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On the Weyl curvature of Deszcz

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Abstract. Geometrical characterizations are given for the $(0,6)$-tensor $R \cdot C$ and the $(0,6)$ Tachibana–Weyl tensor $Q(g,C) := -\wedge_g \cdot C$, whereby C denotes the $(0,4)$ Weyl conformal curvature tensor of a Riemannian manifold (M,g), R denotes the curvature operator acting on C as a derivation, and where the natural metrical endomorphism $\wedge_g$ also acts as a derivation on C. By comparison of these $(0,6)$-tensors $R \cdot C$ and $Q(g,C)$, a new scalar valued Riemannian curvature invariant $L_C(p,\pi,\overline{\pi})$ is determined on (M,g), called the Weyl curvature of Deszcz, which in general depends on two tangent 2-planes π and $\overline{\pi}$ at the same point p, and of which the isotropy determines that M is Weyl pseudo-symmetric in the sense of Deszcz.

1. Introduction

Recently, the *parallel transport* of Riemann curvatures and Ricci curvatures on a (semi-)Riemannian manifold (M,g) around *infinitesimal co-ordinate parallelograms* was studied in [13] and [14]. There, amongst others, new geometrical interpretations of the $(0,6)$ curvature tensor $R \cdot R$, whereby the first R stands for the *curvature operator* acting as a derivation on the second R which stands for the $(0,4)$ *Riemann–Christoffel curvature tensor*, of the $(0,6)$ *Tachibana tensor* $Q(g,R) := -\wedge_g \cdot R$, whereby the *metrical endomorphism* $\wedge_g$ also acts as a derivation on the $(0,4)$ Riemann–Christoffel curvature tensor, as well as of the $(0,4)$ curvature tensor $R \cdot S$, whereby S denotes the $(0,2)$ *Ricci tensor* and of the

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Tachibana-Ricci tensor $Q(g, S) := -\wedge_g \cdot S$ are given. By comparison of the $(0, 6)$ tensors $R \cdot R$ and $Q(g, R)$, a new scalar valued Riemannian invariant curvature function was determined on (M, g), the so-called *double sectional curvature* or *the sectional curvature of Deszcz* $L_R(p, \pi, \overline{\pi})$, which depends on two tangent 2-planes π and $\overline{\pi}$ at any point p of M. The manifolds (M, g) for which the sectional curvature of Deszcz is isotropic, i.e., does not depend on the planes at p, but remains a scalar valued function which at most depends only on the points of M, are the manifolds which are *pseudo-symmetric in the sense of Deszcz* (see e.g. [7], [17]). And similarly, by comparison of the $(0, 4)$-tensors $R \cdot S$ and $Q(g, S)$, another new scalar valued Riemannian invariant curvature function was determined on (M, g), the so-called *Ricci curvature of Deszcz*, $L_S(p, d, \overline{\pi})$, which depends on a tangent direction d and a tangent plane $\overline{\pi}$ at any point p of M. The manifolds (M, g) for which the Ricci curvature of Deszcz is isotropic, i.e., depends at most only on the points of M, are the manifolds which are *Ricci pseudo-symmetric in the sense of Deszcz* (see e.g. [7], [8], [14]).

In the present article, we basically make a similar study concerning the $(0, 4)$ Weyl conformal curvature tensor C on a manifold (M, g) of dimension $n \geq 4$. New geometrical interpretations of the $(0, 6)$-tensors $R \cdot C$ and $Q(g, C) := -\wedge_g \cdot C$ are given, in particular thus characterizing the *Weyl semi-symmetric spaces* ($R \cdot C = 0$) and the *conformally flat spaces* ($C = 0$). Then, the *conformal sectional curvature of Deszcz* or the *Weyl curvature of Deszcz*, $L_C(p, \pi, \overline{\pi})$, is defined. This scalar curvature invariant $L_C(p, \pi, \overline{\pi})$ is isotropic with respect to both planes π and $\overline{\pi}$ at all points p of M if and only if the manifold is *Weyl pseudo-symmetric in the sense of Deszcz* (see e.g. [5], [6], [7]). For dimension $n = 3$, C vanishes identically and therefore hereafter we always assume $n \geq 4$. Further, we recall that when $n \geq 5$, a Riemannian manifold M is pseudo-symmetric if and only if it is Weyl pseudo-symmetric, but that for $n = 4$ the class of Weyl pseudo-symmetric spaces is essentially larger than the class of pseudo-symmetric spaces as shown in [5].

2. A geometrical interpretation of $R \cdot C$

In an n-dimensional ($n \geq 4$) Riemannian manifold M with metric tensor g, let ∇ denote the Levi–Civita connection. Then, the $(1, 1)$-curvature operator $\mathcal{R}(X, Y)$, the $(0, 4)$ curvature tensor R, the $(0, 2)$ Ricci tensor S and the scalar curvature τ of (M, g) are respectively given by:

$$\mathcal{R}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]},$$

$$R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3,X_4),$$
$$S(X,Y) = \sum_{i=1}^{n} R(E_i,X,Y,E_i),, \quad \tau = \sum_{j=1}^{n} S(E_j,E_j), \tag{1}$$

whereby $\{E_1,E_2,\dots,E_n\}$ denotes any local orthonormal frame field on M, $[.,.]$ denotes the Lie bracket of vector fields and X_1, X_2, X_3, X_4, X, Y denote any tangent vector fields on M. And, for $n \geq 4$, the $(0,4)$ Weyl conformal curvature tensor C is then given by

