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On a special diophantine equation $a\binom{x}{n} = by^r + c$

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Abstract. Let a, b, c be given integers. In this paper, we shall prove that apart from $n = 4$, $c/a = -1/24$ or $3/128$, $r = 2$ and b/a is not a square, the diophantine equation $a\binom{x}{n} = by^r + c$ has only finitely many solutions, and all these solutions can be effectively bounded in terms of a, b, c and n.

In 1966, Avanesov [1] has proved that all the positive integral solutions of the diophantine equation

$$\binom{x}{3} = \binom{y}{2}$$

are given by $(x, y) = (3, 2)$, $(5, 5)$, $(10, 16)$, $(22, 56)$, $(36, 120)$.

In 1988, P. Kiss [2] has proved that if p is a given odd prime, then the diophantine equation

$$\binom{x}{p} = \binom{y}{2}$$

has only finitely many positive integral solutions, and all these can be effectively determined.

In 1991, Brindza [3], by using Baker's effective method, has proved that for any given $n \in N$ with $n \geq 3$, the hyperelliptic equation

$$\binom{x}{n} = \binom{y}{2}$$

has only finitely many positive integral solutions, and all these can be effectively computed.

In this paper, we shall discuss the following more general diophantine equation

$$a\binom{x}{n} = by^r + c \tag{1}$$

where $a, b, c \geq 3$ are given integers. We have:

Theorem 1. *Let $a \neq 0$, $b \neq 0$, $c, n \geq 3$ be given integers. Then apart from $n = 4$, $c/a = -1/24$ or $3/128$, $r = 2$ and b/a is not a square, all rational integer solutions x, y, r of the equation (1) with $x, y > 1$, $r > 1$ satisfy*

$$\max(|x|, y, r) < C_1$$

where C_1 is an effectively computable constant depending only on a, b, c and n.

Obviously, $a = 8$, $b = 1$, $c = -1$, $r = 2$ give the result of Brindza [3]. First, we give the following lemmas.

Lemma 1. (1976, Schinzel and Tijdeman). *Let $f(x) \in Z[x]$ be a polynomial with at least two distinct roots. If $b \neq 0$, $m \geq 0, x, y$ with $|y| > 1$ satisfy the equation $f(x) = by^m$, then $m < C_2$, where C_2 is an effectively computable constant depending only on b and f.*

Lemma 2. (1984, Brindza). *Let $f(x) = a_0(x-\alpha_1)^{\gamma_1} \dots (x-\alpha_n)^{\gamma_n} \in Z[x]$, $m \geq 2$, $n \geq 2$ and let $q_i = m/(m, r_i)$ for $i = 1, \dots, n$. Suppose that $(q_1, \dots, q_n)$ is not a permutation of $(q, 1, \dots, 1)$ or $(2, 2, 1, \dots, 1)$ and $y, z \in Z$ satisfy the equation $f(x) = by^m$. Then $\max(|x|, |y|) < C_3$, where C_3 is an effectively computable constant depending only on b, m and f.*

Lemma 3. (1975, Baker). *Let $m = 2$, $f(x) \in Z[x]$ be a polynomial with at least three simple roots. Then there exists an effectively computable constant C_4 depending only on b and f such that for any $x, y \in Z$ satisfying the equation $f(x) = by^m$, we have $\max(|x|, |y|) < C_4$*

Remark. For the proof of Lemmas 1,2 and 3, we refer to Th.10.2, Th.8.3 and Th.6.2 of [4], respectively.

Lemma 4. *Let $k > 1$ be an integer. Then*

(i) $\binom{2k}{k} > 2^{2k}/2k$

(ii) $\binom{2k+1}{k} > 2^{2k+1}/(2k+1)$.

