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Bernstein type theorems for minimal surfaces in (α, β)-space

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Abstract. Let $\mathbb{V}^{n+1}$ be an $(n+1)$-dimensional real vector space and $\tilde{F} = \tilde{\alpha}\phi(s)$, $s = \tilde{\beta}/\tilde{\alpha}$, be an (α, β)-metric, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one form. Minimal surfaces with respect to the Busemann–Hausdorff measure and the Holmes–Thompson measure are called BH-minimal and HT-minimal surfaces, respectively. We give a Bernstein type theorem for minimal graphs in $(\mathbb{V}^{n+1}, \tilde{F})$ with $n \leq 7$. Let $\tilde{F}_b = \tilde{\alpha}\phi(s)$, $s = \tilde{\beta}/\tilde{\alpha}$, be a Minkowski metric with $b := \|\tilde{\beta}\|_{\tilde{\alpha}}$. We use a PDE to characterize the BH-minimal and HT-minimal graph over any hyperplane containing the origin in $(\mathbb{V}^{n+1}, \tilde{F}_b)$. Then we prove that this PDE is an elliptic equation of mean curvature type when $b \in [0, \epsilon)$ for some constant $\epsilon > 0$ and give a Bernstein type theorem for BH-minimal surface in $(\mathbb{V}^3, \tilde{F}_b)$. BH-minimal cones with an isolated singularity at the origin are also given.

1. Introduction

The classical Bernstein theorem states that any (regular) minimal surface in $\mathbb{R}^3$, which is a graph defined by a C^2-function on $\mathbb{R}^2$, is a plane. In higher dimensions, any complete minimal graph in $\mathbb{R}^{n+1}$ with $n \leq 7$ is an affine n-hyperplane. We want to generalize the Bernstein theorem into Finsler space. As is well known, various definitions of mean curvature have been introduced in Finsler geometry because of the uncertainty of the volume form. There are two

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natural volume forms in Finsler geometry, one is the Busemann–Hausdorff volume form, and another is the Holmes–Thompson volume form. Minimal surfaces with respect to the Busemann–Hausdorff measure and the Holmes–Thompson measure will be called BH-minimal and HT-minimal surfaces, respectively. Using the Busemann–Hausdorff volume form, Z. Shen introduced the notion of mean curvature for the submanifold in Finsler space and obtained some local and global results ([8]). A Bernstein theorem was obtained in [12] for BH-minimal surfaces in Minkowski Randers space $(\mathbb{R}^3, \tilde{\alpha}+\tilde{\beta})$ when the norm of the one form $\tilde{\beta}$ satisfies $\|\tilde{\beta}\|_{\tilde{\alpha}} \in [0, 1/\sqrt{3})$. Later, Q. He and Y. B. Shen introduced the notion of mean curvature by calculating the volume variation with respect to the Holmes–Thompson volume form ([5]). Then they gave a Bernstein type theorem ([4]) for hypersurface in Randers space $(\mathbb{R}^{n+1}, \tilde{\alpha}+\tilde{\beta})$ with $\tilde{\alpha}$ an Euclidean metric for $n \leq 7$ and proved that the Bernstein theorem holds for the HT-minimal graph in any 3-dimensional Minkowski space $\mathbb{R}^3$.

The purpose of this paper is to study the minimal hypersurfaces in an (α, β)-space $(\mathbb{V}^{n+1}, \tilde{F})$, where $\mathbb{V}^{n+1}$ is an $(n+1)$-dimensional real vector space, $\tilde{F} = \tilde{\alpha}\phi(s)$, $s = \tilde{\beta}/\tilde{\alpha}$, $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one from.

Set a system of ODE

$$\begin{cases} \phi(\phi - s\phi')^{n-1} = 1 + p(s) + s^2 q(s), \\ \phi(\phi - s\phi')^{n-2}\phi'' = q(s), \end{cases} \tag{1}$$

where $p(s)$ and $q(s)$ are arbitrary odd C^∞ functions.

Theorem 1.1. *Let $\tilde{F} = \tilde{\alpha}\phi(s)$, $s = \tilde{\beta}/\tilde{\alpha}$, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one form, ϕ is given by $\phi(s) = (1+h(s))^{-\frac{1}{n}}$ with $h(s)$ an arbitrary odd C^∞ function (or ϕ satisfies (1)). Then any complete BH-minimal (or HT-minimal) graph in $(n+1)$-space $(\mathbb{V}^{n+1}, \tilde{F})$ with $n \leq 7$ is an affine hyperplane.*

We remark that the only ϕ satisfying (1) we found is $\phi = 1 + s$ (i.e. Randers metric). In this case, Theorem 1.1 was obtained in [4].

We denote by $\tilde{F}_b = \tilde{\alpha}\phi(s)$, $s = \tilde{\beta}/\tilde{\alpha}$, a Minkowski metric with $b := \|\tilde{\beta}\|_{\tilde{\alpha}}$. We shall establish the Bernstein type theorem in $(\mathbb{V}^3, \tilde{F}_b)$. It was obtained in [4] that any complete HT-minimal graph in 3-dimensional Minkowski space $\mathbb{V}^3$ is a plane. So, in the following, we only study the BH-minimal surfaces.

Define

$$\sigma_{BH}(t) := \pi \left[\int_0^\pi \frac{1}{\phi^2(t^{\frac{1}{2}} \cos\theta)} d\theta \right]^{-1} \tag{2}$$

and

$$\Phi_b(t) := 2\sigma'_{BH}(t)(b^2 - t) + \sigma_{BH}(t). \tag{3}$$

Theorem 1.2. *Let $\tilde{F}_b = \tilde{\alpha}\phi(s)$, $s = \tilde{\beta}/\tilde{\alpha}$, be a Minkowski (α,β)-metric, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one form with length $b := \|\tilde{\beta}\|_{\tilde{\alpha}} \in [0, b_o)$ such that $\tilde{F}_b$ is positive definite. Let*

$$\epsilon := \sup\Big\{\epsilon' \in [0, b_o) \;\Big|\; \Phi_b(t) \neq 0,\ \frac{\Phi_b'(t)}{\Phi_b(t)} \geq 0,\ \forall\, t \in [0, b^2],\ \forall\, b \in [0, \epsilon')\Big\}. \tag{4}$$

Then any BH-minimal surface in $(\mathbb{V}^3, \tilde{F}_b)$ with $b \in [0, \epsilon)$, which is the graph of a function defined on $\mathbb{R}^2$, is a plane.

For Randers metric $\tilde{F}_b = \tilde{\alpha} + \tilde{\beta}$, we can compute $\epsilon = \frac{1}{\sqrt{3}}$ (See Example 6.1 in Section 6). This is the main result in [12].

