf{h}}\mathbf{P}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U})) \otimes \delta \overset{(20)}{=} \frac{1}{F^2}\mathbf{\Sigma}(\widetilde{Y}, \widetilde{Z}, \widetilde{X}, \widetilde{U}).$$

This proves (25), whence the statement follows. □

To give a more precise characterization of p-Berwald manifolds, we need the concept of Douglas manifolds. By the *Douglas curvature* of a Finsler manifold we mean the tensor

$$\mathbf{D} := \mathbf{B} - \frac{1}{n+1}(\operatorname{tr}\mathbf{B} \odot \mathbf{1} + (\nabla^{\mathsf{v}} \operatorname{tr}\mathbf{B}) \otimes \delta), \tag{26}$$

where the symbol $\odot$ denotes symmetric product (without any extra numerical factor). An index-free representation of the Douglas curvature was first presented by J. Szilasi and Sz. Vattamány [13]; formula (26) is just a "pull back version" of formula (6.2b) of the cited paper. Finsler manifolds with vanishing Douglas curvature were baptized *Douglas manifolds* by S. Bácsó and M. Matsumoto, who devoted a series of papers to their thorough investigation [1]–[4]. Observe that in weakly Berwald manifolds, and hence in p-Berwald manifolds the Douglas and Berwald curvature coincide.

Lemma 5.6. *The projected tensor of the Douglas curvature is*

$$\mathbf{pD} = \mathbf{pB} - \frac{1}{n+1}\operatorname{tr}\mathbf{B} \odot \mathbf{p} = \mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta - \frac{1}{n+1}\operatorname{tr}\mathbf{B} \odot \mathbf{p}. \tag{27}$$

Proof. First we check that $\mathbf{D}$ satisfies the condition of Corollary 5.2, i.e., $\mathbf{D}(\widetilde{X},\widetilde{Y})\widetilde{Z} = 0$, if $\delta \in \{\widetilde{X},\widetilde{Y},\widetilde{Z}\}$. Let, for example, $\widetilde{X} := \delta$. Then

$$\begin{aligned}\mathbf{D}(\delta,\widetilde{Y},\widetilde{Z}) &:= \mathbf{B}(\delta,\widetilde{Y},\widetilde{Z}) - \frac{1}{n+1}(\operatorname{tr}\mathbf{B}(\delta,\widetilde{Y})\widetilde{Z} + \operatorname{tr}\mathbf{B}(\widetilde{Y},\widetilde{Z})\delta + \operatorname{tr}\mathbf{B}(\widetilde{Z},\delta)\widetilde{Y})\\ &-\frac{1}{n+1}(\nabla_C \operatorname{tr}\mathbf{B})(\widetilde{Y},\widetilde{Z})\delta \overset{(16)}{=} -\frac{1}{n+1}(\operatorname{tr}\mathbf{B}(\widetilde{Y},\widetilde{Z})\delta + \nabla_C \operatorname{tr}\mathbf{B})(\widetilde{Y},\widetilde{Z})\delta).\end{aligned}$$

It is known (see e.g. [13], Proposition 4.4) that $\mathbf{B}$ is homogeneous of degree -1, i.e., $\nabla_C\mathbf{B} = -\mathbf{B}$. Thus $\nabla_C \operatorname{tr}\mathbf{B} = \operatorname{tr}\nabla_C\mathbf{B} = -\operatorname{tr}\mathbf{B}$, and hence $\mathbf{D}(\delta,\widetilde{Y},\widetilde{Z}) = 0$. The other two cases may be handled similarly. Now it follows that

$$\mathbf{pD} = \mathbf{p} \circ \mathbf{D} = \mathbf{pB} - \frac{1}{n+1}(\mathbf{p}(\operatorname{tr}\mathbf{B} \odot \mathbf{1}) + \mathbf{p}(\nabla^{\mathsf{v}} \operatorname{tr}\mathbf{B} \otimes \delta).$$