Proof. (i) From $(1+1)^{2k} = 1 + \binom{2k}{1} + \dots + \binom{2k}{k} + \dots + \binom{2k}{2k-1} + 1 < 2k\binom{2k}{k}$. We get

$$\binom{2k}{k} > 2^{2k}/2k$$

(ii) Similarly $(1+1)^{2k+1} = 1 + \binom{2k+1}{1} + \dots + \binom{2k+1}{k} + \binom{2k+1}{k+1} + \dots + \binom{2k+1}{2k} + 1 < (2k+1)\binom{2k+1}{k}$ implies

$$\binom{2k+1}{k} > 2^{2k+1}/(2k+1)\,.$$

Put

$$f_n(x) = x(x-1)\dots(x(n-1)) - \frac{c}{a}n! \tag{2}$$

It follows from (1) that

$$f_n(x) = \frac{b}{a}n!\,y^r \tag{3}$$

From Lemma 1, if $f_n(x)$ has at least two simple roots, then r is effectively bounded in terms of a, b, c and n; From Lemma 2, if $r \geq 2$ and $(q_1, \dots, q_n)$ is not a permutation of $(q, 1, \dots, 1)$ or $(2, 2, 1, \dots, 1)$, then the equation (3) has only finitely many solutions, and all these can be effectively computed; From Lemma 3, if $r = 2$, and $f(x)$ has at least three simple roots, then (3) has only finitely many solutions, and all these can be effectively determined.

From the discussions above, if we can prove that $f_n(x)$ has at least three simple roots when $a \neq 0$, $c \in Z$, then (3), so (1) has only finitely many solutions, and all these can be effectively determined.

On the simple roots of $f_n(x)$, we give the following theorem:

Theorem 2. *Let $a \neq 0$, $c \neq 0$ be rational integers and $n \geq 3$. Then apart from $f_n(x) = x(x-1)(x-2)(x-3)+1$ and $x(x-1)(x-2)(x-3) - \frac{9}{16}$, $f_n(x)$ has at least three simple roots.*

Proof. We have $f_n(0) = f_n(1) = \dots = f_n(n-1) = -\frac{c}{a}n!$. It is well known that there exist $x_i \in (i-1, i)$, $i = 1, \cdots, n-1$ with $f'_n(x_i) = 0$ by Rolle's Theorem. Since $\deg f'_n(x) = n-1$,

$$f'_n(x) = (x - x_1)\cdots(x - x_{n-1})$$

It is easily seen that the roots of $f'_n(x)$ are real and simple, so the multiple roots of $f_n(x)$ are twofold roots and the imaginary roots of $f_n(x)$ are simple.

Now we consider the following two cases.

Case I. $c/a > 0$.

(i) If $n = 2k+1$ is odd and $x > n-1$, then $f(x)$ is a monotone increasing function and $f_n(n-1) < 0$, $f_n(+\infty) = +\infty$, therefore $f_n(x)$ has a simple root $x_1^* > n-1$.

It is easily seen that $f_n(x)$ reaches its maximal values at $x = x_1, x_3, \dots, x_{2k-1}$. If $f(x_{2j-1}) > 0$, $j \in \{1, \dots, k\}$, then $f_n(x)$ has exactly two real simple roots in the interval $(2j-2, 2j-1)$; If $f(x_{2j-1}) = 0$, $j \in \{1, \dots, k\}$, then x_{2j-1} is the twofold root of $f_n(x)$; And if $f_n(x_{2j-1}) < 0$, then $f_n(x)$ has no real roots in the interval $(2j-2, 2j-1)$, and it is easily seen from the above discussions that we can assume x_{2j-1} as corresponding to two conjugate imaginary simple roots of $f_n(x)$ in this case.

Thus we know from above that if we can prove that $f_n(x_1)$, $f_n(x_3), \ldots,$ $f_n(x_{2k-1})$ are not all zero, then $f_n(x)$ has at least three simple roots. Define

$$f_n^*(x) = f_n(x) + \frac{c}{a}n! = x(x-1)(x-2)\ldots(x-n+1)\,. \tag{4}$$

We know from the above discussions that $f_n(x_1)$ resp. $f_n^*(x_1)$ is the largest of the $f_n(x)$ (resp. the $f_n^*(x)$) in the interval (0,1). Then

$$\begin{aligned}(5)\quad f_n^*(x_1) &= x_1(x_1-1)\ldots(x_1-2k) \geq \\ &\geq \frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-2k\right) = \frac{(4k)!}{2^{4k+1}\cdot(2k)!}\end{aligned}$$