This paper is organized as follows. In Section 2, we give the definitions and notations. In Section 3, we study the Busemann–Hausdorff volume form ([2], [3]) and the Holmes–Thompson volume form ([2]) of an (α,β)-metric. Then we give a Bernstein type theorem for minimal graphs in the (α,β)-space $(\mathbb{V}^{n+1}, \tilde{F})$ with $n \leq 7$. In Section 4, we calculate the mean curvature of a graph over any hyperplane containing the origin in Minkowski space $(\mathbb{V}^{n+1}, \tilde{F}_b)$ and use a PDE to characterize such a BH-minimal or HT-minimal graph. Then we give a local rigidity theorem which can be viewed as a generalization of the result in [15]. In Section 5, we prove that the minimal graph equation obtained in Section 4 is an elliptic equation of mean curvature type when $b \in [0, \epsilon)$ for some constant $\epsilon > 0$. Then we can obtain a Bernstein type theorem for BH-minimal graphs in $(\mathbb{V}^3, \tilde{F}_b)$ (i.e. Theorem 1.2). In Section 6, we give some examples to show that we can find the number ϵ in Theorem 1.2 for a given (α,β)-metric. We also give BH-minimal cones with an isolated singularity at the origin when $b > \epsilon$. It is clear that our result is a generalization of that in [12].

2. Preliminaries

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. The local coordinate system (x^i) on M and the tangent vector field $y = y^i \frac{\partial}{\partial x^i} \in TM$ give a local coordinate system (x^i, y^i) on TM. A Finsler metric on M is a function $F : TM \to [0, \infty)$ satisfying: (i) F is smooth on $TM \setminus \{0\}$; (ii) $F(x, \lambda y) = \lambda F(x, y)$ for $(x, y) \in TM$ and any positive real number λ; (iii) The fundamental form $g := g_{ij} dx^i \otimes dx^j$ is positive definite, where $g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$. A smooth manifold endowed with a Finsler metric is called a Finsler space. The simplest Finsler space is Minkowski space, on which the metric function F is independent of x.

Here and from now on, we shall use the following convention of index ranges:

$$1 \leq i, j, \cdots \leq n; \quad 1 \leq \alpha, \beta, \cdots \leq n+1.$$

Einstein summation convention is also used throughout this paper.

If (M, F) is a Finsler space, the Busemann–Hausdorff volume form induced from F is defined by

$$dV_F^{BH} = \sigma_F^{BH}(x) dx^1 \wedge \cdots \wedge dx^n,$$

where

$$\sigma_F^{BH}(x) = \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \mid F(x, y^i \frac{\partial}{\partial x^i}|_x) \leq 1\}}.$$

It coincides with the Hausdorff measure of M as a metric space when F is reversible. The Holmes–Thompson volume form induced from F is defined by

$$dV_F^{HT} = \sigma_F^{HT}(x) dx^1 \wedge \cdots \wedge dx^n,$$

where

$$\sigma_F^{HT}(x) = \frac{1}{\mathrm{Vol}(\mathbb{S}^{n-1})} \int_{S_x M} \sqrt{\det(g_{ij}(x, y))} \dot{\eta},$$

where $\dot{\eta}$ is the induced volume form of $S_x M := \{y \in T_x M | F_x(y) = 1\}$ from the Riemannian metric $\hat{g} = g_{ij}(y) dy^i \otimes dy^j$ on the punctured tangent space $T_x M \setminus 0$.

Let $(\tilde{M}^{n+p}, \tilde{F})$ be a Finsler manifold and $f : M^n \to (\tilde{M}^{n+p}, \tilde{F})$ be an isometric immersion. The notion of mean curvature with respect to Busemann–Hausdorff volume form was introduced by Z. Shen in [8]. Let $f_t : M^n \to (\tilde{M}^{n+p}, \tilde{F})$, $t \in (-\epsilon_o, \epsilon_o)$ be a variation of f such that f_t are isometric immersions with $f_0 = f$ and $f_t = f$ outside a compact set $\Omega \subset M$. f_t induce a family of Finsler metrics $F_t = f_t^* \tilde{F}$ on M and a variational vector field $X = \frac{\partial f_t}{\partial t}|_{t=0}$ along f. Consider the function $V(t) = \int_M dV_{F_t}^{BH}$, we have

$$V'(0) = \int_M \mathcal{H}_f^{BH}(X) dV_F^{BH},$$

where $\mathcal{H}_f^{BH}$ is called the *BH-mean curvature form* of the immersion f and f is said to be *BH-minimal* when $\mathcal{H}_f^{BH} = 0$.

From now on, we consider a hypersurface isometrically immersed in $(\mathbb{V}^{n+1}, \tilde{F})$. Let $f : (M^n, F) \to (\mathbb{V}^{n+1}, \tilde{F})$ be an isometric immersion. In local coordinates

$$\tilde{x}^\alpha = f^\alpha(x^1, \ldots, x^n).$$

Let $z_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}$ and $z=(z_i^\alpha) \in GL(n, n+1)$. For $\tilde{x} \in \mathbb{V}^{n+1}$, define

$$\mathcal{F}^{BH}(\tilde{x}, z) := \frac{\mathrm{Vol}(\mathbb{B}^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n \mid \tilde{F}_{\tilde{x}}(y^i z_i^\alpha \frac{\partial}{\partial \tilde{x}^\alpha}|_{\tilde{x}}) \leq 1\}}. \tag{5}$$

The Busemann–Hausdorff volume form dV_F^{BH} of the induced metric F is given by

$$dV_F^{BH}|_x = \mathcal{F}^{BH}(\tilde{x}, z) dx^1 \wedge \cdots \wedge dx^n, \tag{6}$$

where $\tilde{x} = f(x)$ and $z = (z_i^\alpha)$. The BH-mean curvature form of f is $\mathcal{H}_f^{BH} = \mathcal{H}_\alpha^{BH} d\tilde{x}^\alpha$, where ([8])

$$\mathcal{H}_\alpha^{BH} = \frac{1}{\mathcal{F}^{BH}} \left\{ -\frac{\partial \mathcal{F}^{BH}}{\partial \tilde{x}^\alpha} + \frac{\partial^2 \mathcal{F}^{BH}}{\partial z_i^\alpha \partial z_j^\beta} \frac{\partial^2 f^\beta}{\partial x^i \partial x^j} + \frac{\partial^2 \mathcal{F}^{BH}}{\partial \tilde{x}^\beta \partial z_i^\alpha} \frac{\partial f^\beta}{\partial x^i} \right\}. \tag{7}$$

Note that when $(\mathbb{V}^{n+1}, \tilde{F})$ is a Minkowski space, $\mathcal{F}^{BH}(\tilde{x}, z)$ is independent of $\tilde{x}$ from (5). In this case, (7) reduces to

$$\mathcal{H}_\alpha^{BH} = \frac{1}{\mathcal{F}^{BH}} \left\{ \frac{\partial^2 \mathcal{F}^{BH}}{\partial z_i^\alpha \partial z_j^\beta} \frac{\partial^2 f^\beta}{\partial x^i \partial x^j} \right\}. \tag{8}$$

It was pointed out in [14] that Z. Shen's method is also valid for Holmes–Thompson volume form. The mean curvature form $\mathcal{H}_f^{HT}$ and the minimal surface with respect to this volume form will be called *HT-mean curvature form* and *HT-minimal surface* respectively. The corresponding quantities will be marked with HT such as $\mathcal{F}^{HT}$, etc.