$$\begin{aligned} C(X_1,X_2,X_3,X_4) &:= R(X_1,X_2,X_3,X_4) \\ &+ \frac{1}{n-2}\big\{g(X_1,X_3)S(X_2,X_4) + g(X_2,X_4)S(X_1,X_3) \\ &- g(X_1,X_4)S(X_2,X_3) - g(X_2,X_3)S(X_1,X_4)\big\} \\ &+ \frac{\tau}{(n-1)(n-2)}\big\{g(X_1,X_4)g(X_2,X_3) - g(X_1,X_3)g(X_2,X_4)\big\}. \end{aligned} \tag{2}$$

The $(0,6)$-tensor $R \cdot C$ is obtained by the action of the curvature operator $\mathcal{R}$ as a derivation on the $(0,4)$ Weyl conformal curvature tensor C:

$$\begin{aligned} (R\cdot C)(X_1,X_2,X_3,X_4;X,Y) &= (\mathcal{R}(X,Y)\cdot C)(X_1,X_2,X_3,X_4) \\ &= -C(\mathcal{R}(X,Y)X_1,X_2,X_3,X_4) - C(X_1,\mathcal{R}(X,Y)X_2,X_3,X_4) \\ &\quad - C(X_1,X_2,\mathcal{R}(X,Y)X_3,X_4) - C(X_1,X_2,X_3,\mathcal{R}(X,Y)X_4). \end{aligned} \tag{3}$$

Now let $\mathcal{P}$ be any co-ordinate parallelogram on the manifold M cornered at the point p for which the co-ordinate values x and y at p change along the sides by amounts Δx and Δy (Figure 1). Let $\vec{x} = \frac{\partial}{\partial x}|_p$ and $\vec{y} = \frac{\partial}{\partial y}|_p$ be the natural tangent vectors at p of the x and y co-ordinate lines, respectively.

Then, as is well known and which goes back to Schouten in 1918 [16], after parallel transport of any vector $\vec{z}$ at p all around an infinitesimal co-ordinate parallelogram $\mathcal{P}$ (Figure 2), the resulting vector $\vec{z}^{\star}$ at p is given by

$$\vec{z}^{\star} = \vec{z} + [\mathcal{R}(\vec{x},\vec{y})\vec{z}]\,\Delta x\Delta y + O^{>2}(\Delta x,\Delta y). \tag{4}$$

For any plane π tangent to M at p the Weyl sectional curvature or, in short, the Weyl curvature, $K_C(p,\pi)$, is given by

$$K_C(p,\pi) = C(\vec{v},\vec{w},\vec{w},\vec{v}), \tag{5}$$

whereby $\vec{v}$ and $\vec{w}$ is any pair of orthonormal tangent vectors at p spanning $\pi = \vec{v} \wedge \vec{w}$. Since C is a curvature tensor, similarly as shown by Cartan for the Riemann–Christoffel tensor R and the Riemann sectional curvatures K, *the knowledge of the*

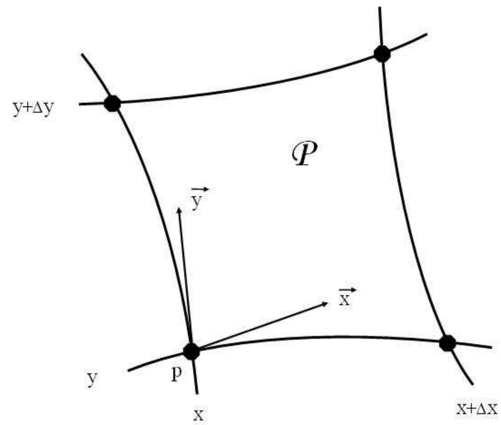

Figure 1. A co-ordinate parallelogram

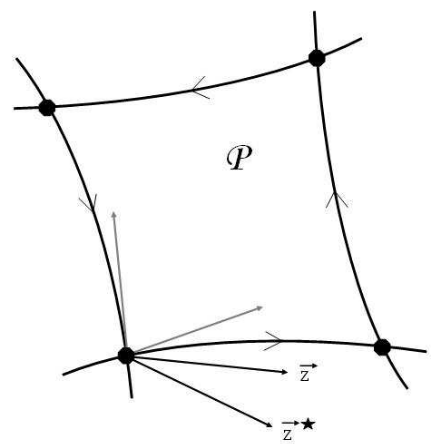

Figure 2. Parallel transport of a vector around a co-ordinate parallelogram

"full" tensor C is equivalent to the knowledge of the Weyl sectional curvatures K_C. Using (2), the Weyl sectional curvature of a plane $\pi = \vec{v} \wedge \vec{w}$ at $p \in M$ can be expressed in terms of the Riemann sectional curvature $K(p, \pi) = R(\vec{v}, \vec{w}, \vec{w}, \vec{v})$ and of the Ricci curvatures of the directions d and $\overline{d}$ corresponding with the vectors $\vec{v}$ and $\vec{w}$, i.e., $\mathrm{Ric}(p, d) = S(\vec{v}, \vec{v})$, $\mathrm{Ric}(p, \overline{d}) = S(\vec{w}, \vec{w})$, as follows,

$$K_C(p, \pi) = K(p, \pi) - \frac{1}{n-2}\big\{\mathrm{Ric}(p, d) + \mathrm{Ric}(p, \overline{d})\big\} + \frac{\tau}{(n-1)(n-2)}.$$