Here, for any vector fields X, Y, Z on M,

$$\begin{aligned}\mathbf{p}(\operatorname{tr}\mathbf{B} \odot \mathbf{1}(\widehat{X},\widehat{Y},\widehat{Z})) &:= \mathbf{p}(\operatorname{tr}\mathbf{B} \odot \mathbf{1}(\mathbf{p}\widehat{X},\mathbf{p}\widehat{Y},\mathbf{p}\widehat{Z})) \overset{(16),\mathrm{Cor.5.2}}{=} \mathbf{p}(\operatorname{tr}\mathbf{B}(\widehat{X},\widehat{Y})\mathbf{p}(\widehat{Z})\\ &+ \operatorname{tr}\mathbf{B}(\widehat{Y},\widehat{Z})\mathbf{p}(\widehat{X}) + \operatorname{tr}\mathbf{B}(\widehat{Z},\widehat{X})\mathbf{p}(\widehat{Y})) = \operatorname{tr}\mathbf{B}(\widehat{X},\widehat{Y})\mathbf{p}(\widehat{Z})\\ &+ \operatorname{tr}\mathbf{B}(\widehat{Y},\widehat{Z})\mathbf{p}(\widehat{X}) + \operatorname{tr}\mathbf{B}(\widehat{Z},\widehat{X})\mathbf{p}(\widehat{Y})\\ &= (\operatorname{tr}\mathbf{B} \odot \mathbf{P})(\widehat{X},\widehat{Y},\widehat{Z}),\end{aligned}$$

while
$$\mathbf{p}(\nabla^{\mathsf{v}} \operatorname{tr} \mathbf{B} \otimes \delta)(\widehat{X}, \widehat{Y}, \widehat{Z}) = \mathbf{p}((\nabla^{\mathsf{v}}_{\mathbf{p}\widehat{X}} \operatorname{tr} \mathbf{B})(\mathbf{p}\widehat{Y}, \mathbf{p}\widehat{Z})\delta) = 0,$$
since $\mathbf{p}(\delta) = 0$.

This concludes the proof of (27). $\square$

Theorem 5.7. *If (M, F) is a Finsler manifold of dimension $n > 2$, then (M, F) is a p-Berwald manifold, if and only if, it is a weakly Berwald Douglas manifold.*

Proof. If (M, F) is a p-Berwald manifold, then it is weakly Berwald by Proposition 5.4, therefore (27) reduces to $\mathbf{pD} = 0$. However, by a theorem of T. Sakaguchi [11] (see also [18]), $\mathbf{pD} = 0$ is equivalent to the vanishing of the Douglas tensor under the condition $n > 2$.

Conversely, if (M, F) is a weakly Berwald Douglas manifold, then $\mathbf{D} = \mathbf{pD} = 0$ and $\operatorname{tr} \mathbf{B} = 0$ imply by (27) that (M, F) is a p-Berwald manifold. $\square$

Finally, we have a look at the "exceptional case" $\dim M = 2$. Then one can choose a section $m \in \Gamma(\overset{\circ}{\pi})$ such that
$$g(\ell, m) = 0, \quad g(m, m) = 1;$$
the pair (ℓ, m) is said to be a *Berwald frame* on (M, F). An immediate calculation shows that the only non vanishing component of the tensor $\nabla^{\mathsf{v}} g$ with respect to (ℓ, m) is the function
$$I := \frac{1}{2}\nabla^{\mathsf{v}} g(m, m, m),$$
it is called the *main scalar* of (M, F). For the Landsberg tensor of (M, F) we have the expression
$$2\mathbf{P} = \frac{SI}{I}\nabla^{\mathsf{v}} g, \tag{28}$$
where S is the canonical spray. By (16), the only surviving component of the Berwald curvature is $\mathbf{B}(m, m)m$. It may be shown that
$$\mathbf{B}(m, m)m = -\frac{2SI}{F}\ell + \big((\mathbf{i}m)(SI) + (\mathcal{H}m)I\big)m, \tag{29}$$
where $\mathcal{H}$ is the Barthel connection arising from S according to (9). By (28) and (29), condition $\mathbf{B} + \frac{1}{E}\mathbf{P} \otimes \delta = 0$ takes the form
$$\mathbf{B}(m, m)m + \frac{1}{2E}\frac{SI}{I}\nabla^{\mathsf{v}} g(m, m, m)\delta = 0. \tag{30}$$
Since $\frac{1}{2E}\frac{SI}{I}\nabla^{\mathsf{v}} g(m, m, m)\delta = \frac{1}{E}(SI)\delta = \frac{2}{F}(SI)\ell$, (29) and (30) yield
$$(\mathbf{i}m)SI + (\mathcal{H}m)I = 0. \tag{31}$$
Thus we obtain:

Theorem 5.8. *A two-dimensional Finsler manifold is a p-Berwald manifold, if and only if, the main scalar satisfies relation* (31).