In Finsler geometry, (α,β)-metric is an important class of Finsler metrics which are defined by a Riemannian metric $\alpha = \sqrt{a_{ij} y^i y^j}$ and a one form $\beta = b_i y^i$. They are expressed in the form

$$F = \alpha \phi(s), \quad s = \beta/\alpha,$$

where $\phi(s)$ is a positive C^∞ function on $(-b_o, b_o)$. It can be proved ([9]) that F is a positive definite Finsler metric for any α and β with $\|\beta_x\|_\alpha < b_o$ if and only if ϕ satisfies

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_o). \tag{9}$$

We note that (α,β)-metric contains many important Finsler metrics. If $\phi = 1+s$, the Finsler metric $F = \alpha + \beta$ is Randers metric. If $\phi = \frac{1}{1-s}$, $F = \frac{\alpha^2}{\alpha - \beta}$ is Matsumoto metric which was first introduced in [6]. If $\phi = (1+s)^2$, $F = \frac{(\alpha+\beta)^2}{\alpha}$ is called *two order metric.*

3. Bernstein type theorem in high dimensions

For an (α,β)-metric $F=\alpha\phi(\beta/\alpha)$, the Busemann–Hausdorff volume form was calculated in [3] and the Holmes–Thompson volume form together with Busemann–Hausdorff volume form were calculated in [2] independently. Define

$$\sigma_{BH}(t):=\frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}\left[\int_0^\pi \frac{\sin^{n-2}\theta}{\phi^n(t^{\frac{1}{2}}\cos\theta)}d\theta\right]^{-1}. \tag{10}$$

(10) reduces to (2) when $n=2$. Thanks to the computation in [2], we define

$$\sigma_{HT}(t):=\frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n-1}{2})}\int_0^\pi H(t^{\frac{1}{2}}\cos\theta)\sin^{n-2}\theta d\theta, \tag{11}$$

where

$$H(s):=\phi(\phi-s\phi')^{n-2}[\phi-s\phi'+(t-s^2)\phi'']$$

and $\Gamma(t)=\int_0^{+\infty}x^{t-1}e^{-x}dx$ is the Euler function.

Lemma 3.1 ([2], [3])**.** *For an (α,β)-metric $F=\alpha\phi(s)$, $s=\beta/\alpha$, the Busemann–Hausdorff volume form and the Holmes–Thompson volume form of F are given by $dV_F^{BH}=\sigma_{BH}(\|\beta\|_\alpha^2)dV_\alpha$ and $dV_F^{HT}=\sigma_{HT}(\|\beta\|_\alpha^2)dV_\alpha$ respectively.*

Proposition 3.2. *Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric. If ϕ is given by $\phi(s)=(1+h(s))^{-\frac{1}{n}}$ with $h(s)$ an arbitrary odd C^∞ function, then $dV_F^{BH}=dV_\alpha$. If ϕ satisfies (1), then $dV_F^{HT}=dV_\alpha$.*

Proof. Note that $\int_0^\pi h(t^{\frac{1}{2}}\cos\theta)\sin^{n-2}\theta d\theta=0$ for any odd function $h(s)$ and

$$\int_0^\pi \sin^{n-2}\theta d\theta=\frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}.$$

Plugging $\phi(s)=(1+h(s))^{-\frac{1}{n}}$ into (10) yields $\sigma_{BH}(t)=1$. Similarly, if ϕ satisfies (1), then $\sigma_{HT}(t)=1$. $\square$

Now we consider an isometric immersion $f:(M^n,F)\to(\tilde{M}^{n+1},\tilde{F})$, where $\tilde{F}=\tilde{\alpha}\phi(s)$, $s=\tilde{\beta}/\tilde{\alpha}$ with Riemannian metric $\tilde{\alpha}=\sqrt{\tilde{a}_{\alpha\beta}d\tilde{x}^\alpha d\tilde{x}^\beta}$ and one form $\tilde{\beta}=\tilde{b}_\alpha d\tilde{x}^\alpha$. Then the induced metric F is also an (α,β)-metric of the same type $F=\alpha\phi(\beta/\alpha)$ with Riemannian metric $\alpha=\sqrt{a_{ij}y^iy^j}$ and one form $\beta=b_i(x)y^i$, where

$$a_{ij}(x)=z_i^\alpha z_j^\beta\tilde{a}_{\alpha\beta},\quad b_i(x)=\tilde{b}_\alpha z_i^\alpha,\quad z_i^\alpha=\frac{\partial f^\alpha}{\partial x^i}. \tag{12}$$

If ϕ is given in Proposition 3.2, then the Busemann–Hausdorff volume form (or Holmes–Thompson volume form) of (M,F) is that of the Riemannian manifold (M,α).

From the formula of mean curvature (7), we have immediately

Proposition 3.3. *Let $\tilde{F}=\tilde{\alpha}\phi(s)$, $s=\tilde{\beta}/\tilde{\alpha}$, be an (α,β)-metric, where ϕ is given by $\phi(s)=(1+h(s))^{-\frac{1}{n}}$ with $h(s)$ an arbitrary odd C^∞ function (or ϕ satisfies* (1)*), then the BH-mean curvature form (or HT-mean curvature form) $\mathcal{H}$ of the submanifold (M,F) isometrically immersed in $(\tilde{M},\tilde{F})$ is just that of the submanifold (M,α) isometrically immersed in the Riemannian manifold $(\tilde{M},\tilde{\alpha})$.*

Let $\tilde{M}^{n+1}$ be a real vector space $\mathbb{V}^{n+1}$, $\tilde{\alpha}$ be an Euclidean metric. By Bernstein theorem on minimal graphs in the Euclidean space ([10]), we have immediately

Theorem 3.4. *Let $\tilde{F}=\tilde{\alpha}\phi(s)$, $s=\tilde{\beta}/\tilde{\alpha}$, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one form, ϕ is given by $\phi(s)=(1+h(s))^{-\frac{1}{n}}$ with $h(s)$ an arbitrary odd C^∞ function (or ϕ satisfies* (1)*). Then any complete BH-minimal (or HT-minimal) graph in $(n+1)$-space $(\mathbb{V}^{n+1},\tilde{F})$ with $n\le 7$ is an affine hyperplane.*

Theorem 3.5. *Let $\tilde{F}=\tilde{\alpha}\phi(s)$, $s=\tilde{\beta}/\tilde{\alpha}$, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one form, ϕ is given by $\phi(s)=(1+h(s))^{-\frac{1}{2}}$ with $h(s)$ an arbitrary odd C^∞ function (or ϕ satisfies* (1) *for $n=2$). Then any complete stable BH-minimal (or HT-minimal) surface in 3-space $(\mathbb{V}^3,\tilde{F})$ is a plane.*

4. Minimal graph over any hyperplane in $(\mathbb{V}^{n+1},\tilde{F}_b)$

In this section, we study the BH-minimal and HT-minimal graph over any hyperplane containing the origin in Minkowski (α,β)-space $(\mathbb{V}^{n+1},\tilde{F}_b)$ with $\tilde{F}_b=\tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha})$, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one form of constant length $b=\|\tilde{\beta}\|_{\tilde{\alpha}}$ with respect to $\tilde{\alpha}$.