By the metrical character of the Levi–Civita connection ∇, in particular, any pair of orthonormal vectors $\vec{v}$ and $\vec{w}$ at p after parallel transport around any co-ordinate parallelogram $\mathcal{P}$ yields again a pair of orthonormal vectors $\vec{v}^\star$ and $\vec{w}^\star$ at p. These vectors span the plane $\pi^\star = \vec{v}^\star \wedge \vec{w}^\star$ which is the parallel transported plane around $\mathcal{P}$ of the plane $\pi = \vec{v} \wedge \vec{w}$ (Figure 3).

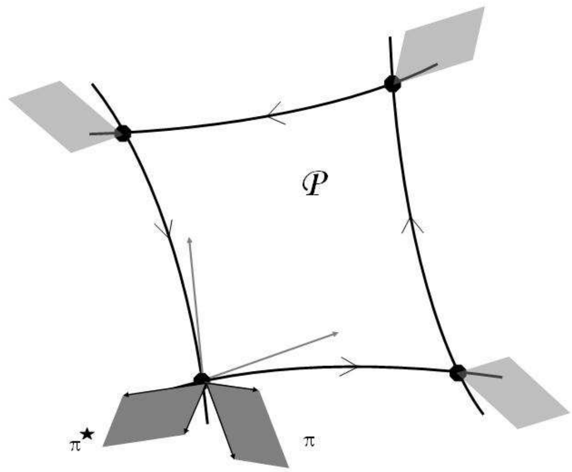

Figure 3. Parallel transport of a plane around a co-ordinate parallelogram

Hence, by (3), (4) and (5) and by the fact that C is a curvature tensor, it follows that

$$\begin{aligned} K_C(p,\pi^\star) &= C(\vec{v}^\star, \vec{w}^\star, \vec{w}^\star, \vec{v}^\star) \\ &= K_C(p,\pi) - [(R\cdot C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})]\,\Delta x \Delta y + O^{>2}(\Delta x, \Delta y). \end{aligned}$$

We recall that a Riemannian manifold is said to be *Weyl semi-symmetric* if $R \cdot C = 0$. Then, denoting by $\Delta^\star_\pi K_C(p,\pi) = K_C(p,\pi) - K_C(p,\pi^\star)$ the change in Weyl sectional curvature $K_C(p,\pi)$ under the parallel transport of the plane π around an infinitesimal parallelogram $\mathcal{P}$, we can formulate the following.

Theorem 1. *In second order approximation, the tensor $R{\cdot}C$ of a Riemannian manifold (of dimension ≥ 4) measures the change of the Weyl sectional curvature $K_C(p,\pi)$ of a plane $\pi = \vec{v} \wedge \vec{w}$ at any point p under parallel transport around any infinitesimal co-ordinate parallelogram $\mathcal{P}$ cornered at p and tangent to $\vec{x}$ and $\vec{y}$, i.e.,*

$$\Delta^\star_\pi K_C(p,\pi) \approx [(R\cdot C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})]\,\Delta x \Delta y, \tag{6}$$

where $\overline{\pi}$ is the tangent plane at p spanned by $\vec{x}$ and $\vec{y}$.

Corollary 2. *A Riemannian manifold (of dimension ≥ 4) is Weyl semi-symmetric if and only if, up to second order, the Weyl sectional curvature for any 2-plane π at any point p is invariant under the parallel transport of π around any infinitesimal co-ordinate parallelogram $\mathcal{P}$ cornered at p.*

The next properties readily follow from the algebraic symmetries of the Weyl tensor C.

Lemma 3. *The tensor $R\cdot C$ has the following algebraic symmetry properties:*

(i)
$$\begin{aligned}(R\cdot C)(X_1,X_2,X_3,X_4;X,Y) &= -(R\cdot C)(X_2,X_1,X_3,X_4;X,Y)\\ &= -(R\cdot C)(X_1,X_2,X_4,X_3;X,Y) = (R\cdot C)(X_3,X_4,X_1,X_2;X,Y)\\ &= -(R\cdot C)(X_1,X_2,X_3,X_4,Y,X),\end{aligned}$$

(ii)
$$\begin{aligned}(R\cdot C)(X_1,X_2,X_3,X_4;X,Y) + (R\cdot C)(X_1,X_3,X_4,X_2;X,Y)&\\ +(R\cdot C)(X_1,X_4,X_2,X_3;X,Y) = 0.&\end{aligned}$$

3. On the Tachibana–Weyl tensor

The simplest $(0,6)$-tensor on a Riemannian manifold which has the same algebraic symmetry properties as the $(0,6)$-tensor $R\cdot C$ may well be the $(0,6)$-tensor $Q(g,C) := -\wedge_g\cdot C$, defined by the action as a derivation on C of the metrical endomorphism $X\wedge_g Y$. This endomorphism is defined by sending a tangent vector field Z to the tangent vector field given by

$$(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y.$$

Then,

$$\begin{aligned}Q(g,C)(X_1,X_2,X_3,X_4;X,Y) &:= -\left[(X\wedge_g Y)\cdot C\right](X_1,X_2,X_3,X_4)\\ &= C((X\wedge_g Y)X_1,X_2,X_3,X_4) + C(X_1,(X\wedge_g Y)X_2,X_3,X_4)\\ &+ C(X_1,X_2,(X\wedge_g Y)X_3,X_4) + C(X_1,X_2,X_3,(X\wedge_g Y)X_4).\end{aligned}$$

By analogy with the action of the natural metrical endomorphism as a derivation on the $(0,4)$ curvature tensor R, i.e., $Q(g,R) := -\wedge_g\cdot R$, which is called the Tachibana tensor of the Riemannian manifold (M,g), we will call $Q(g,C) := -\wedge_g\cdot C$ the *Tachibana–Weyl tensor* of (M,g).