If k is even, then $f_n^*(x_{k+1}) = x_{k+1}(x_{k+1}-1)\ldots(x_{k+1}-k-1)\ldots(x_{k+1}-2k)$ If k is odd, then $f_n^*(x_k) = x_k(x_k-1)\ldots(x_k-k)\ldots(x_k-2k)$. Below we shall prove that if $n > 3$, then

$$f_n^*(x_1) > \begin{cases} f_n^*(x_{k+1}), & \text{if } k \text{ is even.} \\ f_n^*(x_k), & \text{if } k \text{ is odd.} \end{cases}$$

It follows from Lemma 4 that

$$f_n^*(x_1) > \frac{2^{2k}\cdot k!\cdot k!}{2\cdot(4k)\cdot(2k)}$$

Hence, if $2^{2k} > 4^2k^2(k+1)$, i.e. $k > 4$, then

$$f_n^*(x_1) > \frac{2^{2k}\cdot k!\cdot k!}{2\cdot(4k)\cdot(2k)} > \frac{k!(k+1)!}{4} \geq \begin{cases} f_n^*(x_{k+1}), & \text{if } k \text{ is even,} \\ f_n^*(x_k), & \text{if } k \text{ is odd.} \end{cases}$$

If $k=4$, then $f_9^*(x_1) > f_9^*(x_5)$, since $\frac{16!}{2^{17}.8!} > \frac{4!\cdot 5!}{4}$; if $k=3$, then $f_7^*(x_1) > f_n^*(x_3)$, since $\frac{12!}{2^{13}.6!} > \frac{3!\cdot 4!}{4}$, if $k=2$, then $f_5^*(x_1) > f_5^*(x_3)$, since $\frac{8!}{2^9\cdot 4!} > \frac{2!\cdot 3!}{4}$; if $k=1$, then $n=3$, since $f_3^*(x) = x(x-1)(x-2)$; then $x_1 = \frac{3-\sqrt{3}}{3}$, $x_2 = \frac{3+\sqrt{3}}{3}$, $f_3^*(x_1) = \frac{2\sqrt{3}}{9}$ is not rational number, so $f_3(x_1) \neq 0$. Which proves that $f_n(x)$ has at least three simple roots in this case.

(ii) Let $n = 2k$ be even. Since $f_n(x)$ is a monotone decreasing function as $x < 0$, and a monotone increasing function as $x > n-1$, and $f_n(0) = f_n(n-1) = -\frac{c}{a}n!$, $f_n(-\infty) = f_n(+\infty) = +\infty$, in this case $f_n(x)$ has two simple roots $x_1^* < 0$, $x_2^* > n-1$.

It is easily seen that $f_n(x)$, so $f_n^*(x)$ reaches its maximal values at $x = x_2, x_4, \dots, x_{2k-2}$, since

$$f_n^*(x_2) > \frac{3}{2} \cdot \frac{1}{2} \cdot \left(-\frac{1}{2}\right)\left(\frac{3}{2} - 3\right) \cdots \left(\frac{3}{2} - 2k + 1\right) =$$

$$= \frac{3 \cdot 1 \cdot 3 \cdot \dots \cdot (4k-5)}{2^{2k}} = \frac{3 \cdot (4k-4)!}{2^{4k-2} \cdot (2k-2)!}$$

If k is even, then $f_n^*(x_k) \le \frac{k! \cdot k!}{4}$, if k is odd, then $f_n^*(x_{k+1}) \le \frac{(k-1)! \cdot (k+1)!}{4}$ It follows from Lemma 4 that

$$f_n^*(x_2) > \frac{3 \cdot 2^{2k-2} \cdot (k-1)! \cdot (k-1)!}{4 \cdot (4k-2)(2k-2)}$$

Hence, if $3 \cdot 2^{2k-5} > (k-1)^2 k(k+1)$, i.e. $k \ge 8$, then

$$f_n^*(x_2) > \begin{cases} f_n^*(x_k), & \text{if } k \text{ is even,} \\ f_n^*(x_{k+1}), & \text{if } k \text{ is odd.} \end{cases}$$