Given a hyperplane $\mathbf{P}$ containing the origin, we can use an $\tilde{\alpha}$-orthonormal basis $\{e_1,e_2,\dots,e_{n+1}\}$ such that $\mathbf{P}=\operatorname{span}\{e_1,e_2,\dots,e_n\}$. This basis gives a coordinate system $\{\tilde{x}^1,\tilde{x}^2,\dots,\tilde{x}^{n+1}\}$ for $\mathbb{V}^{n+1}$. Note that $\tilde{\beta}$ is a constant one form. We denote by $\theta_\alpha:=\angle_{\tilde{\alpha}}(\tilde{\beta}^\sharp,e_\alpha)$ the angle of $\tilde{\beta}^\sharp$ and e_α with respect to the Euclidean metric $\tilde{\alpha}$ and $\lambda_\alpha:=\cos\theta_\alpha$, where $\tilde{\beta}^\sharp$ is the dual vector of $\tilde{\beta}$ with respect to $\tilde{\alpha}$. Then $\tilde{F}_b$ can be expressed by

$$\tilde{F}_b=\sqrt{\delta_{\alpha\beta}d\tilde{x}^\alpha d\tilde{x}^\beta}\,\phi\left(\frac{b\lambda_\alpha d\tilde{x}^\alpha}{\sqrt{\delta_{\alpha\beta}d\tilde{x}^\alpha d\tilde{x}^\beta}}\right).$$

For an isometric immersion $f:(M,F)\to(\mathbb{V}^{n+1},\tilde{F}_b)$ locally given by

$$\tilde{x}^\alpha=f^\alpha(x^1,\dots,x^n),$$

the induced metric $F = f^*\tilde{F}_b$ has the form $F = \alpha\phi(\beta/\alpha)$ with Riemannian metric $\alpha = \sqrt{a_{ij}y^iy^j}$ and one form $\beta = b_iy^i$, where

$$a_{ij}(x) = z_i^\alpha z_j^\beta \delta_{\alpha\beta}, \quad b_i = b\lambda_\alpha z_i^\alpha, \quad z_i^\alpha = \frac{\partial f^\alpha}{\partial x^i}. \tag{13}$$

Note that $(\mathbb{V}^{n+1}, \tilde{F}_b)$ is a Minkowski space, then (8) holds. By Lemma 3.1 and (6), $\mathcal{F}(z)$ is given by

$$\mathcal{F}(z) = \sigma(\|\beta\|^2)\sqrt{\det(a_{ij})}. \tag{14}$$

Here and from now on, $\mathcal{F}(z)$ will denote $\mathcal{F}^{BH}(z)$ or $\mathcal{F}^{HT}(z)$, $\sigma(t)$ will be given by (10) or (11). We denote by $\|\beta\|^2 = b^2a^{ij}\lambda_\alpha\lambda_\beta z_i^\alpha z_j^\beta$ the square length of β with respect to α.

Theorem 4.1. *Let $f : (M, F) \to (\mathbb{V}^{n+1}, \tilde{F}_b)$ be an isometric immersion locally given by*

$$\tilde{x}^\alpha = f^\alpha(x^1, \ldots, x^n).$$

Then the BH-mean curvature form (or HT-mean curvature form) $\mathcal{H}_f = \mathcal{H}_\gamma d\tilde{x}^\gamma$ is given by

$$\begin{aligned}\mathcal{H}_\gamma = \frac{1}{\sigma(\|\beta\|^2)}\Big\{&a^{ij}\Big[2b^2\lambda_\alpha\lambda_\beta\sigma'(\|\beta\|^2)(\delta^{\alpha\eta} - B^{\alpha\eta})(\delta^{\beta\gamma} - B^{\beta\gamma})\\&+ \sigma(\|\beta\|^2)(\delta^{\gamma\eta} - B^{\gamma\eta})\Big]\\&+ 2b^2\lambda_\delta\lambda_\tau A^{i\delta}A^{j\tau}\Big[2b^2\lambda_\alpha\lambda_\beta\sigma''(\|\beta\|^2)(\delta^{\alpha\eta} - B^{\alpha\eta})(\delta^{\beta\gamma} - B^{\beta\gamma})\\&- \sigma'(\|\beta\|^2)(\delta^{\gamma\eta} - B^{\gamma\eta})\Big]\Big\}\frac{\partial^2 f^\eta}{\partial x^i\partial x^j},\end{aligned} \tag{15}$$

where $A^{i\alpha} := a^{ij}z_j^\alpha$, $B^{\alpha\beta} := a^{ij}z_i^\alpha z_j^\beta$, $\|\beta\|^2 = b^2\lambda_\alpha\lambda_\beta B^{\alpha\beta}$, $\lambda_\alpha = \cos\angle_{\tilde{\alpha}}(\tilde{\beta}^\sharp, e_\alpha)$ and $\sigma(t)$ is given by (10) or (11).

Proof. We denote

$$A^{i\alpha} := a^{ij}z_j^\alpha, \quad B^{\alpha\beta} := a^{ij}z_i^\alpha z_j^\beta. \tag{16}$$

Then the square length of β with respect to α can be expressed by

$$\|\beta\|^2 = b^2\lambda_\alpha\lambda_\beta B^{\alpha\beta}. \tag{17}$$

By (13) and 16) we can compute

$$\frac{\partial}{\partial z_i^\gamma}a^{kl} = -(a^{ki}A^{l\gamma} + a^{li}A^{k\gamma}), \tag{18}$$

$$\frac{\partial}{\partial z_i^\gamma} B^{\alpha\beta} = A^{i\alpha}(\delta^{\beta\gamma} - B^{\beta\gamma}) + A^{i\beta}(\delta^{\alpha\gamma} - B^{\alpha\gamma}). \tag{19}$$

Differentiating (17) we have

$$\frac{\partial}{\partial z_i^\gamma} \|\beta\|^2 = 2b^2 \lambda_\alpha \lambda_\beta A^{i\alpha}(\delta^{\beta\gamma} - B^{\beta\gamma}). \tag{20}$$

It is easy to show

$$\frac{\partial}{\partial z_i^\gamma} \sqrt{\det(a_{ij})} = \sqrt{\det(a_{ij})} A^{i\gamma}. \tag{21}$$

Since $\mathcal{F}(z) = \sigma(\|\beta\|^2)\sqrt{\det(a_{ij})}$, from (20) and (21) we get

$$\frac{\partial}{\partial z_i^\gamma} \mathcal{F} = \Big\{ 2b^2 \lambda_\alpha \lambda_\beta \sigma'(\|\beta\|^2) A^{i\alpha}(\delta^{\beta\gamma} - B^{\beta\gamma}) + \sigma(\|\beta\|^2) A^{i\gamma} \Big\} \sqrt{\det(a_{ij})}. \tag{22}$$

By (18), we can compute easily

$$\frac{\partial}{\partial z_j^\eta} A^{i\alpha} = a^{ij}(\delta^{\alpha\eta} - B^{\alpha\eta}) - A^{j\alpha} A^{i\eta}. \tag{23}$$