Concerning the geometrical meaning of this tensor we first state the following.

Theorem 4. *A Riemannian manifold (M, g) of dimension $n \geq 4$ is conformally flat if and only if its Tachibana–Weyl tensor vanishes identically.*

Proof. By a classical result of Weyl, a Riemannian manifold of dimension ≥ 4 is conformally flat if and only if its conformal curvature tensor C vanishes identically [18]. And, of course, $C \equiv 0$ automatically implies that $Q(g, C) \equiv 0$.

Conversely, if $Q(g, C) \equiv 0$ we need to show that $C \equiv 0$. Algebraically, just like the fact that $Q(g, R) \equiv 0$ implies that the sectional curvature K of the Riemannian manifold (M, g) is constant (see e.g. [9]), $Q(g, C) \equiv 0$ straightforwardly implies that K_C is constant. And, since the trace of C is zero, the result follows. □

A different kind of geometrical meaning of the tensor $Q(g, C)$ corresponds somewhat to the one given in Theorem 1 for the tensor $R \cdot C$. It is related to the geometrical meaning of the endomorphism $\wedge_g$ according to which

$$\widetilde{\vec{z}} = \vec{z} - [(\vec{x} \wedge_g \vec{y})\vec{z}]\,\Delta\varphi + O^{>1}(\Delta\varphi),$$

whereby $\widetilde{\vec{z}}$ is the vector obtained from a tangent vector $\vec{z}$ at p after the rotation over an angle $\Delta\varphi$ of the projection of $\vec{z}$ on $\overline{\pi} = \vec{x} \wedge \vec{y}$, while keeping the component of $\vec{z}$ perpendicular to $\overline{\pi}$ fixed (Figure 4) [13].

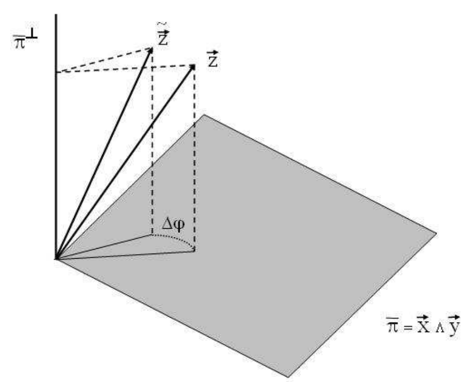

Figure 4. A geometrical interpretation of the vector $(\vec{x} \wedge_g \vec{y})\vec{z}$

Now, consider at p any two orthonormal vectors $\vec{v}$ and $\vec{w}$ and let $\widetilde{\vec{v}}$ and $\widetilde{\vec{w}}$ be the vectors obtained from $\vec{v}$ and $\vec{w}$ after such a rotation over an infinitesimal angle $\Delta\varphi$ of the projections of $\vec{v}$ and $\vec{w}$ on $\overline{\pi} = \vec{x} \wedge \vec{y}$. Comparing the Weyl sectional curvatures of the planes $\pi = \vec{v} \wedge \vec{w}$ and $\widetilde{\pi} = \widetilde{\vec{v}} \wedge \widetilde{\vec{w}}$, we find that

$$K_C(p, \widetilde{\pi}) = K_C(p, \pi) - [Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})]\, \Delta\varphi + O^{>1}(\Delta\varphi).$$

Then, denoting by $\widetilde{\Delta_{\overline{\pi}}} K_C(p, \pi) = K_C(p, \pi) - K_C(p, \widetilde{\pi})$ the change in Weyl sectional curvature $K_C(p, \pi)$ under the above kind of rotations over an infinitesimal angle $\Delta\varphi$, we can formulate the following.

Theorem 5. *In first order approximation, the Tachibana–Weyl tensor $Q(g, C)$ of a Riemannian manifold (of dimension ≥ 4) measures the change of the Weyl sectional curvature $K_C(p, \pi)$ of a plane $\pi = \vec{v} \wedge \vec{w}$ at any point p under an infinitesimal rotation over an angle $\Delta\varphi$ of the projections of $\vec{v}$ and $\vec{w}$ on $\overline{\pi} = \vec{x} \wedge \vec{y}$, i.e.,*

$$\widetilde{\Delta_{\overline{\pi}}} K_C(p, \pi) \approx [Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})]\, \Delta\varphi. \tag{7}$$

4. Definition and properties of the Weyl curvature of Deszcz

Let (M, g) be an $n(\geq 4)$-dimensional Riemannian manifold which is not conformally flat and denote by $\mathcal{U}_C$ the set of points where the Tachibana–Weyl tensor $Q(g, C)$ is not identically zero, i.e., $\mathcal{U}_C = \{p \in M \mid Q(g, C)_p \neq 0\}$. Then, at a point $p \in \mathcal{U}_C$, a plane $\pi = \vec{v} \wedge \vec{w}$ is said to be *Weyl curvature-dependent* with respect to a plane $\overline{\pi} = \vec{x} \wedge \vec{y}$ when $Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) \neq 0$. This definition is independent of the choice of bases for π and $\overline{\pi}$.