References

[1] S. Bácsó and M. Matsumoto, On Finsler spaces of Douglas type. A generalization of the notion of Berwald space, *Publicationes Mathematicae Debrecen* **51** (1997), 385–406.

[2] S. Bácsó and M. Matsumoto, On Finsler spaces of Douglas type II. Projectively flat spaces, *Publicationes Mathematicae Debrecen* **53** (1998), 423–438.

[3] S. Bácsó and M. Matsumoto, On Finsler spaces of Douglas type III., in: Finslerian Geometries, (by P. Antonelli, ed.), *Kluwer Academic Publishers*, 2000, 89–94.

[4] S. Bácsó and M. Matsumoto, On Finsler spaces of Douglas type IV. Projectively flat Kropina spaces, *Publicationes Mathematicae Debrecen* **56** (2000), 213–221.

[5] S. Bácsó and Z. Szilasi, On the direction independence of two remarkable Finsler tensors, In: Differential Geometry and its Applications – Proceedings of the 10th International Conference on DGA2007, *World Scientific*, 2008, 385–394.

[6] S. Bácsó and R. Yoshikawa, Weakly-Berwald spaces, *Publicationes Mathematicae Debrecen* **61** (2002), 219–231.

[7] D. Bao, S. S. Chern and Z. Shen, An Introduction to Riemann–Finsler Geometry, *Springer Verlag*, 2000.

[8] M. Matsumoto, On the indicatrices of a Finsler space, *Periodica Mathematica Hungarica* **8** (1977), 185–191.

[9] M. Matsumoto, Foundations of Finsler Geometry and special Finsler spaces, *Kaiseisha Press*, 1986.

[10] M. Matsumoto, Finsler Geometry in the 20th Century, in: Handbook of Finsler Geometry, (by P. Antonelli, ed.), *Kluwer Academic Publishers, Dordrecht*, 2003.

[11] T. Sakaguchi, On Finsler spaces of scalar curvature, *Tensor, N. S.* **38** (1982), 211–219.

[12] Z. Shen, Differential Geometry of Spray and Finsler spaces, *Kluwer Academic Press*, 2001.

[13] J. Szilasi and Sz. Vattamány, Erratum to "On the projective geometry of sprays", *Differential Geom. Appl.* **13** (2000), 95–118.

[14] J. Szilasi, A Setting for Spray and Finsler Geometry, in: Handbook of Finsler Geometry, (by P. Antonelli, ed.), *Kluwer Academic Publishers, Dordrecht*, 2003, 1185–1426.

[15] J. Szilasi, Á. Győry, A generalization of Weyl's theorem on projectively related affine connections, *Reports on Mathematical Physics* **53** (2004), 261–273.

[16] J. Szilasi and R. L. Lovas, Some aspects of Differential Theories, in: Handbook of Global Analysis, *Elsevier*, 2007, 1071–1116.

[17] J. Szilasi, Calculus along the tangent bundle projection and projective metrizability, In: Differential Geometry and its Applications – Proceedings of the 10th International Conference on DGA2007, *World Scientific*, 2008, 527–546.

[18] Sz. Vattamány, Projection onto the indicatrix bundle of a Finsler manifold, *Publicationes Mathematicae Debrecen* **58** (2001), 193–221.

[19] F. W. Warner, The Conjugate Locus of a Riemannian Manifold, *American Journal of Mathematics* **87** (1965), 575–604.

SÁNDOR BÁCSÓ
INSTITUTE OF INFORMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN
HUNGARY

E-mail: bacsos@inf.unideb.hu

ZOLTÁN SZILASI
INSTITUTE OF INFORMATICS
UNIVERSITY OF DEBRECEN
H-4010 DEBRECEN
HUNGARY

(Received September 9, 2008; revised February 16, 2009)