Differentiating (22), using (19) (20) (21) (23) and by a direct computation, we get

$$\begin{aligned}
\frac{\partial^2 \mathcal{F}}{\partial z_i^\gamma \partial z_j^\eta} = &\Big\{ \sigma(\|\beta\|^2)(A^{i\gamma} A^{j\eta} - A^{j\gamma} A^{i\eta}) \\
&+ 2b^2 \lambda_\alpha \lambda_\beta \sigma'(\|\beta\|^2)(\delta^{\beta\gamma} - B^{\beta\gamma})(A^{i\alpha} A^{j\eta} - A^{j\alpha} A^{i\eta}) \\
&+ 2b^2 \lambda_\alpha \lambda_\beta \sigma'(\|\beta\|^2)(\delta^{\beta\eta} - B^{\beta\eta})(A^{j\alpha} A^{i\gamma} - A^{i\alpha} A^{j\gamma}) \\
&+ a^{ij} \big[2b^2 \lambda_\alpha \lambda_\beta \sigma'(\|\beta\|^2)(\delta^{\alpha\eta} - B^{\alpha\eta})(\delta^{\beta\gamma} - B^{\beta\gamma}) + \sigma(\|\beta\|^2)(\delta^{\gamma\eta} - B^{\gamma\eta}) \big] \\
&+ 2b^2 \lambda_\delta \lambda_\tau A^{i\delta} A^{j\tau} \big[2b^2 \lambda_\alpha \lambda_\beta \sigma''(\|\beta\|^2)(\delta^{\alpha\eta} - B^{\alpha\eta})(\delta^{\beta\gamma} - B^{\beta\gamma}) \\
&- \sigma'(\|\beta\|^2)(\delta^{\gamma\eta} - B^{\gamma\eta}) \big] \Big\} \sqrt{\det(a_{ij})}.
\end{aligned} \tag{24}$$

From (8) and (14) we can obtain (15) immediately. □

In the following, we will use Theorem 4.1 to study the graph over a connected domain Ω in the hyperplane $\mathbf{P}$. In this case, we shall denote the coordinates for $\mathbb{V}^{n+1}$ by $\{x^1, x^2, \ldots, x^{n+1}\}$ rather than $\{\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^{n+1}\}$. Define a function for $0 \le t \le b^2$ by

$$\Phi_b(t) := 2\sigma'(t)(b^2 - t) + \sigma(t), \tag{25}$$

where $\sigma(t)$ is given by (10) or (11).

Theorem 4.2. *Let $f:\Omega\to(\mathbb{V}^{n+1},\tilde{F}_b)$ be a graph over a connected domain Ω in a hyperplane $\boldsymbol{P}$ which is given by*

$$f(x^1,\dots,x^n)=(x^1,\dots,x^n,u(x^1,\dots,x^n)). \tag{26}$$

Then the BH-mean curvature form (or HT-mean curvature form) is given by

$$\mathcal{H}_f=\mathcal{H}_{n+1}\Big(-\sum_k u_k dx^k+dx^{n+1}\Big), \tag{27}$$

where

$$\begin{aligned}\mathcal{H}_{n+1}=\frac{1}{\sigma(\|\beta\|^2)W^2}\sum_{ij}\Big\{&\Phi_b(\|\beta\|^2)\left(\delta_{ij}-\frac{u_iu_j}{W^2}\right)\\&+2b^2\Phi_b'(\|\beta\|^2)\left(\lambda_i+\omega\frac{u_i}{W^2}\right)\left(\lambda_j+\omega\frac{u_j}{W^2}\right)\Big\}u_{ij}.\end{aligned} \tag{28}$$

Thus f is BH-minimal (or HT-minimal) if and only if

$$\begin{aligned}\sum_{ij}\Big\{&\Phi_b(\|\beta\|^2)\left(\delta_{ij}-\frac{u_iu_j}{W^2}\right)\\&+2b^2\Phi_b'(\|\beta\|^2)\left(\lambda_i+\omega\frac{u_i}{W^2}\right)\left(\lambda_j+\omega\frac{u_j}{W^2}\right)\Big\}u_{ij}=0,\end{aligned} \tag{29}$$

where $u_i:=\frac{\partial u}{\partial x^i}, u_{ij}:=\frac{\partial^2 u}{\partial x^i x^j}$, $W^2:=1+\sum_i u_i^2$, $\omega:=\lambda_{n+1}-\sum_k\lambda_k u_k$, $\|\beta\|^2=b^2\big(1-\frac{\omega^2}{W^2}\big)$, $\lambda_\alpha=\cos\angle_{\tilde{\alpha}}(\tilde{\beta}^\sharp,e_\alpha)$, $\Phi_b(t)$ is defined by (25).

Proof. Since f is a graph given by (26), we get $z_i^k=\delta_i^k$, $z_i^{n+1}=u_i$. From (13) and (16), we can compute

$$a_{ij}=\delta_{ij}+u_iu_j,\quad a^{ij}=\delta^{ij}-\frac{u_iu_j}{W^2},\quad A^{i\alpha}=B^{i\alpha} \tag{30}$$

and

$$B^{ij}=a^{ij},\quad B^{i,n+1}=\frac{u_i}{W^2},\quad B^{n+1,n+1}=1-\frac{1}{W^2}, \tag{31}$$

where $W^2:=1+\sum_i u_i^2$. Let $\omega:=\lambda_{n+1}-\sum_k\lambda_k u_k$. We can compute directly

$$\begin{gathered}\lambda_\alpha(\delta^{\alpha,n+1}-B^{\alpha,n+1})=\frac{\omega}{W^2},\quad \lambda_\beta(\delta^{k\beta}-B^{k\beta})=-\omega\frac{u_k}{W^2},\\ \lambda_\delta A^{i\delta}=\lambda_i+\omega\frac{u_i}{W^2}.\end{gathered} \tag{32}$$

Plugging (30)–(32) into (15), we obtain

$$\mathcal{H}_{n+1}=\frac{1}{\sigma(\|\beta\|^2)W^2}\sum_{ij}\Big\{\Big[2b^2\sigma'(\|\beta\|^2)\frac{\omega^2}{W^2}+\sigma(\|\beta\|^2)\Big]\left(\delta_{ij}-\frac{u_iu_j}{W^2}\right)$$

$$+ 2b^2\Big[2b^2\sigma''(\|\beta\|^2)\frac{\omega^2}{W^2} - \sigma'(\|\beta\|^2)\Big]\left(\lambda_i + \omega\frac{u_i}{W^2}\right)\left(\lambda_j + \omega\frac{u_j}{W^2}\right)\Big\}u_{ij} \tag{33}$$

and

$$\mathcal{H}_k = -u_k\mathcal{H}_{n+1}. \tag{34}$$

From (17) (30) (31) and (32) we get

$$\|\beta\|^2 = b^2(1 - \frac{\omega^2}{W^2}). \tag{35}$$

By (35) and the definition of $\Phi_b(t)$ in (25), we obtain (27) and (28) from (33) and (34) immediately. □