At a point $p \in \mathcal{U}_C$, let a plane $\pi = \vec{v} \wedge \vec{w}$ be Weyl curvature-dependent with respect to a plane $\overline{\pi} = \vec{x} \wedge \vec{y}$. Then, we define the *Weyl curvature of Deszcz* of the planes π and $\overline{\pi}$ at the point p as the scalar

$$L_C(p, \pi, \overline{\pi}) = \frac{(R \cdot C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})}{Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})}.$$

This definition is again independent of the choice of bases for the planes π and $\overline{\pi}$.

Theorem 6. *At any point $p \in \mathcal{U}_C$, the tensor $R \cdot C$ of a Riemannian manifold M is completely determined by the Weyl curvatures of Deszcz $L_C(p, \pi, \overline{\pi})$.*

Proof. Assume there exists a $(0, 6)$-tensor W with the same algebraic symmetries as $R \cdot C$ and so that for any two Weyl curvature-dependent planes $\pi = \vec{v} \wedge \vec{w}$ and $\overline{\pi} = \vec{x} \wedge \vec{y}$ at p,

$$\frac{(R \cdot C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})}{Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})} = \frac{W(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})}{Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})}.$$

We have to prove that $\forall \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4, \vec{x}_5, \vec{x}_6 \in T_pM$,

$$(R \cdot C)(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6) = W(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6).$$

Let T be the $(0,6)$-tensor $T = R \cdot C - W$. Obviously, T has the same algebraic symmetries as $R \cdot C$ and W. Further, for every pair of Weyl curvature-dependent planes $\pi = \vec{v} \wedge \vec{w}$ and $\overline{\pi} = \vec{x} \wedge \vec{y}$,

$$T(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0. \tag{8}$$

When two planes $\pi = \vec{v} \wedge \vec{w}$ and $\overline{\pi} = \vec{x} \wedge \vec{y}$ are not Weyl curvature-dependent there holds that $Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0$. Using the following argument from [10] we show that also in this case $T(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0$. Namely, since $Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y})$ is a polynomial in the components of $\vec{v}, \vec{w}, \vec{x}$ and $\vec{y}$, the zero set does not contain any open subset (for otherwise $Q(g, C)_p \equiv 0$, which would be a contradiction with $p \in \mathcal{U}_C$). We can then choose series of tangent vectors $\vec{v}_l \to \vec{v}$, $\vec{w}_l \to \vec{w}$, $\vec{x}_l \to \vec{x}$ and $\vec{y}_l \to \vec{y}$ such that for any l, $\vec{v}_l \wedge \vec{w}_l$ is Weyl curvature-dependent with respect to $\vec{x}_l \wedge \vec{y}_l$. We have for every l that $T(\vec{v}_l, \vec{w}_l, \vec{w}_l, \vec{v}_l; \vec{x}_l, \vec{y}_l)=0$ and thus in the limit we find that also $T(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0$. Hence, (8) holds for all $\vec{v}, \vec{w}, \vec{x}, \vec{y} \in T_pM$. Using polarization and the symmetric properties of $R \cdot C$ completes the proof. □

Corollary 7. *The Weyl semi-symmetric spaces are characterized by the vanishing of the Weyl curvatures of Deszcz.*

A Riemannian manifold M $(n \geq 4)$ is said to be Weyl pseudo-symmetric at a point $p \in \mathcal{U}_C$ if there exists a scalar $L_C(p)$ such that,

$$R \cdot C\,|_p = L_C(p)\, Q(g, C)\,|_p\ .$$

The manifold (M, g) is called *Weyl pseudo-symmetric in the sense of Deszcz* if it is Weyl pseudo-symmetric at every point of $\mathcal{U}_C \subset M$.

Theorem 8. *A Riemannian manifold (M, g) $(n \geq 4)$ is Weyl pseudo-symmetric in the sense of Deszcz if and only if at all of its points $p \in \mathcal{U}_C$ all the Weyl curvatures of Deszcz are the same, i.e., for all Weyl curvature-dependent planes π and $\overline{\pi}$ at p, $L_C(p, \pi, \overline{\pi}) = L_C(p)$.*

Proof. If $R \cdot C\,|_p = L_C(p)\, Q(g, C)\,|_p$ at p, the statement is obvious. So assume that $L_C(p, \pi, \overline{\pi}) = L_C(p)$ for every two Weyl curvature-dependent planes π and $\overline{\pi}$. Then,

$$(R \cdot C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = L_C(p)\, Q(g, C)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}).$$