If $k = 7$, then $f_{14}^*(x_2) > f_{14}^*(x_8)$, since $\frac{3 \cdot 24!}{2^{26} \cdot 12!} > \frac{6! \cdot 8!}{4}$; if $k = 6$, then $f_{12}^*(x_2) > f_{12}^*(x_6)$, since $\frac{3 \cdot 20!}{2^{22} \cdot 10!} > \frac{6! \cdot 6!}{4}$; if $k = 5$, then $f_{10}^*(x_2) > f_{10}^*(x_6)$, since $\frac{3 \cdot 16!}{2^{18} \cdot 8!} > \frac{4! \cdot 6!}{4}$; if $k = 4$, put $u = x - \frac{7}{2}$, then

$$f_8^*(u) = \left(u + \frac{7}{2}\right)\left(u + \frac{5}{2}\right)\left(u + \frac{3}{2}\right)\left(u + \frac{1}{2}\right)\left(u - \frac{1}{2}\right)\left(u - \frac{3}{2}\right) \cdot$$

$$\cdot \left(u - \frac{5}{2}\right)\left(u - \frac{7}{2}\right)$$

It is easy to prove that $f_8'(u)$ has a root $u = 0$, this implies that $f_8'(x)$ has a solution $x = \frac{7}{2} \in (3,4)$, and so $x_4 = \frac{7}{2}$. Then

$$f_8^*(x_2) \ge \frac{3}{2} \cdot \frac{1}{2}\left(-\frac{1}{2}\right) \cdot \left(-\frac{3}{2}\right) \cdot \left(-\frac{5}{2}\right) \cdot \left(-\frac{7}{2}\right) \cdot \left(-\frac{9}{2}\right) \cdot \left(-\frac{11}{2}\right) >$$

$$> \left(\frac{7}{2}\right)^2 \cdot \left(\frac{5}{2}\right)^2 \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 = f_8^*(x_4)$$

If $k = 3$, then $f_6^*(x) = (x^2 - 5x)(x^2 - 5x + 4)(x^2 - 5x + 6)$ and $f_6'(x) = (2x - 5)(3(x^2 - 5x) + (x^2 - 5x) + 26)$. Hence $x_3 = \frac{5}{2}$, and x_2 is the root of $x^2 - 5x + \frac{10 + 2\sqrt{7}}{3} = 0$ or $x^2 - 5x + \frac{10 - 2\sqrt{7}}{3} = 0$. So $f_6^*(x_2) = \frac{10 + 2\sqrt{7}}{3} \cdot \frac{-2 + 2\sqrt{7}}{3} \cdot \frac{-8 + 2\sqrt{7}}{3} \cdot (-1)$ is not a rational number, hence $f_6(x_2) \ne 0$. If $k = 2$, then $f_n^*(x) = f_4^*(x) = x(x-1)(x-2)(x-3) = (x^2 - 3x)^2 + 2(x^2 - 3x)$, $x_2 = \frac{3}{2}$, since $f_4^*\left(\frac{3}{2}\right) = 9/16$, so if $\frac{c}{a} n! = 9/16$, that is $\frac{c}{a} = 3/128$, $f_4(x) =$

$(x-\frac{3}{2})^2(x^2-3x+\frac{1}{4})$ has only two simple roots and if b/a is not a square, and $(2x-3)^2-8=b/ay^2$ has a solution, then $f_4(x)=by^2$ has infinitely many solutions.

Case II. $c/a<0$.

(i) If $n=2k+1$ is odd, then $f_n^*(x)$ has a simple root x_1^* with $x_1^*<0$, and $f_n^*(x)$ reaches its minimal values at $x=x_2,x_4,\dots,x_{2k}$, since $|f_n^*(x_{2k})|>\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\dots\cdot\frac{4k-1}{2}=\frac{(4k)!}{2^{4k+1}\cdot(2k)!}$.
All the remaining cases are similar to the case of $c/a>0$, and $n=2k+1$.