If the graph is defined over a hyperplane **P** which is perpendicular to $\tilde{\beta}^\sharp$, then $\tilde{\beta}^\sharp$ is in the direction of x^{n+1}-axis. In this case, $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0, \lambda_{n+1} = 1$. Theorem 4.2 reduces to

Theorem 4.3. *Let $f : \Omega \to (\mathbb{V}^{n+1}, \tilde{F}_b)$ be a graph over a connected domain Ω in the hyperplane $\boldsymbol{P}$ which is perpendicular to $\tilde{\beta}^\sharp$ and given by* (26)*. Then the BH-mean curvature form (or HT-mean curvature form) is given by*

$$\mathcal{H}_f = \mathcal{H}_{n+1}\Big(- \sum_k u_k dx^k + dx^{n+1}\Big),$$

where

$$\mathcal{H}_{n+1} = \frac{1}{\sigma(\|\beta\|^2)W^2}\sum_{ij}\Big\{\Phi_b(\|\beta\|^2)\left(\delta_{ij} - \frac{u_iu_j}{W^2}\right) + 2b^2\Phi_b'(\|\beta\|^2)\frac{u_iu_j}{W^4}\Big\}u_{ij}.$$

Thus f is BH-minimal (or HT-minimal) if and only if

$$\sum_{ij}\Big\{\Phi_b(\|\beta\|^2)\left(\delta_{ij} - \frac{u_iu_j}{W^2}\right) + 2b^2\Phi_b'(\|\beta\|^2)\frac{u_iu_j}{W^4}\Big\}u_{ij} = 0, \tag{36}$$

where $u_i = \frac{\partial u}{\partial x^i}, u_{ij} = \frac{\partial^2 u}{\partial x^i x^j}, W^2 = 1 + \sum_i u_i^2, \|\beta\|^2 = b^2(1 - \frac{1}{W^2})$, $\Phi_b(t)$ is defined by (25).

Remark 4.4. The equations (29) and (36) for a BH-minimal graph in Randers space have been obtained in [12] for $n = 2$(see [12], p. 298, Theorem 4 and p. 296, Theorem 3).

Remark 4.5. The BH-minimal and HT-minimal graphs in Randers space were simultaneously studied in [15] for dimension n (see [15], p. 380, Proposition 3.1 and 3.2, p. 381, Proposition 3.5). The equations (29) and (36) can be viewed as the generalization of corresponding formulas in [15].

An interesting problem is to study surfaces which are both BH-minimal and HT-minimal in 3-dimensional Minkowski space. In [15], the author obtained a local rigidity theorem for surfaces which are both BH-minimal and HT-minimal in Minkowski Randers space $(\mathbb{V}^3, \tilde{\alpha}+\tilde{\beta})$ with $\tilde{\beta}^\sharp$ in the direction of x^3-axis. We shall point out that this result is also valid in Minkowski (α, β)-space $(\mathbb{V}^3, \tilde{F}_b)$ with $\tilde{\beta}^\sharp$ in the direction of x^3-axis. Obviously, the plane and helicoid generated by lines screwing about x^3-axis are both BH-minimal and HT-minimal. Conversely, we note that a surface can be locally viewed as a graph over a connected domain in certain plane $\mathbf{P}$. Without loss of generality, we can assume that $\mathbf{P} = \{x^3 = 0\}$ and $\mathbf{P} = \{x^1 = 0\}$. We assume that $\frac{\Phi_b'(t)}{\Phi_b(t)}$ is different for $\sigma = \sigma_{BH}$ and $\sigma = \sigma_{HT}$. If $\mathbf{P} = \{x^3 = 0\}$, from (36) we get a system of PDE:

$$\begin{cases} u_{11} + u_{22} = 0, \\ u_1^2 u_{11} + u_2^2 u_{22} + 2u_1 u_2 u_{12} = 0. \end{cases} \tag{37}$$

If $\mathbf{P} = \{x^1 = 0\}$, this is equivalent to the case that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = 0$ in Theorem 4.2. In this case, from (29) one can get a system of PDE:

$$\begin{cases} \displaystyle\sum_{ij} \left(\delta_{ij} - \frac{u_i u_j}{W^2}\right) u_{ij} = 0, \\ \displaystyle u_{11} + \frac{u_1^2}{W^4} \sum_{ij} u_i u_j u_{ij} - \frac{2}{W^2} \sum_j u_1 u_j u_{1j} = 0. \end{cases} \tag{38}$$

Here, $1 \le i,\ j \le 2$. Note that (37) and (38) have been explicitly solved in [15] (see [15], p. 382, Theorem 4.1 and p. 383, Theorem 4.2). We get immediately

Corollary 4.6. *Let $(\mathbb{V}^3, \tilde{F}_b = \tilde{\alpha}\phi(\tilde{\beta}/\tilde{\alpha}))$ be a Minkowski (α, β)-space with $\tilde{\beta}^\sharp$ in the direction of x^3-axis. Then the helicoid generated by lines screwing about x^3-axis is both BH-minimal and HT-minimal. Moreover, if $\frac{\Phi_b'(t)}{\Phi_b(t)}$ is different for $\sigma = \sigma_{BH}$ and $\sigma = \sigma_{HT}$, then any local surface which is both BH-minimal and HT-minimal must be either a piece of palne or a piece of helicoid generated by lines screwing about x^3-axis.*

5. Bernstein type theorem in $(\mathbb{V}^3, \tilde{F}_b)$

Lemma 5.1. *$\Phi_b(t)$ defined by (25) is smooth for $t \in [0, b^2]$.*

Proof. We prove this lemma for $\sigma_{BH}(t)$ which is defined by (10). Since $\phi(s)$ is a positive smooth function on $(-b_o, b_o)$, we only need to show that $\sigma_{BH}(t)$ is smooth at $t=0$. Denote $\tilde{\phi}(s) := \phi^{-n}(s)$ and by Taylor expansion

$$\tilde{\phi}(s) = a_0 + a_1 s + a_2 s^2 + \cdots + a_k s^k + o(s^k) \tag{39}$$

holds for arbitrary large order k. Plugging (39) into (10), we get

$$\sigma_{BH}(t) = \frac{\sqrt{\pi}\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}\Big[\tilde{a}_0 + \tilde{a}_2 t + \cdots + \tilde{a}_{2k} t^k + o(t^k)\Big]^{-1}.$$

Since $a_0 = \phi^{-n}(0) \neq 0$, then $\tilde{a}_0 \neq 0$, we can see that $\sigma_{BH}(t)$ is smooth at $t=0$. Similarly, we can prove this lemma for (11). $\square$

Theorem 5.2. *Let $\tilde{F}_b = \tilde{\alpha}\phi(s)$, $s = \tilde{\beta}/\tilde{\alpha}$, be a Minkowski metric, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one form with length $b := \|\tilde{\beta}\|_{\tilde{\alpha}} \in [0, b_o)$ such that $\tilde{F}_b$ is positive definite. Define $\Phi_b(t)$ by (25) and let*

$$\epsilon := \sup\left\{\epsilon' \in [0, b_o) \mid \Phi_b(t) \neq 0,\ \frac{\Phi_b'(t)}{\Phi_b(t)} \geq 0,\quad \forall\, t \in [0, b^2],\ \forall\, b \in [0, \epsilon')\right\}. \tag{40}$$

If $b \in [0,\epsilon)$, then the minimal equation (29) is an elliptic equation of mean curvature type.