The tensor $T = R \cdot C - L_C\, Q(g, C)$ has the same algebraic symmetries as $R \cdot C$. For two Weyl curvature-dependent planes $\pi = \vec{v} \wedge \vec{w}$ and $\overline{\pi} = \vec{x} \wedge \vec{y}$, there holds that $T(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, \vec{y}) = 0$. If both planes are not Weyl curvature-dependent, an argument as in the proof of Theorem 6 shows that $T(\vec{x}_1, \vec{x}_2, \vec{x}_2, \vec{x}_1; \vec{x}_5, \vec{x}_6) = 0$, $\forall \vec{x}_1, \vec{x}_2, \vec{x}_5, \vec{x}_6 \in T_pM$, and by polarization it then follows that

$$T(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6) = 0, \quad \forall \vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4; \vec{x}_5, \vec{x}_6 \in T_pM. \qquad \square$$

5. Pseudo-symmetry and the squaroids of Levi–Civita

Next, we give a geometrical interpretation of the Weyl curvature of Deszcz L_C in terms of the *squaroids of Levi–Civita* (see e.g. [2], [9], [13, 15]). Starting from any two tangent vectors $\vec{v}$ and $\vec{w}$ at any point p of M, Levi–Civita constructed in 1917 his so-called *parallelogramoids* as follows. Consider through p the geodesic α with tangent $\vec{v}$ and let q be the point on this geodesic at an infinitesimal distance A from p. Denote by $\vec{w}^\star$ the vector obtained after parallel transport of $\vec{w}$ from p to q along α. Then, through p and q consider the geodesics β_p and β_q which are tangent to $\vec{w}$ and $\vec{w}^\star$, respectively. Fix on them the points $\overline{p}$ and $\overline{q}$ at a same infinitesimal distance B from p and q, respectively. The parallelogramoid cornered at p with sides tangent to $\vec{v}$ and $\vec{w}$ is then completed by the geodesic $\overline{\alpha}$ through $\overline{p}$ and $\overline{q}$. Let A' be the geodesic distance between $\overline{p}$ and $\overline{q}$. Levi–Civita showed that, in first order approximation, the sectional curvature $K(p, \pi)$ of the plane $\pi = \vec{v} \wedge \vec{w}$ can be expressed as

$$K(p, \pi) \approx \frac{A^2 - A'^2}{A^2 B^2 \sin^2(\theta)}.$$

whereby θ is the angle between the vectors $\vec{v}$ and $\vec{w}$.

Let $\vec{v}$ and $\vec{w}$ be orthonormal vectors at any point $p \in M$. Consider the Levi–Civita squaroid based on $\vec{v}$ and $\vec{w}$ with side ε, i.e., the parallelogramoid for which $A = B = \varepsilon$ (Figure 5). Then, when ε' is the length of the closing geodesic, the sectional curvature $K(p, \pi)$ is given by

$$K(p, \pi) \approx \frac{\varepsilon^2 - \varepsilon'^2}{\varepsilon^4}.$$

Consider at $p \in M$ an orthonormal basis $\{\vec{v} = \vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_n\}$ of T_pM and construct for every plane $\vec{v} \wedge \vec{e}_j$ $(j \neq 1)$ the squaroid of Levi–Civita, all with the same sides ε. Let us denote the lengths of the completing geodesics by ε'_j. The

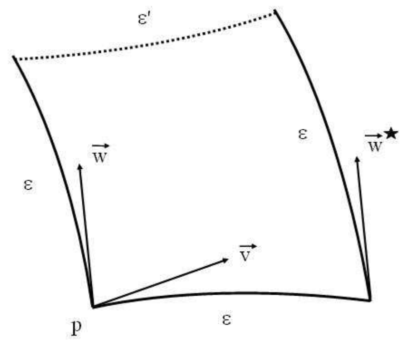

Figure 5. A squaroid of Levi–Civita

Ricci curvatures $\mathrm{Ric}(p,d)$, with d the direction of the vector $\vec{v}$, can then, up to first order approximation, be expressed as

$$\mathrm{Ric}(p,d) \approx \sum_{j\neq 1} \frac{\varepsilon^2 - \varepsilon_j'^2}{\varepsilon^4}.$$

Now, consider two planes $\pi = \vec{v} \wedge \vec{w}$ and $\overline{\pi} = \vec{x} \wedge \vec{y}$ at a point p of M and parallely transport the frame $\{\vec{v} = \vec{e}_1, \vec{w} = \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_n\}$ to $\{\vec{v}^\star = \vec{e}_1^\star, \vec{w}^\star = \vec{e}_2^\star, \vec{e}_3^\star, \ldots, \vec{e}_n^\star\}$ around an infinitesimal co-ordinate parallelogram $\mathcal{P}$. We construct for every plane $\vec{v}^\star \wedge \vec{e}_j^\star$ $(j \neq 1)$ and for every plane $\vec{w}^\star \wedge \vec{e}_k^\star$ $(k \neq 2)$, the squaroids of Levi–Civita, all with the same sides ε and denote the lengths of the completing geodesics by $\varepsilon_j^{\star\prime}$ and $\varepsilon_k^{\star\prime}$, respectively.
Then, according to the formulas for the tensors $R\cdot R$ and $R \cdot S$ which are analogous to formula (6) for the tensor $R \cdot C$ [13], [14], we find, up to second order approximation with respect to Δx and Δy, that

$$(R\cdot R)(\vec{v}, \vec{w}, \vec{w}, \vec{v}; \vec{x}, vecy) \approx \frac{(\varepsilon_2^{\star\prime})^2 - (\varepsilon_2')^2}{\varepsilon^4} \, \frac{1}{\Delta x \Delta y},$$

and

$$(R\cdot S)(\vec{v}, \vec{v}; \vec{x}, \vec{y}) \approx \sum_{j\neq 1} \frac{(\varepsilon_j^{\star\prime})^2 - (\varepsilon_j')^2}{\varepsilon^4} \, \frac{1}{\Delta x \Delta y}.$$