(ii) If $n=2k$ is even, then $f_n^*(x)$ reaches its minimal values at $x=x_1,x_3,\dots,x_{2k-1}$. Put $x=u+\frac{2k-1}{2}$, then it is easily seen that $f_n(x_i)=f_n(x_{2k-i})$ for $i=1,\dots,k$. Therefore, if we can prove that

$$f_n^*(x_1)<\begin{cases} f_n^*(x_{k-2}), & \text{if } k \text{ is odd,}\\ f_n^*(x_{k-1}), & \text{if } k \text{ is even,}\end{cases}$$

then $f_n(x)$ has at least four simple roots, since

$$f_n^*(x_1)<-\frac{1}{2}\cdot\frac{1}{2}\cdot(2-\frac{1}{2})\cdot\dots\cdot(2k-1-\frac{1}{2})=-\frac{(4k-2)!}{2^{4k-1}\cdot(2k-1)!}$$

$$0>f_n^*(x_{k-2})>-\frac{(k-2)!\,(k+2)!}{4},\quad 0>f_n^*(x_{k-1})\geq\frac{(k-1)!\,(k+1)!}{4}.$$

It follows from Lemma 4 that

$$|f_n^*(x_1)|>\frac{2^{2k-1}\cdot k!\cdot(k-1)!}{2\cdot(4k-2)\cdot(2k-1)}.$$

If $2^{2k-1}(k-1)>(2k-1)!(k+1)(k+2)$, i.e. $k\geq 7$, then

$$|f_n^*(x_1)|>|f_n^*(x_{k-2})| \quad\text{or}\quad |f_n^*(x_{k-1})|.$$

If $k=6$, then $|f_n^*(x_1)|>|f_n^*(x_5)|$, since $\frac{22!}{2^{23}\cdot 11!}>\frac{5!\cdot 7!}{4}$; if $k=5$, then $|f_n^*(x_1)|>|f_n^*(x_3)|$, since $\frac{18!}{2^{19}\cdot 9!}>\frac{3!\cdot 7!}{4}$; if $k=4$, then $|f_n^*(x_1)|>|f_n^*(x_3)|$, since $\frac{14!}{2^{15}\cdot 7!}>\frac{3!\cdot 5!}{4}$; if $k=3$, then $n=6$, this case is similar to the case of $c/a>0$ and $n=6$, $f_6^*(x_1)$, $f_6^*(x_3)$ are not rational numbers, and $f_6(x_1)\neq 0$, $f_6(x_3)\neq 0$. If $k=2$, then $n=4$, $f_4^*(x)=(x^2-3x)^2+2(x^2-3x)$, $x_1=\frac{3-\sqrt{5}}{2}$, $f_4^*(x_1)=-1$. Hence if $\frac{c}{a}n!=-1$. i.e., $c/a=-1/24$, then $f_4(x)=(x^2-3x+1)^2$. It is easily seen that if $\frac{b}{a}n!=a_1^2$, and $x^2-3x+1\equiv 0 \pmod{a_1^*}$ (here a_1^* is the numberator p of a_1 as a_1 is represented by p/q, $(p,q)=1$, $p,q\in Z$) has a solution, then $f_4(x)=n!\,y^2$ has infinitely many solutions. This completes the proof of Theorem 2.

Proof of Theorem 1. It follows from Theorem 2 that apart from the two cases described in Theorem 1, $f(x)$ has at least three simple roots. Then

$$\max(|x|, y, r) < C_1(a, b, c, n),$$

where $C_1(a, b, c, n)$ is an effectively computable constant depending only on a, b, c and n. This completes the proof of Theorem 1.

Remarks. It is easily seen from the proof of Theorem 2 that if a, b, c are given algebraic integers, $K = Q(a, b, c)$, then apart from $n = 4$, $c/a = -1/24$ or $3/128$; $n = 3$, $c/a = \pm\frac{\sqrt{3}}{108}$; $n = 6$, $c/a = -\frac{1}{6!} \cdot \frac{10-2\sqrt{7}}{3} \cdot \frac{2+2\sqrt{7}}{3} \cdot \frac{8+2\sqrt{7}}{3}$ or $c/a = \frac{1}{6!} \cdot \frac{10+2\sqrt{7}}{3} \cdot \frac{2\sqrt{7}-2}{3} \cdot \frac{8-2\sqrt{7}}{3}$, $f_n(x) = \binom{x}{n} - \frac{c}{a}$, has at least three simple roots. Then there are only finitely many $x, y \in K$ satisfying the equation

$$a\binom{x}{n} = by^r + c, \quad n \geq 3$$

and all these can be effectively determined.

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