Proof. By (40), $\Phi_b(t) \neq 0$ for any $t \in [0, b^2]$ when $b \in [0,\epsilon)$, then the minimal equation (29) can be written as

$$\sum_{ij} E_{ij}(x, u, \nabla u) u_{ij} = 0, \tag{41}$$

where

$$E_{ij}(x,u,\nabla u) := \left(\delta_{ij} - \frac{u_i u_j}{W^2}\right) + 2b^2 \frac{\Phi_b'}{\Phi_b}(\|\beta\|^2)\left(\lambda_i + \omega\frac{u_i}{W^2}\right)\left(\lambda_j + \omega\frac{u_j}{W^2}\right).$$

To prove that (41) is of mean curvature type, we need to show

$$\left(\delta_{ij} - \frac{p_i p_j}{W^2}\right)\xi^i\xi^j \leq E_{ij}(x,z,p)\xi^i\xi^j \leq (1+C)\left(\delta_{ij} - \frac{p_i p_j}{W^2}\right)\xi^i\xi^j \tag{42}$$

for any p and ξ with some constant $C > 0$, where $(x,z,p) \in U \times \mathbb{R} \times \mathbb{R}^n$, $W^2 = 1 + |p|^2$ and $\xi := (\xi^1, \ldots, \xi^n) \in \mathbb{R}^n$.

By (40), we have $\frac{\Phi_b'(t)}{\Phi_b(t)} \geq 0$ for $t \in [0, b^2]$ when $b \in [0,\epsilon)$, then the LHS of (42) holds. To prove the RHS of (42), we are aimed to show

$$\frac{\Phi_b'}{\Phi_b}(\|\beta\|^2)\left(\langle\lambda,\xi\rangle + \omega\frac{\langle p,\xi\rangle}{W^2}\right)^2 \leq C\left(|\xi|^2 - \frac{\langle p,\xi\rangle^2}{W^2}\right)$$

for any p and ξ with some constant $C > 0$, where $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$, $\omega = \lambda_{n+1} - \langle \lambda, p \rangle$.

Since $\|\beta\|^2 = b^2(1 - \frac{\omega^2}{W^2}) \in [0, b^2]$, we can see that $\frac{\Phi_b'}{\Phi_b}(\|\beta\|^2)$ is bounded from Lemma 5.1. It suffices to show

$$\left(\langle \lambda, \xi \rangle + \omega \frac{\langle p, \xi \rangle}{W^2}\right)^2 \leq C \left(|\xi|^2 - \frac{\langle p, \xi \rangle^2}{W^2}\right).$$

If $\xi = 0$, this is obvious. If $\xi \neq 0$, we denote $\angle_{\tilde{\alpha}}(\lambda, \xi) = \gamma$ and $\angle_{\tilde{\alpha}}(p, \xi) = \vartheta$, then

$$\frac{\left(\langle \lambda, \xi \rangle + \omega \frac{\langle p, \xi \rangle}{W^2}\right)^2}{|\xi|^2 - \frac{\langle p, \xi \rangle^2}{W^2}} = \frac{\left(W^2|\lambda| \cos\gamma + \omega |p| \cos\vartheta\right)^2}{W^2(1 + |p|^2 \sin^2\vartheta)}. \tag{43}$$

Case(i): If $\sin\vartheta \neq 0$, for the RHS of (43), the degree of $|p|$ in the numerator is less than or equal to 4, while the degree of $|p|$ in the denominator is equal to 4, then the RHS of (43) is bounded when $|p| \to \infty$.

Case(ii): If $\sin\vartheta = 0$, then p is parallel to ξ. It has two possibilities. If p and ξ are in the same direction, then $\cos\vartheta = 1$ and $\omega = \lambda_{n+1} - |\lambda| \cdot |p| \cos\gamma$. If p and ξ are in the opposite direction, then $\cos\vartheta = -1$ and $\omega = \lambda_{n+1} + |\lambda| \cdot |p| \cos\gamma$. By a direct computation, the RHS of (43) becomes

$$\frac{\left(W^2|\lambda| \cos\gamma + \omega |p| \cos\vartheta\right)^2}{W^2(1 + |p|^2 \sin^2\vartheta)} = \frac{\left(|\lambda| \cos\gamma + \lambda_{n+1} |p| \cos\vartheta\right)^2}{1 + |p|^2}. \tag{44}$$

Noticing the degree of $|p|$ in the numerator and denominator, the RHS of (44) is bounded when $|p| \to \infty$. This completes the proof of (42).

From the LHS of (42), we have

$$E_{ij}(x, z, p)\xi^i \xi^j \geq |\xi|^2 - \frac{\langle p, \xi \rangle^2}{W^2} = \frac{|\xi|^2}{W^2}\left(1 + |p|^2 \sin^2\vartheta\right) > 0$$

for any $\xi \neq 0$. Thus (41) is elliptic. We complete the proof. □

Theorem 5.3. *Let $f : \boldsymbol{P} \to (\mathbb{V}^{n+1}, \tilde{F}_b)$ be a graph over the hyperplane $\boldsymbol{P}$ which is perpendicular to $\tilde{\beta}^\sharp$ and given by (26). Define $\Phi_b(t)$ by (25). If there exists $t_o \in (0, b^2)$ such that $\Phi_b(t_o) = 0$, then*

$$u(x^1, \ldots, x^n) = \sqrt{\frac{t_o}{b^2 - t_o}} \sqrt{(x^1)^2 + \cdots + (x^n)^2}$$

is a BH-minimal (or HT-minimal) graph with an isolated singularity at the origin.

Proof. Form Theorem 4.3, the minimal equation is given by (36). For any cone

$$u(x^1,\dots,x^n)=k\sqrt{(x^1)^2+\cdots+(x^n)^2}$$

with constant k, a direct computation shows that

$$\sum_{ij}u_iu_ju_{ij}=0.$$

Note that $W^2=1+k^2$, (36) becomes

$$\Phi_b\left(b^2\frac{k^2}{1+k^2}\right)=0.$$

We have $t_o=b^2\frac{k^2}{1+k^2}$, then $k=\sqrt{\frac{t_o}{b^2-t_o}}$. We complete the proof. □

We now use Theorem 5.2 to establish the Bernstein type theorem in $(\mathbb{V}^3,\tilde{F}_b)$. It was proved in [4] that any complete HT-minimal graph in the 3-dimensional Minkowski space $(\mathbb{V}^3,\tilde{F})$ with any Minkowski metric $\tilde{F}$ is a plane. So we only consider BH-minimal surfaces. From Theorem 5.2 and using the theory in [11] we get

Theorem 5.4. *Let $\tilde{F}_b=\tilde{\alpha}\phi(s)$, $s=\tilde{\beta}/\tilde{\alpha}$, be a Minkowski (α,β)-metric, where $\tilde{\alpha}$ is an Euclidean metric and $\tilde{\beta}$ is a one form with length $b:=\|\tilde{\beta}\|_{\tilde{\alpha}}\in[0,b_o)$ such that $\tilde{F}_b$ is positive definite. Define $\Phi_b(t)$ by (3) and let*

$$\epsilon:=\sup\left\{\epsilon'\in[0,b_o)\mid\Phi_b(t)\neq0,\ \frac{\Phi_b'(t)}{\Phi_b(t)}\geq0,\ \forall\ t\in[0,b^2],\ \forall\ b\in[0,\epsilon')\right\}.$$

Then any BH-minimal surface in $(\mathbb{V}^3,\tilde{F}_b)$ with $b\in[0,\epsilon)$, which is the graph of a function defined on $\mathbb{R}^2$, is a plane.