Let $\{\widetilde{\vec{v}} = \widetilde{\vec{e}}_1, \widetilde{\vec{w}} = \widetilde{\vec{e}}_2, \widetilde{\vec{e}}_3, \ldots, \widetilde{\vec{e}}_n\}$ be the frame which is obtained after an infinitesimal rotation as before of the frame $\{\vec{v} = \vec{e}_1, \vec{w} = \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_n\}$ with respect to the plane $\overline{\pi} = \vec{x} \wedge \vec{y}$, and construct for every plane $\widetilde{\vec{v}} \wedge \widetilde{\vec{e}}_j$ $(j \neq 1)$ and for every

plane $\widetilde{\vec{w}} \wedge \widetilde{\vec{e}}_k$ $(k \neq 2)$ the squaroids of Levi–Civita, all with the same side ε, and denote the lengths of the completing geodesics by $\widetilde{\varepsilon}'_j$ and $\widetilde{\varepsilon}'_k$, respectively.
In this case, according to the formulas for the Tachibana tensors $Q(g, R)$ and $Q(g, S)$ which are analogous to formula (7) for the Tachibana–Weyl tensor $Q(g, C)$, we find, up to first order with respect to the angle $\Delta\varphi$ of infinitesimal rotation, that

$$Q(g,R)(\vec{v},\vec{w},\vec{w},\vec{v};\vec{x},\vec{y}) \approx \frac{(\widetilde{\varepsilon}'_2)^2 - (\varepsilon'_2)^2}{\varepsilon^4}\,\frac{1}{\Delta\varphi},$$

and

$$Q(g,S)(\vec{v},\vec{v};\vec{x},\vec{y}) \approx \sum_{j\neq 1} \frac{(\widetilde{\varepsilon}'_j)^2 - (\varepsilon'_j)^2}{\varepsilon^4}\,\frac{1}{\Delta\varphi}.$$

We recall from [13], [14] that a plane $\pi = \vec{v} \wedge \vec{w}$ is said to be *curvature-dependent* with respect to a plane $\overline{\pi} = \vec{x} \wedge \vec{y}$ if $Q(g,R)(\vec{v},\vec{w},\vec{w},\vec{v};\vec{x},\vec{y}) \neq 0$, and that a direction d spanned by a vector $\vec{v}$ is *Ricci curvature-dependent* with respect to a plane $\overline{\pi} = \vec{x} \wedge \vec{y}$ if $Q(g,S)(\vec{v},\vec{v};\vec{x},\vec{y}) \neq 0$. These definitions are independent of the choices of bases for the planes π and $\overline{\pi}$ and the direction d, respectively.
Then, the *sectional curvature of Deszcz* $L_R(p,\pi,\overline{\pi})$ of the plane π which is curvature-dependent with respect to $\overline{\pi}$ at $p \in \mathcal{U}_R = \{x \in M \mid Q(g,R)_x \neq 0\}$, and the *Ricci curvature of Deszcz* $L_S(p,d,\overline{\pi})$ of the direction d which is Ricci curvature-dependent with respect to the plane $\overline{\pi}$ at a point $p \in \mathcal{U}_S = \{x \in M \mid Q(g,S)_x \neq 0\}$, can respectively be expressed as

$$L_R(p,\pi,\overline{\pi}) \approx \frac{(\varepsilon^{\star\prime}_2)^2 - (\varepsilon'_2)^2}{(\widetilde{\varepsilon}'_2)^2 - (\varepsilon'_2)^2}\,\frac{\Delta\varphi}{\Delta x \Delta y},$$

and

$$L_S(p,d,\overline{\pi}) \approx \frac{\sum_{j\neq 1}\left[(\varepsilon^{\star\prime}_j)^2 - (\varepsilon'_j)^2\right]}{\sum_{k\neq 1}\left[(\widetilde{\varepsilon}'_k)^2 - (\varepsilon'_k)^2\right]}\,\frac{\Delta\varphi}{\Delta x \Delta y}.$$

Because the tensor $R \cdot C$ can be written in terms of the tensors $R \cdot R$ and $R \cdot S$ as,

$$\begin{aligned}(R\cdot C)(X_1,X_2,X_3,X_4;X,Y) &= (R\cdot R)(X_1,X_2,X_3,X_4;X,Y)\\ &- \frac{1}{n-2}\{g(X_1,X_4)(R\cdot S)(X_2,X_3;X,Y) + g(X_2,X_3)(R\cdot S)(X_1,X_4;X,Y)\\ &- g(X_1,X_3)(R\cdot S)(X_2,X_4;X,Y) - g(X_2,X_4)(R\cdot S)(X_1,X_3;X,Y)\},\end{aligned}$$

the Weyl curvature of Deszcz $L_C(p,\pi,\overline{\pi})$ of the plane π which is Weyl curvature-dependent with respect to the plane $\overline{\pi}$ at a point $p \in \mathcal{U}_C$ can be expressed as

$$L_C(p,\pi,\overline{\pi}) \approx \frac{(\varepsilon^{\star\prime}_2)^2 - (\varepsilon'_2)^2 - \frac{1}{n-2}\Big[\sum_{j\neq 1}\Big((\varepsilon^{\star\prime}_j)^2 - (\varepsilon'_j)^2\Big) + \sum_{k\neq 2}\big((\varepsilon^{\star\prime}_k)^2 - (\varepsilon'_k)^2\big)\Big]}{(\widetilde{\varepsilon}'_2)^2 - (\varepsilon'_2)^2}\,\frac{\Delta\varphi}{\Delta x \Delta y}.$$

