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A note on the exponential diophantine equation $(2^n - 1)(b^n - 1) = x^2$

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Abstract. Let b be a fixed positive integer with $b > 2$. In this paper, using some elementary methods, we prove that if $3 \mid b$, then the equation $(2^n - 1)(b^n - 1) = x^2$ has no positive integer solution (n, x).

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. Let a and b be fixed positive integers with $1 < a < b$. Recently, there were many works concerned the equation

$$(a^n - 1)(b^n - 1) = x^2, \quad n, x \in \mathbb{N} \tag{1}$$

(see [1], [2], [3], [4], [6]). In this paper we consider the case that $a = 2$. Then, equation (1) can be written as

$$(2^n - 1)(b^n - 1) = x^2, \quad n, x \in \mathbb{N}. \tag{2}$$

In this respect, L. Szalay [6] proved that if $b = 3$, then (2) has no solution (n, x). L. Hajdu and L. Szalay [3] proved that if $b = 6$, then (2) has no solution (n, x). In this paper we prove a general result as follows.

Theorem. *If* $3 \mid b$, *then* (2) *has no solution* (n, x).

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In addition, we notice that (2) has solutions for infinitely many b. In fact, (1) has solutions if b satisfies one of the following conditions.

(i) If $b = 1 + c^2$, where c is a positive integer with $c > 1$, then (2) has the solution $(n, x) = (1, c)$.

(ii) Let $\alpha = 2+\sqrt{3}$ and $\beta = 2-\sqrt{3}$. For any positive integer k, if $b = (\alpha^k+\beta^k)/2$, then (2) has the solution $(n, x) = (2, 3(\alpha^k - \beta^k)/2\sqrt{2}\,)$.

(iii) If $b = 4$ or 22, then (2) has the solutions $(n, x) = (3, 21)$ and $(n, x) = (3, 273)$, respectively.

By the above mentioned observations, we propose the following conjecture.

Conjecture. *Excepting the above cases* (i), (ii) *and* (iii), (2) *has no solution* (n, x).

2. Proof of Theorem

Let d be a positive integer which is not a square. It is a well known fact that the Pell equation

$$u^2 - dv^2 = 1, \quad u, v \in \mathbb{N} \tag{3}$$

has solution (u, v).

Lemma ([5, Lemma 3])**.** *Let* $(u, v) = (u_1, v_1)$ *denote the least solution of* (3)*. Then we have*

(i) *For any solution* (u, v) *of* (3)*, we have* $v_1 \mid v$.

(ii) *If* $(u, v) = (u', v')$ *is a solution of* (3) *such that* u' *is a power of* 2*, then* (u', v') *is the least solution of* (3).

Proof of Theorem. Let b be a positive integer with $3 \mid b$. If (2) has a solution (n, x), then we have

$$2^n - 1 = dy^2, \tag{4}$$

and

$$b^n - 1 = dz^2, \tag{5}$$

where d, y and z are positive integers satisfying $dyz = x$, and d is square free. Since $3 \mid b$, we see from (5) that $dz^2 \equiv -1 \equiv 2 \pmod 3$. It implies that $3 \nmid d$, $3 \nmid z$, $z^2 \equiv 1 \pmod 3$ and $d \equiv 2 \pmod 3$.

If $3 \nmid y$, then $y^2 \equiv 1 \pmod 3$. Further, since $d \equiv 2 \pmod 3$, we get $dy^2+1 \equiv d+1 \equiv 0 \pmod 3$. But, by (4), it is impossible. Therefore, we have

$$3 \mid y \tag{6}$$

Then, by (4), we get $2^n \equiv 1 \pmod 3$. It implies that n must be even.

Since $2 \mid n$, we see from (4) that the Pell equation (3) has the solution $(u, v) = (2^{n/2}, y)$. Therefore, by (ii) of Lemma, $(u_1, v_1) = (2^{n/2}, y)$ is the least solution of (3).

On the other hand, we find from (5) that (3) has an other solution $(u, v) = (b^{n/2}, z)$. By (i) of Lemma, we get $y \mid z$. Further, by (6), we obtain $3 \mid z$. But, since $3 \mid b$, it is impossible by (5). Thus, if $3 \mid b$, then (2) has no solution (n, x). The theorem is proved. □

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