6. Examples

In this section, we study BH-minimal surfaces in $(\mathbb{V}^3,\tilde{F}_b)$ for some (α,β)-metrics including Randers metric, Matsumoto metric and the two order metric.

Example 6.1 ([12], Randers metric). Let $f:\mathbb{R}^2\to\left(\mathbb{V}^3,\tilde{F}_b=\sqrt{\sum_{\alpha=1}^3(y^\alpha)^2}+by^3\right)$ be a graph over the x^1x^2-plane which is given by $f(x^1,x^2)=(x^1,x^2,u(x^1,x^2))$. By (2), we have $\sigma_{BH}(t)=(1-t)^{\frac{3}{2}}$. Then from (3) we get

$$\Phi_b(t)=(1-t)^{\frac{1}{2}}(1-3b^2+2t),\quad \Phi_b'(t)=\frac{3[1-t+(b^2-t)]}{2(1-t)^{\frac{1}{2}}}.$$

It can be proved from (9) that $\tilde{F}_b$ is positive definite if and only if $b \in [0,1)$ (cf. [9]). We can see that $\Phi_b'(t) \geq 0$ for any $t \in [0,b^2]$.

(i) If $0 \leq b < \frac{1}{\sqrt{3}}$, then $\frac{\Phi_b'(t)}{\Phi_b(t)} \geq 0$ for $\forall t \in [0,b^2]$. We conclude from Theorem 1.2 that any BH-minimal surface in such a Minkowski Randers space with $0 \leq b < \frac{1}{\sqrt{3}}$, which is the graph of a function defined on $\mathbb{R}^2$, is a plane.

(ii) If $\frac{1}{\sqrt{3}} < b < 1$, we have $\Phi_b(t_o) = 0$ for $t_o = \frac{3b^2-1}{2}$. By Theorem 5.3, the cone

$$u(x^1,x^2) = \sqrt{\frac{3b^2-1}{1-b^2}}\sqrt{(x^1)^2+(x^2)^2}$$

over $\Omega = \{x^3 = 0\} \setminus \{0\}$ is a BH-minimal surface. In particular, when $b = \frac{1}{\sqrt{2}}$, the cone $u(x^1,x^2) = \sqrt{(x^1)^2+(x^2)^2}$ is a BH-minimal graph. It is a BH-minimal graph with an isolated singularity at the origin.

Example 6.2 (Matsumoto metric). Let $f : \mathbb{R}^2 \to \left(\mathbb{V}^3, \tilde{F}_b = \frac{\tilde{\alpha}^2}{\tilde{\alpha}-\tilde{\beta}}\right)$ be a graph with $\tilde{\alpha} = \sqrt{\sum_{\alpha=1}^3 (y^\alpha)^2}$ and $\tilde{\beta} = by^3$ over the x^1x^2-plane. For Matsumoto metric, by (2), we have $\sigma_{BH}(t) = (2+t)^{-1}$. Then from (3) we get

$$\Phi_b(t) = \frac{2(1-b^2)+3t}{(2+t)^2}, \quad \Phi_b'(t) = \frac{2+b^2+3(b^2-t)}{(2+t)^3}.$$

It can be proved from (9) that $\tilde{F}_b$ is positive definite if and only if $b \in [0,\frac{1}{2})$. We can see $\frac{\Phi_b'(t)}{\Phi_b(t)} > 0$ for $\forall\, t \in [0,b^2]$. Then we conclude from Theorem 1.2 that any BH-minimal surface in such a Minkowski Matsumoto space, which is the graph of a function defined on $\mathbb{R}^2$, is a plane.

Example 6.3 (Two order metric). Let $f : \mathbb{R}^2 \to \left(\mathbb{V}^3, \tilde{F}_b = \frac{(\tilde{\alpha}+\tilde{\beta})^2}{\tilde{\alpha}}\right)$ be a graph with $\tilde{\alpha} = \sqrt{\sum_{\alpha=1}^3 (y^\alpha)^2}$ and $\tilde{\beta} = by^3$ over the x^1x^2-plane, which is given by $f(x^1,x^2) = (x^1,x^2,u(x^1,x^2))$. By (2), we have $\sigma_{BH}(t) = \frac{2(1-t)^{\frac{7}{2}}}{2+3t}$. Then from (3) we get

$$\Phi_b(t) = \frac{2(1-t)^{\frac{5}{2}}[2(1-10b^2)+3t(7-5b^2+4t)]}{(2+3t)^2},$$

$$\Phi_b'(t) = \frac{5(1-t)^{\frac{3}{2}}[(76+72t+27t^2)(b^2-t)+(1-t)(4+3t)(2+3t)]}{(2+3t)^3}.$$

It can be proved from (9) that $\tilde{F}_b$ is positive definite if and only if $b \in [0,1)$(cf. [9]). We can see that $\Phi_b'(t) \geq 0$ for any $t \in [0,b^2]$.

(i) If $0 \leq b < \frac{1}{\sqrt{10}}$, then $\frac{\Phi_b'(t)}{\Phi_b(t)} \geq 0$ for $\forall\, t \in [0,b^2]$. We conclude from Theorem 1.2 that any BH-minimal surface in such a Minkowski space $\big(\mathbb{V}^3, \tilde{F}_b =$

$\frac{(\tilde{\alpha}+\tilde{\beta})^2}{\tilde{\alpha}}$) with $0 \leq b < \frac{1}{\sqrt{10}}$, which is the graph of a function defined on $\mathbb{R}^2$, is a plane.

(ii) If $\frac{1}{\sqrt{10}} < b < 1$, then $\Phi_b(t_o) = 0$ for

$$t_o = \frac{1}{24}\big[15b^2 - 21 + \sqrt{15(15b^4 + 22b^2 + 23)}\,\big].$$

By Theorem 5.3, the cone

$$u(x^1, x^2) = \sqrt{\frac{15b^2 - 21 + \sqrt{15(15b^4 + 22b^2 + 23)}}{9b^2 + 21 - \sqrt{15(15b^4 + 22b^2 + 23)}}}\sqrt{(x^1)^2 + (x^2)^2}$$

defined over $\Omega = \{x^3 = 0\} \setminus \{0\}$ is a BH-minimal surface. In particular, when $b^2 = \frac{\sqrt{505}-19}{18}$, the cone $u(x^1, x^2) = \sqrt{(x^1)^2 + (x^2)^2}$ is a BH-minimal surface. It is a BH-minimal graph with an isolated singularity at the origin.

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