Thus, calibrating the changes of the Riemann, Ricci and Weyl curvatures under parallel translation ($\star$) around a parallelogram $\mathcal{P}$ with infinitesimal parameter growths Δx and Δy by the changes of the same curvatures under rotation ($\sim$) over an infinitesimal angle $\Delta\varphi = \Delta x \Delta y$ with respect to $\overline{\pi}$, we find the following approximate geometrical expressions in terms of the squaroids of Levi–Civita of sides ε, for, respectively, the Riemann sectional curvature of Deszcz L_R, the Ricci curvature of Deszcz L_S and the Weyl curvature of Deszcz L_C.

Theorem 9. *Let (M,g) be a Riemannian manifold, $p \in \mathcal{U}_R$ and $\pi \subset T_pM$ curvature-dependent with respect to a tangent plane $\overline{\pi} \subset T_pM$. Under the above calibration of infinitesimal parallel translation by associated infinitesimal rotation, the sectional curvature of Deszcz $L_R(p,\pi,\overline{\pi})$ can be expressed in terms of the lengths of closing sides in squaroids of Levi–Civita as follows:*

$$L_R(p,\pi,\overline{\pi}) \approx \frac{(\varepsilon_2^{\star\prime})^2 - (\varepsilon_2^{\prime})^2}{(\widetilde{\varepsilon}_2^{\prime})^2 - (\varepsilon_2^{\prime})^2}.$$

Theorem 10. *Let (M,g) be a Riemannian manifold, $p \in \mathcal{U}_S$ and d a tangent direction which is Ricci curvature-dependent with respect to a tangent plane $\overline{\pi} \subset T_pM$. Under the above calibration of infinitesimal parallel translation by associated infinitesimal rotation, the Ricci curvature of Deszcz $L_S(p,d,\overline{\pi})$ can be expressed in terms of the lengths of closing sides in squaroids of Levi–Civita as follows:*

$$L_S(p,d,\overline{\pi}) \approx \frac{\sum_{j\neq 1}\left[(\varepsilon_j^{\star\prime})^2 - (\varepsilon_j^{\prime})^2\right]}{\sum_{k\neq 1}\left[(\widetilde{\varepsilon}_k^{\prime})^2 - (\varepsilon_k^{\prime})^2\right]}.$$

Theorem 11. *Let (M,g) be a Riemannian manifold, $p \in \mathcal{U}_R$ and $\pi \subset T_pM$ Weyl curvature-dependent with respect to a tangent plane $\overline{\pi} \subset T_pM$. Under the above calibration of infinitesimal parallel translation by associated infinitesimal rotation, the Weyl curvature of Deszcz $L_C(p,\pi,\overline{\pi})$ can be expressed in terms of the lengths of closing sides in squaroids of Levi–Civita as follows:*

$$L_C(p,\pi,\overline{\pi}) \approx \frac{(\varepsilon_2^{\star\prime})^2 - (\varepsilon_2^{\prime})^2 - \frac{1}{n-2}\left[\sum_{j\neq 1}\left((\varepsilon_j^{\star\prime})^2 - (\varepsilon_j^{\prime})^2\right) + \sum_{k\neq 2}\left((\varepsilon_k^{\star\prime})^2 - (\varepsilon_k^{\prime})^2\right)\right]}{(\widetilde{\varepsilon}_2^{\prime})^2 - (\varepsilon_2^{\prime})^2}.$$

Remarks 12. If a manifold (M,g) is pseudo-symmetric, then it is automatically Ricci pseudo-symmetric as well as Weyl pseudo-symmetric, but the converse statements are not true in general. The warped products of a 1-dimensional manifold and a non pseudo-symmetric Einstein manifold of dimension ≥ 3, are non pseudo-symmetric, Ricci pseudo-symmetric manifolds. All Cartan hypersurfaces in the spheres $\mathbb{S}^{n+1}$ with $n = 6, 12$ or 24 are non pseudo-symmetric, Ricci pseudo-symmetric manifolds. The warped products of Riemannian spheres of dimension

≥ 2 with Einstein spaces of dimension ≥ 4 are non conformally flat, non pseudo-symmetric, non Einstein, but Ricci pseudo-symmetric manifolds. Examples of non pseudo-symmetric, non conformally flat, but Weyl pseudo-symmetric Riemannian manifolds were obtained in [5] by applying suitable conformal deformations on a non semi-symmetric, non conformally flat, but Weyl semi-symmetric, Riemannian 4-dimensional manifold given by Derdziński in [4]. Also, of course, every conformally flat manifold of dimension ≥ 4 is Weyl pseudo-symmetric, but there do exist conformally flat manifolds of dimension ≥ 4 which are not pseudo-symmetric. For more information on various pseudo-symmetries, see e.g. [1], [3], [5], [7], [11], [12].

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